Sharp Bounds for the Oriented Diameters of Interval Graphs and 2-Connected Proper Interval Graphs

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Abstract. The diameter $diam(H)$ of a (directed) graph H is the maximum value computed over the lengths of all shortest (directed) paths in H . Given a bridgeless connected graph G , the oriented diameter $OD(G)$ is given by $OD(G) = \min\{diam(H) : H \text{ is an orientation of } G\}$. In this paper, we show that $OD(G) \leq \lceil \frac{3}{2}diam(G) \rceil + 1$ for every connected bridgeless interval graph G . Further, for every 2-connected proper interval graph G , we show $OD(G) \leq \lceil \frac{5}{4}diam(G) \rceil + 1$ if $diam(G) \leq 3$ and $OD(G) \leq \lceil \frac{5}{4}diam(G) \rceil$, otherwise. All the bounds are sharp and improve earlier estimations obtained by Fomin et al.

Keywords: Diameter, orientation, oriented diameter, interval graph, proper interval graph.

1 Introduction

Given a (directed) graph H , the *distance* from a vertex x to a vertex y in H , denoted by $d_H(x, y)$, is the length of a shortest (directed) path from x to y in H . The maximum value of distances, denoted by $diam(H)$, is called the *diameter* of H . When H is a graph, $diam(H)$ is finite if and only if H is connected; when H is a directed graph, $diam(H)$ is finite if and only if H is strongly connected.

Let G be a graph. An *orientation* of G is a directed graph obtained from G by assigning a direction to each edge of G . The *oriented diameter* of G , $OD(G)$, is given by

$$OD(G) = \min\{diam(H) : H \text{ is an orientation of } G\}.$$

Clearly if G is not connected or has a bridge (i.e., a cut-edge) then $OD(G)$ is not finite. A celebrated theorem of Robbins [11] asserts $OD(G)$ is finite if G is both connected and bridgeless. Any orientation H of G with $diam(H) = OD(G)$ is called a *minimum diameter orientation* of G . The problem of evaluating the

value $OD(G)$ for an arbitrary graph G is very difficult. In fact it is NP-complete to determine whether a graph G satisfies $OD(G) \leq 2$, [2].

Chvátal and Thomassen [2] initiated the investigation of the relation of the two parameters $OD(G)$ and $diam(G)$. They proved that $OD(G) \leq 2(diam(G))^2 + 2diam(G)$ for every connected bridgeless graph G . Minimum diameter orientations of various families of graphs have already been studied by Gutin [6], Koh and Tan [7], Soltés [12], Plesnik [10], Boesch and Tindell [1], and others. We direct the reader to the survey of Koh and Tay [8] for a comprehensive discussion on the minimum diameter orientations of graphs.

A graph G is an *interval graph* if there is a family of intervals I_v , $v \in V(G)$, in a real line such that two vertices u, v are adjacent in G if and only if I_u, I_v overlap. If such a family of intervals can be chosen so that no interval is completely contained in another, then the graph G is called a *proper interval graph*. Recently, Fomin et al. [3] proved that, for every connected bridgeless interval graph G , $OD(G) \leq \frac{3}{2}diam(G) + \frac{25}{2}$ and, for every 2-connected proper interval graph G , $OD(G) \leq \frac{5}{4}diam(G) + \frac{29}{2}$.

The purpose of this paper is to sharpen these two upper bounds. More precisely, we show that $OD(G) \leq \lceil \frac{3}{2}diam(G) \rceil + 1$ for every connected bridgeless interval graph G . There are (proper) interval graphs of arbitrarily larger diameter achieving this bound. Further, we show that, for every 2-connected proper interval graph G , $OD(G) \leq \lceil \frac{5}{4}diam(G) \rceil + 1$ if $diam(G) \leq 3$ and $OD(G) \leq \lceil \frac{5}{4}diam(G) \rceil$, otherwise. Again the bounds can be attained by appropriate 2-connected proper interval graphs.

2 Connected Bridgeless Interval Graphs

Interval graphs can be characterized in many ways, cf. [5]. According to Gilmore and Hoffman [4], a graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered so that, for every vertex v of G , the maximal cliques containing vertex v occur consecutively in the ordering.

