A Class of Graphs of f -Class 1*

Xia Zhang** and Guizhen Liu

School of Mathematics and System Science, Shandong University, 250100, P.R. China
xzhang@math.sdu.edu.cn

Abstract. An f -coloring of a graph G is an edge-coloring of G such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f -color G is called the f -chromatic index of G , and denoted by $\chi'_f(G)$. Any simple graph G has f -chromatic index equal to $\Delta_f(G)$ or $\Delta_f(G) + 1$, where $\Delta_f(G) = \max_{v \in V(G)} \{\lceil \frac{d(v)}{f(v)} \rceil\}$. If $\chi'_f(G) = \Delta_f(G)$, then G is of f -class 1; otherwise G is of f -class 2. In this paper, we show that if $f(v)$ is positive and even for all $v \in V_0^*(G) \cup N_G(V_0^*(G))$, then G is of f -class 1, where $V_0^*(G) = \{v \in V(G) : \frac{d(v)}{f(v)} = \Delta_f(G)\}$ and $N_G(V_0^*(G)) = \{v \in V(G) : uv \in E(G), u \in V_0^*(G)\}$. This result improves the simple graph version of a result of Hakimi and Kariv [4].

Keywords: Edge-coloring, simple graph, f -coloring, f -chromatic index.

1 Introduction

Our terminologies and notations will be standard, except where indicated. The reader is referred to [1] for the undefined terms. Throughout this paper, a *graph* G allows multiple edges but no loops and has a finite nonempty vertex set $V(G)$ and a finite edge set $E(G)$. If G has neither loops nor multiple edges, we will call G a *simple graph*.

An *edge-coloring* of G is an assignment of colors to all the edges of G . A *proper edge-coloring* of G is an edge-coloring of G such that no two adjacent edges are assigned the same color. The minimum number of colors for which such a proper edge-coloring of G exists is called the *chromatic index* of G , and denoted by $\chi'(G)$. Hakimi and Kariv [4] generalized the proper edge-coloring and obtained many interesting results. Let G be a graph and f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An f -coloring of G is an edge-coloring of G such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f -color G is called the f -chromatic index of G , and denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V(G)$, the f -coloring problem, which is to determine $\chi'_f(G)$ of a given graph G , is reduced to the proper edge-coloring problem.

* This research is supported by NSFC(10471078, 60673047) and RSDP(20040422004) and NSF of Hebei(A2007000002) of China.

** Corresponding author.

f -colorings have many applications in scheduling problems such as the file transfer problem in a computer network (see [2, 3, 6]). Since the proper edge-coloring problem is NP-complete [5], the f -coloring problem is also NP-complete. Hakimi and Kariv [4] studied the f -coloring problem and obtained some upper bounds on $\chi'_f(G)$. Nakano et al. [8] obtained another upper bound on $\chi'_f(G)$.

In the proper edge-coloring, one of the most celebrated results, which is due to Vizing [9], is that $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ for any simple graph G . This result naturally partitions all simple graphs into two classes. We say that simple graph G is of *class 1* if $\chi'(G) = \Delta(G)$, and of *class 2* otherwise. Significantly, we have a similar result to simple graphs on f -colorings. We define

$$\Delta_f(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{f(v)} \right\rceil,$$

where $\lceil \frac{d(v)}{f(v)} \rceil$ is the smallest integer not smaller than $\frac{d(v)}{f(v)}$. It is trivial that $\chi'_f(G) \geq \Delta_f(G)$.

Lemma 1. [10, 11] *Let G be a simple graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \left\lceil \frac{d(v) + 1}{f(v)} \right\rceil \leq \Delta_f(G) + 1.$$

For the sake of self-completeness, a proof of Lemma 1 can be obtained by Lemma 3 later. Similarly, we can classify simple graphs into two classes according to their f -chromatic indices. We say that simple graph G is of *f -class 1* if $\chi'_f(G) = \Delta_f(G)$, and of *f -class 2* otherwise. Zhang and Liu [10, 11], Zhang, Wang and Liu [12] studied the classification of complete graphs, simple regular graphs and some other special classes of simple graphs on f -colorings. Liu, Hou and Cai [7] studied the properties of f -critical graphs (i.e. connected simple graphs G of f -class 2 satisfying that $\chi'_f(G - e) < \chi'_f(G)$ for any edge $e \in E(G)$).