Let G be a connected bridgeless interval graph and let $\mathcal{C} : C_1, C_2, \dots, C_r$ be an ordering of the maximal cliques of G that satisfies the above-mentioned property. Since G is bridgeless, every C_i has at least 3 vertices. When $r = 1$, G is a complete graph, i.e., $G = K_n$ where n is the number of vertices of G . Plesnik [10] has proved that $OD(K_4) = 3$, and $OD(K_n) = 2$ for all $n \geq 3$ and $n \neq 4$ (cf. also [1,9]). A minimum diameter orientation of K_4 is depicted in Figure 1.

Fig. 1. K_4 and its minimum diameter orientation

Suppose that G is not a complete graph, i.e., $r \geq 2$. Let $\mathcal{C}' : C_{i_1}, C_{i_2}, \dots, C_{i_d}$ with $1 = i_1 < i_2 < \dots < i_d = r$ be chosen such that $C_{i_j} \cap C_{i_{j+1}} \neq \emptyset$ and $C_{i_j} \cap C_{i_{j+1}+1} = \emptyset$ for $1 \leq j \leq d-1$. Clearly, $d = \text{diam}(G)$. Denote $S_j = C_{i_j} \cap C_{i_{j+1}}$, $1 \leq j \leq d-1$. Then S_j is contained in C_m for $i_j \leq m \leq i_{j+1}$. Since $C_{i_j} \cap C_{i_{j+1}+1} = \emptyset$, $S_j \cap S_{j+1} = \emptyset$ for $1 \leq j \leq d-1$.

When $|S_j| = 1$, denote by z_j the only vertex in S_j ; otherwise when $|S_j| \geq 2$, choose arbitrarily two vertices z_j, z'_j in S_j . Call C_{i_j} *bad* if $|S_{j-1}| = |S_j| = 1$ ($2 \leq j \leq d-1$), and *good* otherwise. Denote by ℓ the number of bad maximal cliques (in $\mathcal{C}'$). Then $\ell \leq d-2 = \text{diam}(G) - 2$.

We orient G using the following steps:

- (1) If $|S_1| = 1$, arbitrarily choose a vertex $w \in C_1 - \{z_1\}$ let $w \rightarrow x \rightarrow z_1 \rightarrow w$ for all $x \in C_1 - \{w, z_1\}$; if $|S_1| \geq 2$, let $z_1 \rightarrow z'_1 \rightarrow x \rightarrow z_1$ for all $x \in C_1 - \{z_1, z'_1\}$. Note that this gives a partial orientation of C_1 . Similarly, if $|S_{d-1}| = 1$, arbitrarily choose a vertex $w \in C_r - \{z_{d-1}\}$ let $w \rightarrow x \rightarrow z_{d-1} \rightarrow w$ for all $x \in C_r - \{w, z_{d-1}\}$; if $|S_{d-1}| \geq 2$, let $z_{d-1} \rightarrow z'_{d-1} \rightarrow x \rightarrow z_{d-1}$ for all $x \in C_r - \{z_{d-1}, z'_{d-1}\}$. Again this gives a partial orientation of C_r .
- (2) For the first $\lfloor \frac{\ell}{2} \rfloor$ bad cliques C_{i_j} , orient $v \rightarrow z_{j-1} \rightarrow z_j \rightarrow v$ for $v \in C_{i_j} - \{z_{j-1}, z_j\}$ and for the remaining $\lceil \frac{\ell}{2} \rceil$ bad maximal cliques C_{i_j} , orient $v \rightarrow z_j \rightarrow z_{j-1} \rightarrow v$ for $v \in C_{i_j} - \{z_{j-1}, z_j\}$;
- (3) For each good clique C_{i_j} with $|S_{j-1}| = 1$ and $|S_j| \geq 2$, orient $v \rightarrow z_j \rightarrow z'_j \rightarrow v$ for any $v \in C_{i_j} - \{z_j, z'_j\}$. For each good clique with $|S_j| = 1$ and $|S_{j-1}| \geq 2$, orient $v \rightarrow z_{j-1} \rightarrow z'_{j-1} \rightarrow v$ for $v \in C_{i_j} - \{z_{j-1}, z'_{j-1}\}$;
- (4) For each good clique C_{i_j} with $|S_{j-1}| \geq 2$ and $|S_j| \geq 2$, we let $z_{j-1} \rightarrow z'_{j-1}$, $z_{j-1} \rightarrow z_j \rightarrow z'_j \rightarrow z'_{j-1}$, and $\{z'_{j-1}, z'_j\} \rightarrow v \rightarrow \{z_{j-1}, z_j\}$ for all $v \in C_{i_j} - \{z_j, z'_j, z_{j-1}, z'_{j-1}\}$;
- (5) For each $C_m \in \mathcal{C} - \mathcal{C}'$, we must have $i_j < m < i_{j+1}$ for some j . (Note that the ordering of maximal cliques ensures that C_m contains S_j and in particular z_j and z'_j .) When $|S_j| \geq 2$, for each $v \in C_m - (C_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_d})$, let $v \rightarrow z_j \rightarrow z'_j \rightarrow v$. When $|S_j| = 1$, we select a vertex $w \in C_m - \{z'_j\}$ in such a way that the preference is given to a vertex in $C_{i_j} \cup C_{i_{j+1}}$. Note that the edge wz_j may or may not be oriented by the previous steps. If it has not been oriented, then we orient it arbitrarily. Now for each vertex $v \in C_m - (C_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_d})$, we assign directions to edges vw, vz_j in such a way that v, w, z_j form a directed triangle;
- (6) Finally, for each edge which has not been oriented by the above steps, we assign arbitrarily a direction to it.