In Section 2 we present a useful lemma, by which we can easily deduce two results of Hakimi and Kariv [4]. In Section 3, we give a new sufficient condition for a simple graph to be of f -class 1, which generalizes the simple graph version of a result of Hakimi and Kariv [4]. In Section 4 we conclude that this condition is best in a sense, and briefly discuss some problems for further research.

2 A Useful Lemma

Let G be a graph, $Q \subseteq E(G)$ and $Q \neq \emptyset$. We use $G[Q]$ to denote the subgraph of G induced by Q . A *partial edge-coloring* of G is an edge-coloring of a subgraph $G[Q]$ of G . When $Q = E(G)$, a partial edge-coloring of G is exactly an edge-coloring of G . Let C denote the set of colors available to color the edges of a graph G . An edge colored with color $\alpha \in C$ is called an α -edge. We denote by $|\alpha(v)|$ the number of α -edges of G incident with the vertex $v \in V(G)$. Given a partial edge-coloring of G with the colors in C , a vertex v of G is called *f -proper* if $|c(v)| \leq f(v)$ for each $c \in C$. All vertices of G are *f -proper* if and only if G is

given a partial f -coloring. Note that any subgraph G' of G has $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

A *circuit* is a connected graph in which each vertex has even degree. Denote $|E(G)|$ by $\varepsilon(G)$. A circuit G is *even*, or *odd*, if $\varepsilon(G)$ is even, or odd, respectively. A *walk* W is a sequence of distinct edges $v_0v_1, v_1v_2, \dots, v_{k-1}v_k$, where the vertices $v_0, v_1, \dots, v_k$ are not necessarily distinct. An *Eulerian circuit* of a graph G is a closed walk (i.e. $v_0 = v_k$) that traverses every edge in G . It is well known that every circuit has an Eulerian circuit. Given a partial edge-coloring of graph G , we call a connected subgraph H of G an *obstruction* (to a partial f -coloring), if $\varepsilon(H)$ is odd and $d_H(v) = 2f(v)$ for each $v \in V(H)$. Clearly, any obstruction is an odd circuit. Denote by $G(v; \alpha, \beta)$ the component containing v of the subgraph of G induced by all α -edges and all β -edges.

Lemma 2. *Let G be a graph and $w \in V(G)$. Let $C = \{c_1, c_2, \dots, c_{\Delta_f(G)}\}$. Suppose that G is given a partial edge-coloring with the colors in C such that all vertices but w are f -proper. Let $\alpha, \beta \in C$ such that $|\alpha(w)| = \max_{c \in C} \{|c(w)|\}$ and $|\beta(w)| = \min_{c \in C} \{|c(w)|\}$. Also, suppose that the vertex w has*

$$\begin{cases} |\alpha(w)| = f(w) + 1, \\ |c(w)| \leq f(w) \end{cases} \quad \text{for each } c \in C \text{ and } c \neq \alpha.$$

If $H = G(w; \alpha, \beta)$ is not an obstruction, then we can recolor H such that all vertices of G are f -proper.

Proof. By the choice of β , we claim that $|\beta(w)| \leq f(w) - 1$. Otherwise, the number of the edges colored and incident with w is $(\Delta_f(G) - 1)f(w) + f(w) + 1$. Clearly, $(\Delta_f(G) - 1)f(w) + f(w) + 1 > \Delta_f(G)f(w) \geq d(w)$. A contradiction.

Let m' represent the set m with respect to the new partial edge-coloring of G after recoloring H . We will recolor H with α and β . Obviously, all vertices in $V(G) \setminus V(H)$ remain f -proper, and the edges colored with colors other than α and β remain unchanged.

Next, we consider three cases as follows.

Case 1. H is not a circuit.

Then it has even vertices with odd degree. Pair off the vertices with odd degree, and join the two vertices of each pair by an extra edge. The graph H^* so formed is a circuit, and then it has an Eulerian circuit. Starting with an extra edge and following an Eulerian circuit, we color the edges of H^* with α and β alternately. Restrict H^* to H . Obviously, we have $||(\alpha(v))'| - |(\beta(v))'|| \leq 1$ for all $v \in V(H)$. $||(\alpha(v))'| - |(\beta(v))'|| = 1$ if and only if $d_H(v)$ is odd. So $||(\alpha(v))'| - |(\beta(v))'|| \leq ||\alpha(v)| - |\beta(v)||$ for all $v \in V(H)$. Then we have $\max\{|(\alpha(v))'|, |(\beta(v))'|\} \leq \max\{|\alpha(v)|, |\beta(v)|\}$ for all $v \in V(H)$. So all vertices but w are still f -proper. For vertex w , since $|\alpha(w)| - |\beta(w)| \geq 2$, we have $||(\alpha(w))'| - |(\beta(w))'|| \leq 1 < |\alpha(w)| - |\beta(w)|$. Then $\max\{|(\alpha(w))'|, |(\beta(w))'|\} < |(\alpha(w))| = f(w) + 1$. So w is f -proper now.

Case 2. H is an even circuit.

Then it has an Eulerian circuit. Starting at w and following an Eulerian circuit, color the edges of H with α and β alternately. Now we have $||(\alpha(v))'| - |(\beta(v))'||$,

i.e. $|(\alpha(v))'| - |(\beta(v))'| = 0$, for all $v \in V(H)$. Note that $|\alpha(w)| - |\beta(w)| \geq 2$. we can see that all vertices of G are f -proper now.

Case 3. H is an odd circuit, but not an obstruction.

By the assumption about H , there is at least one vertex in it, say v_1 , at which $|\alpha(v_1)| + |\beta(v_1)| \leq 2f(v_1) - 2$. Find an Eulerian circuit and color the edges in it with α and β alternately starting at v_1 . Then we have $|(\alpha(v_1))'| - |(\beta(v_1))'| = 2$ and $|(\alpha(v))'| = |(\beta(v))'|$ for each $v \neq v_1$ in $V(H)$. For vertex v_1 , $\max\{|(\alpha(v_1))'|, |(\beta(v_1))'|\} \leq f(v_1)$. Then all vertices of G are f -proper now. $\square$

We define $M(v) = \{c \in C : |c(v)| < f(v)\}$ and $E_j = \{e_1, e_2, \dots, e_j\}$ ($1 \leq j \leq \varepsilon(G)$). By Lemma 2, we have the following theorem.

Theorem 1. *Let G be a graph. If there is no obstruction in G , then $\chi'_f(G) = \Delta_f(G)$.*

Proof. To see this point, we present an f -coloring of G with colors in $C = \{c_1, c_2, \dots, c_{\Delta_f(G)}\}$ by coloring the edges of G in an arbitrary order $e_1, e_2, \dots, e_{\varepsilon(G)}$. For the first edge $e_1 = w_1v_1$, there are $M(w_1) = M(v_1) = C$. Thus $G[E_1]$ can be f -colored with an arbitrary color in C . Suppose that $G[E_{j-1}]$ is f -colored so far, $2 \leq j \leq \varepsilon(G)$. When color the j th edge $e_j = w_jv_j$, we consider two cases. If $M(w_j) \cap M(v_j) \neq \emptyset$, color e_j with a color in $M(w_j) \cap M(v_j)$, and then $G[E_j]$ is f -colored. If $M(w_j) \cap M(v_j) = \emptyset$, then color e_j with a color $\alpha \in M(v_j)$. Now, all vertices but w_j are f -proper. In particular, for vertex w_j , we have $|\alpha(w_j)| = f(w_j) + 1$ and $|c(w_j)| \leq f(w_j)$ for each $c \in C$ and $c \neq \alpha$. By Lemma 2, we can obtain an f -coloring of $G[E_j]$ with colors in C , $2 \leq j \leq \varepsilon(G)$. So $\chi'_f(G) = \Delta_f(G)$. $\square$

Obviously, a bipartite graph contains no odd circuit. A graph G , in which $f(v)$ is positive and even for all $v \in V(G)$, contains no obstruction. The reason is that, for any circuit H of G with $d_H(v) = 2f(v)$ for each $v \in V(H)$, $\varepsilon(H) = (\sum_{v \in V(H)} 2f(v))/2 = \sum_{v \in V(H)} f(v)$ is even. Therefore, two results of Hakimi and Kariv [4] are immediately derived from Theorem 1.

Corollary 2. [4] *Let G be a bipartite graph. Then $\chi'_f(G) = \Delta_f(G)$.*

Corollary 3. [4] *Let G be a graph. If $f(v)$ is positive and even for all $v \in V(G)$, then $\chi'_f(G) = \Delta_f(G)$.*

3 Main Results

We denote the neighbor set of vertex set S in G by $N_G(S)$ for $S \subseteq V(G)$. Let

$$V_0^*(G) = \{v \in V(G) : \frac{d(v)}{f(v)} = \Delta_f(G)\}.$$

Then $N_G(V_0^*(G)) = \{v \in V(G) : uv \in E(G), u \in V_0^*(G)\}$. The f -core of a graph G is the subgraph of G induced by the vertices of $V_0^*(G)$ and is denoted by G_{Δ_f} .