Lemma 1. *Let G be a connected bridgeless interval graph which is not complete and let H be an orientation of G obtained using the above steps. Then*

$$\text{diam}(H) \leq \lceil \frac{3}{2} \text{diam}(G) \rceil + 1.$$

Proof. Let v be a vertex of G . Then $v \in C_i$ for some $1 \leq i \leq r$. Assume $i_j \leq i \leq i_{j+1}$, i.e., $v \in C_{i_j} \cup C_{i_{j+1}} \cup \dots \cup C_{i_{j+1}}$. When $|S_j| = 1$, it follows from

Fig. 2. A minimum diameter orientation of an interval graph G of diameter n showing $OD(G) = \lceil \frac{3}{2}n \rceil + 1$, where $k = \lfloor \frac{n}{2} \rfloor$

steps 1, 2, 3, and 5 that $d_H(v, z_j) \leq 2$ and $d_H(z_j, v) \leq 2$; when $|S_{j+1}| \geq 2$, by steps 1, 3, 4, and 5, $d_H(v, z_j) \leq 1$, $d_H(z'_j, v) \leq 1$, and $d_H(z_j, z'_j) = 1$.

Let u, v be any two vertices of G . Then $u \in C_a$ and $v \in C_b$ for some $1 \leq a, b \leq r$. If $i_j \leq a, b \leq i_{j+1}$ for some $j = 1, 2, \dots, d-1$, then

$$d_H(u, v) \leq \begin{cases} d_H(u, z_j) + d_H(z_j, v) \leq 4 & \text{if } |S_j| = 1; \\ d_H(u, z_j) + d_H(z_j, z'_j) + d_H(z'_j, v) \leq 3 & \text{if } |S_j| \geq 2. \end{cases}$$

Suppose that $i_j \leq a < i_{j+1} \leq i_k \leq b < i_{k+1}$ where $1 \leq j < k \leq d-1$. By step 2, $d_H(z_j, z_k) \leq \text{diam}(G) - 2 + \lceil \frac{\ell}{2} \rceil$ and $d_H(z_k, z_j) \leq \text{diam}(G) - 2 + \lfloor \frac{\ell}{2} \rfloor$. Hence

$$\begin{aligned} d_H(u, v) &\leq d_H(u, z_j) + d_H(z_j, z_k) + d_H(z_k, v) \leq 2 + \text{diam}(G) - 2 + \lceil \frac{\ell}{2} \rceil + 2 \\ &\leq 2 + \text{diam}(G) + \lceil \frac{\text{diam}(G) - 2}{2} \rceil = \lceil \frac{3}{2} \text{diam}(G) \rceil + 1, \end{aligned}$$

and

$$\begin{aligned} d_H(v, u) &\leq d_H(v, z_k) + d_H(z_k, z_j) + d_H(z_j, u) \leq 2 + \text{diam}(G) - 2 + \lfloor \frac{\ell}{2} \rfloor + 2 \\ &\leq 2 + \text{diam}(G) + \lfloor \frac{\text{diam}(G) - 2}{2} \rfloor \leq \lceil \frac{3}{2} \text{diam}(G) \rceil + 1. \end{aligned}$$