There has been great interest in properties of the f -core G_{Δ_f} of a simple graph G , which would guarantee that G is of f -class 1. For example, if G_{Δ_f} is a forest, then G is of f -class 1 [11]. The *quasi- f -core* of a graph G is the subgraph of G induced by the vertices of $V_0^*(G) \cup N_G(V_0^*(G))$. In fact, the f -chromatic index of a simple graph G has very close relations with the quasi- f -core of G . Later, we can see that if the quasi- f -core of G can be f -colored with $r(\geq \Delta_f(G))$ colors, then the following Lemma 3 guarantees that G can be f -colored with r colors.

Before proceeding, we need some new notations and definitions.

We define $m(v, \alpha) = f(v) - |\alpha(v)|$ for each $v \in V(G)$ and each $\alpha \in C$. Clearly, given an edge-coloring of G with the colors in C , G is f -colored if and only if $m(v, \alpha) \geq 0$ for each $v \in V(G)$ and each $\alpha \in C$. Color α is *available* at vertex v if $m(v, \alpha) \geq 1$. Obviously $M(v) = \{\alpha \in C : m(v, \alpha) \geq 1\}$ for each $v \in V(G)$. Walk W is often denoted simply by $v_0v_1 \dots v_k$. We call v_0 the *start vertex* of W and v_k the *end vertex*. The *length* of W is the number k of edges in W , and denoted by $|W|$.

We now describe two tools commonly used for f -colorings of simple graphs.

(1) *ab*-alternating walks

For two distinct colors $a, b \in C$, a walk W of length one or more is called an *ab*-alternating walk if W satisfies the following conditions:

- (a) The edges of W are colored alternately with a and b , and the first edge of W is colored with b ;
- (b) $m(v_0, a) \geq 1$ if $v_0 \neq v_k$,
 $m(v_0, a) \geq 2$ if $v_0 = v_k$ and $|W|$ is odd;
- (c) $m(v_k, b) \geq 1$ if $v_0 \neq v_k$ and $|W|$ is even,
 $m(v_k, a) \geq 1$ if $v_0 \neq v_k$ and $|W|$ is odd.

Note that any closed walk W of even length whose edges are colored with a and b alternately is an *ab*-alternating walk. If G is given a partial f -coloring and W is an *ab*-alternating walk in G , then interchanging the colors a and b of the edges in walk W yields another partial f -coloring of G . This operation is called *switching* W . When W is switched, $m(v_i, a)$ and $m(v_i, b)$ remain as they were if $i \neq 0, k$, while $m(v_0, b) \geq 1$ if W is not a closed walk of even length. We denote by $W(a, b; v_0)$ an *ab*-alternating walk which starts with vertex v_0 .

(2) Fan Sequences

Let $e_0 = wv_0$ be an uncolored edge of G . Then a *fan* F is a sequence of distinct edges $e_0, e_1, \dots, e_k$ incident with the vertex w such that there is a sequence of distinct colors $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ satisfying the following conditions (a) and (b), where w is called the *pivot* of F , v_i ($0 \leq i \leq k$) is the end of e_i other than w and $c(e)$ is the color assigned to the edge e .

- (a) $\alpha_i \in M(v_i)$, $0 \leq i \leq k-1$;
- (b) $c(e_i) = \alpha_{i-1}$, $1 \leq i \leq k$.

When G is a simple graph, vertices $v_0, v_1, \dots, v_k$ are distinct. *Shifting a fan* F means to recolor e_i with α_i for each i , $0 \leq i \leq k-1$, and erase the color α_{k-1} of e_k . If G is given a partial f -coloring, then shifting F yields another partial

f -coloring of G in which e_k instead of e_0 is uncolored. A *maximal fan* is the one whose length k cannot be increased.

Lemma 3. *Let C denote the set of colors available to color the edges of a simple graph G . Suppose that $e_0 = wv_0$ is an uncolored edge in G , and graph $G - \{e_0\}$ is f -colored with the colors in C now. If every neighbor v of either w or v_0 has $M(v) \neq \emptyset$, then we can f -color e_0 with a color in C .*

Proof. Without loss of generality, we assume that all neighbors v of w have $M(v) \neq \emptyset$. If $M(w) \cap M(v_0) \neq \emptyset$, clearly e_0 can be f -colored with a color in $M(w) \cap M(v_0)$. Otherwise, construct a fan $F = e_0, e_1, \dots, e_m$ with a sequence of distinct colors $\alpha_0, \alpha_1, \dots, \alpha_{m-1}$. If $M(w) \cap M(v_i) \neq \emptyset$ for some $i \in \{1, 2, \dots, m\}$, then there exists a color $\gamma \in M(w) \cap M(v_i)$ such that γ is available at vertices w and v_i . By shifting the subfan $F' = e_0, e_1, \dots, e_i$ and subsequently coloring e_i with γ , we f -color e_0 with a color in C . Thus one can assume that $M(w) \cap M(v_i) = \emptyset$ for all $i \in \{0, 1, \dots, m\}$. Furthermore, assume that F is a maximal fan. Obviously, F has $M(v_m) \subseteq \{\alpha_0, \alpha_1, \dots, \alpha_{m-1}\}$.

Let $\beta \in M(w)$ and $\alpha_m \in M(v_m)$. With the assumptions above, F does not have a β -edge and has exactly one $\alpha_m (= \alpha_j)$ -edge e_{j+1} , $0 \leq j \leq m-1$. Note that $\alpha_0, \alpha_1, \dots, \alpha_{m-1}$ are all distinct. Thus any $\alpha_m \beta$ -alternating walk $P = W(\alpha_m, \beta; v_m)$ of G passes through none of edges of F except for the α_m -edge e_{j+1} . We consider four cases depending on the end vertex of P .

Case 1. If P ends at $v \in V(G) \setminus \{w, v_j, v_m\}$, then switch P and subsequently shift $F = e_0, e_1, \dots, e_m$. Since β becomes an available color at both ends of the uncolored edge $e_m = wv_m$, we can color e_m with β .

Case 2. If P ends at w , then switch P and subsequently shift the subfan $F' = e_0, e_1, \dots, e_j$. Since $\alpha_m = \alpha_j$ becomes an available color at both ends of the uncolored edge $e_j = wv_j$, we can color e_j with α_m .

Case 3. If P ends at v_j , then switch P and subsequently shift the subfan $F' = e_0, e_1, \dots, e_j$. Since β becomes an available color at both ends of the uncolored edge $e_j = wv_j$, we can color e_j with β .

Case 4. If P ends at v_m , then P is a closed $\alpha_m \beta$ -alternating walk and $|P|$ is odd because $\beta \notin M(v_m)$. Furthermore, $m(v_m, \alpha_m) \geq 2$. Switching P and subsequently shifting the fan $F = e_0, e_1, \dots, e_m$ make β an available color at both ends of the uncolored edge $e_m = wv_m$. We can color e_m with β .

In any case, e_0 can be f -colored with a color in C . $\square$

Let $q(G) = \max_{v \in V(G)} \{\lceil \frac{d(v)+1}{f(v)} \rceil\}$. For any $v \in V(G)$, since $f(v)q(G) > d(v)$, there are always $M(v) \neq \emptyset$. By Lemma 3, we can obtain an f -coloring of G with $q(G)$ colors by coloring the edges of G one by one. Hence Lemma 1 is proved. Similarly, when $V_0^*(G) = \emptyset$, there are also $M(v) \neq \emptyset$ for all $v \in V(G)$. Hence an earlier result of the authors [11] can be derived from Lemma 3.

Theorem 4. [11] *Let G be a simple graph. If $V_0^*(G) = \emptyset$, then G is of f -class 1.*

Holding above result, we only need to consider the situation with $V_0^*(G) \neq \emptyset$. Now we give the main result as follows, which improves Corollary 3 on simple graphs.

Theorem 5. *Let G be a simple graph. Suppose that $V_0^*(G) \neq \emptyset$. If $f(v)$ is positive and even for all $v \in V_0^*(G) \cup N_G(V_0^*(G))$, then G is of f -class 1.*

Proof. We will give an f -coloring of G with $\Delta_f(G)$ colors. Let C denote the set of $\Delta_f(G)$ colors available to color the edges of G .

Here we denote the quasi- f -core of G by G^* . Simply, we write $E(G^*)$ as E^* . First, we color the edges in E^* . By Corollary 3, we know that G^* is of f -class 1. Also, by $\Delta_f(G^*) = \Delta_f(G)$, we have $\chi'_f(G^*) = \Delta_f(G)$. Hence we can f -color G^* by the colors in C .