Therefore $\text{diam}(H) \leq \lceil \frac{3}{2} \text{diam}(G) \rceil + 1$. □

Theorem 1. Let G be a connected bridgeless interval graph. Then

$$OD(G) \leq \lceil \frac{3}{2} \text{diam}(G) \rceil + 1.$$

Proof. The statement follows from Lemma 1 and the fact that $OD(K_n) \leq 3$. □

The sharpness of the upper bound given in Theorem 1 is shown by Figure 2, where an orientation of diameter $\lceil \frac{3}{2}n \rceil + 1$ is given for an interval graph of diameter n .

3 2-Connected Proper Interval Graphs

In this section, we focus on the minimum diameter orientations of 2-connected proper interval graphs G . We shall adopt the same notations as in Section 2.

In particular, $C_1, C_2, \dots, C_r$ is an ordering of the maximal cliques of G in the sense of Gilmore and Hoffman. Since G is a proper interval graph, the ordering is essentially unique. We also let $C_{i_1}, C_{i_2}, \dots, C_{i_d}$ be the cliques as defined in Section 2. We begin with the lemma (which does not assume G is 2-connected).

Lemma 2. *Let G be a connected proper interval graph. Then for $1 \leq j \leq d-1$ and $i_j < m < i_{j+1}$, $C_m \subseteq C_{i_j} \cup C_{i_{j+1}}$.*

Proof. Suppose that C_m contains a vertex w that is not in $C_{i_j} \cup C_{i_{j+1}}$. Since w is not in $C_{i_j} \cup C_{i_{j+1}}$, w is not adjacent to a vertex $x \in C_{i_j}$ and not adjacent to a vertex $y \in C_{i_{j+1}}$. Let z be any vertex in $C_{i_j} \cap C_{i_{j+1}}$. Since $C_{i_j} \cap C_{i_{j+1}} \subseteq C_m$, z is in C_m and hence adjacent to w . So z is adjacent to w, x , and y . Since G is a proper interval graph, it does not contain an induced $K_{1,3}$. This implies that x and y are adjacent (as otherwise the vertices w, x, y, z induce a $K_{1,3}$). In particular, x, y, z are pairwise adjacent.

Consider a maximal clique $C_{m'}$ containing x, y, z . Note $C_{m'}$ is distinct from C_{i_j}, C_m and $C_{i_{j+1}}$. If $m' > m$, then x must be in C_m and hence adjacent to w , a contradiction. On the other hand, if $m' < m$, then y is in C_m and adjacent to w , also a contradiction. $\square$

From now on, we assume G is a 2-connected proper interval graph. Lemma 2 implies that $V(G) = C_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_d}$. As in Section 2, if $|S_j| = |C_{i_j} \cap C_{i_{j+1}}| \geq 2$, let z_j, z'_j be any two fixed vertices in S_j , and if $|S_j| = 1$, let z_j be the only vertex in S_j . Since G is 2-connected, in the case when $|S_j| = 1$, there exist $x_j \in C_{i_j} - \{z_j\}$ and $x'_{j+1} \in C_{i_{j+1}} - \{z_j\}$ such that x_j and x'_{j+1} are adjacent.

We next show how to orient G and to do so we consider several cases. In each case, we only give a partial orientation of G and the orientations of the remaining edges may be given arbitrarily.

Case 1. $\text{diam}(G) = 2$.

- (1) If $|S_1| = 1$, let $x'_2 \rightarrow x_1, v \rightarrow z_1 \rightarrow w$ for all $v \in C_{i_1} - z_1$ and $w \in C_{i_2} - z_1, x_1 \rightarrow v$ for all $v \in C_{i_1} - \{x_1, z_1\}$, and $w \rightarrow x'_2$ for all $w \in C_{i_2} - \{x'_2, z_1\}$.
- (2) If $|S_1| \geq 2$, let $v \rightarrow z_1 \rightarrow z'_1 \rightarrow v$ for all $v \in V(G) \setminus \{z_1, z'_1\}$.