Let $\overline{E^*} = E(G) \setminus E^*$. Next, we color the edges in $\overline{E^*}$. Clearly, for any edge $e \in \overline{E^*}$, both ends of e are not in $V_0^*(G)$. Furthermore, at least one end of e is not in $N_G(V_0^*(G))$. (Otherwise, $e \in E^*$.) That is to say, for any edge $e \in \overline{E^*}$, there is at least one end of e which is not adjacent to the vertices in $V_0^*(G)$. Note that, for any vertex $v \in V(G) \setminus V_0^*(G)$, we always have $M(v) \neq \emptyset$. Let $e = wu$. Thus every neighbor v of either w or u has $M(v) \neq \emptyset$. e can be f -colored with a color in C by Lemma 3. $\square$

In fact, a more general sufficient condition can be obtained in the similar way. Obviously, it is necessary also.

Theorem 6. *Let G be a simple graph. Suppose that $V_0^*(G) \neq \emptyset$. G is of f -class 1 if and only if the quasi- f -core of G is of f -class 1.*

4 Conclusion and Further Research

Let G and H be disjoint graphs. We use $G + H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H . Let G_1 be a subgraph of G . We use $G - G_1$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(G_1)$. If $k \geq 3$ and $v_0, v_1, \dots, v_{k-1}$ are distinct, then we call a closed walk $v_0v_1 \dots v_{k-1}v_0$ a *cycle* and denote it by C_k .

In this paper, we give a new sufficient condition for a simple graph G to be of f -class 1 based on vertex set $V_0^*(G) \cup N_G(V_0^*(G))$. We attempt to find a more general sufficient condition based on $V_0^*(G)$. However, the condition that $f(v)$ is positive and even for all $v \in V_0^*(G)$ is not sufficient for a simple graph G to be of f -class 1. A counterexample is $G = (K_{14} - C_{14}) \vee K_1$, in which each vertex v with $d(v) = 12$ has $f(v) = 4$ and the unique vertex w with $d(w) = 14$ has $f(w) = 5$. It is easy to see that $\Delta_f(G) = 3$ and all vertices but w belong to $V_0^*(G)$. Let $C = \{c_1, c_2, c_3\}$. We say that G is of f -class 2, for otherwise, there will exist some color $\gamma \in C$ such that the number of edges colored with γ , i.e. $(4 \times 14 + 5)/2$, is not an integer.

In the further research, we may try to find new even conditions on function f for a simple graph to be of f -class 1 based on some other special vertex subsets. Also, we may study that which classes of simple graphs G are of f -class 1 when $f(v)$ is positive and even for all $v \in V_0^*(G)$. In many edge-coloring problems, some special subgraphs such as obstructions in f -colorings are rather difficult to deal with. Therefore, we often seek some techniques to avoid them. Applying even conditions on function f to other edge-coloring problems, e.g. equitable edge-coloring problem, maybe we can obtain some interesting results.

References

1. Bondy, J. A., Murty, U. S. R.: Graph Theory with Applications. Macmillan, London, 1976
2. Choi, H., Hakimi, S.L.: Scheduling file transfers for trees and odd cycles. SIAM J. Comput. **16**(1) (1987) 162-168
3. Coffman, Jr E.G., Garey, M.R., Johnson, D.S., LaPaugh, A.S.: Scheduling file transfers. SIAM J. Comput. **14**(3) (1985) 744-780
4. Hakimi, S.L., Kariv, O.: A generalization of edge-coloring in graphs. J. Graph Theory **10** (1986) 139-154
5. Holyer, I.J.: The NP-completeness of edge-coloring. SIAM J. Comput. **10** (1981) 718-720
6. Krawczyk, H., Kubale, M.: An approximation algorithm for diagnostic test scheduling in multicomputer systems. IEEE Trans. Comput. **C-34** (1985) 869-872
7. Liu, G.Z., Hou, J.F., Cai, J.S.: Some results about f -critical graphs. Networks (accepted)
8. Nakano, S., Nishizeki, T., Saito, N.: On the f -coloring of multigraphs. IEEE Trans. Circuit and Syst. **35**(3) (1988) 345-353
9. Vizing, V.G.: On an estimate of the chromatic class of a p -graph. Discret Analiz **3**, (1964) 25-30 (Russian)
10. Zhang, X., Liu, G.Z.: The classification of complete graphs K_n on f -coloring. J. Applied Mathematics & Computing **19**(1-2) (2005) 127-133.
11. Zhang, X., Liu, G.Z.: Some sufficient conditions for a graph to be C_f 1. Applied Mathematics Letters **19** (2006) 38-44.
12. Zhang, X., Wang, J.H., Liu, G.Z.: The classification of regular graphs on f -colorings. Ars Combinatoria (to appear)