Case 2. $\text{diam}(G) = 3$.

- (1) Suppose $|S_1| = |S_2| = 1$. Note that it is possible to select x'_2, x_2 from $C_{i_2} - \{z_1, z_2\}$. Let $x_1 \rightarrow z_1 \rightarrow x'_2 \rightarrow x_1, z_1 \rightarrow z_2$, and $x_2 \rightarrow x'_3 \rightarrow z_2 \rightarrow x_2$. For all $u \in C_{i_1} - \{x_1, z_1\}$ let $x_1 \rightarrow u \rightarrow z_1$, for all $v \in C_{i_2} - \{z_1, z_2, x'_2, x_2\}$ let $z_2 \rightarrow v \rightarrow z_1$, and for all $w \in C_{i_3} - \{z_2, x'_3\}$ let $x'_3 \rightarrow w \rightarrow z_2$.
- (2) Suppose that exactly one of $|S_1|$ and $|S_2|$ equals 1. Assume $|S_1| = 1$ (the case when $|S_2| = 1$ can be handled in a similar way. Let $v \rightarrow z_1 \rightarrow x_1 \rightarrow v$ for $v \in C_{i_1} - \{z_1, x_1\}$, $u \rightarrow z_2 \rightarrow z'_2 \rightarrow u$ for $u \in C_{i_2} \cup C_{i_3} - \{z_2, z'_2\}$.
- (3) Suppose $|S_1| \geq 2$ and $|S_2| \geq 2$. Let $z_1 \rightarrow z_2$ and $z'_2 \rightarrow z'_1$. Let $v \rightarrow z_1 \rightarrow z'_1 \rightarrow v$ for $v \in C_{i_1} - \{z_1, z'_1\}$, and $u \rightarrow z_2 \rightarrow z'_2 \rightarrow u$ for $u \in C_{i_2} \cup C_{i_3} - S_1 - \{z_2, z'_2\}$.

Case 3. $\text{diam}(G) \geq 4$.

Let C_{i_k} be a bad clique with $3 \leq k \leq d-3$. Call C_{i_k} 2-bad if $C_{i_{k+1}}$ is also bad. Choose a set $\mathcal{L}$ of 2-bad cliques $C_{i_{j_1}}, C_{i_{j_2}}, \dots, C_{i_{j_\eta}}$ so that the following properties are satisfied:

(P1) the set contains as many 2-bad cliques as possible;

(P2) $i_{j_{k+1}} \neq i_{j_k+1}$;

(P3) $i_{j_1} + i_{j_2} + \dots + i_{j_\eta}$ is minimum.

Note that $\mathcal{L}$ does not contain all 2-bad cliques. For instance, if $C_{i_{k-1}}$ is in $\mathcal{L}$, then C_{i_k} cannot be in $\mathcal{L}$ even though it may be 2-bad. Denote by $\mathcal{M}$ be the set of all bad but not 2-bad cliques C_{i_k} with $3 \leq k \leq d-2$ such that $C_{i_{k-1}}$ is not in $\mathcal{L}$ and $|\mathcal{M}| = \ell$. Then the number of good cliques in $\mathcal{C}' - \{C_{i_1}, C_{i_2}, C_{i_{d-1}}, C_{i_d}\}$ is $d-4-2\eta-\ell$. When $C_{i_k} \in \mathcal{M}$, we have $C_{i_{k+1}}$ is either good or $C_{i_{d-1}}$ and hence $\ell \leq (d-4-2\eta-\ell)+1$.

– (1) If $|S_1| = 1$, let $x'_2 \rightarrow x_1 \rightarrow z_1$, $x_1 \rightarrow v \rightarrow z_1$ for all $v \in C_{i_1} - \{z_1, x_1\}$, and $u \rightarrow x'_2 \rightarrow z_1 \rightarrow u$ for all $u \in C_{i_2} - \{z_2, x'_2\}$; If $|S_1| \geq 2$, let $z'_1 \rightarrow v \rightarrow z_1$ for all $v \in C_{i_1} - \{z'_1, z_1\}$, and $u \rightarrow z'_1 \rightarrow z_1 \rightarrow u$ for all $u \in C_{i_2} - S_1$.

– (2) If $|S_{d-1}| = 1$, let $x_{d-1} \rightarrow x'_d \rightarrow z_{d-1}$, $v \rightarrow x_{d-1} \rightarrow z_{d-1} \rightarrow v$ for all $v \in C_{i_{d-1}} - \{z_{d-1}, x'_{d-1}\}$, and $x'_d \rightarrow u \rightarrow z_{d-1}$ for all $u \in C_{i_d} - \{z_{d-1}, x'_d\}$; If $|S_{d-1}| \geq 2$, let $v \rightarrow z'_{d-1} \rightarrow z_{d-1} \rightarrow v$ for all $v \in C_{i_{d-1}} - \{z_{d-1}, z'_{d-1}\}$ and $z'_{d-1} \rightarrow u \rightarrow z_{d-1}$ for all $u \in C_{i_d} - S_{d-1}$.

– (3) Suppose $C_{i_j} \in \mathcal{L}$. We only show how to give orientations of the edges in C_{i_j} and $C_{i_{j+1}}$ for $i_{j_1} \leq i_j \leq i_{j_{\lceil \frac{\eta}{2} \rceil}}$. (When $i_{j_{\lceil \frac{\eta}{2} \rceil}} < i_j \leq i_{j_\eta}$, the edges are oriented reversely.) Let $x'_{i_{j+1}} \rightarrow x_{i_j}$, $v \rightarrow z_{j-1} \rightarrow z_j \rightarrow v$ for all $v \in C_{i_j} - \{z_j, z_{j-1}\}$, and $u \rightarrow z_j \rightarrow z_{j+1} \rightarrow u$ for all $u \in C_{i_{j+1}} - \{z_j, z_{j+1}\}$.

– (4) For the first (with respect to the order of $\mathcal{C}$) $\lfloor \frac{\ell}{2} \rfloor$ cliques C_{i_k} in $\mathcal{M}$, let $v \rightarrow z_{k-1} \rightarrow z_k \rightarrow v$ for all $v \in C_{i_k} - \{z_{k-1}, z_k\}$; for the other cliques in $\mathcal{M}$, we orient the edges reversely.

– (5) Suppose that C_{i_j} with $3 \leq j \leq d-2$ is a good clique. If $|S_{j-1}| = 1$ and $|S_j| \geq 2$, let $v \rightarrow z_j \rightarrow z'_j \rightarrow v$ for any $v \in C_{i_j} - \{z_j, z'_j\}$; if $|S_j| = 1$ and $|S_{j-1}| \geq 2$, let $v \rightarrow z_{j-1} \rightarrow z'_{j-1} \rightarrow v$ for $v \in C_{i_j} - \{z_{j-1}, z'_{j-1}\}$; if $|S_{j-1}| \geq 2$ and $|S_j| \geq 2$, let $z_{j-1} \rightarrow z'_{j-1}$, $z_{j-1} \rightarrow z_j \rightarrow z'_j \rightarrow z'_{j-1}$, and $\{z'_{j-1}, z'_j\} \rightarrow v \rightarrow \{z_{j-1}, z_j\}$ for all $v \in C_{i_j} - \{z_j, z'_j, z_{j-1}, z'_{j-1}\}$.

Theorem 2. Let G be a 2-connected proper interval graph. Then

$$OD(G) \leq \begin{cases} \lceil \frac{5}{4} \text{diam}(G) \rceil + 1 & \text{if } \text{diam}(G) \leq 3; \\ \lceil \frac{5}{4} \text{diam}(G) \rceil & \text{if } \text{diam}(G) \geq 4. \end{cases}$$

Proof. When G is complete, $OD(G) \leq 3 \leq \lceil \frac{5}{4} \text{diam}(G) \rceil + 1$ by Theorem 1. When G is not complete and satisfies $\text{diam}(G) \leq 3$, it can be verified easily from the orientations given in Case 1 and Case 2 that $OD(G) \leq \lceil \frac{5}{4} \text{diam}(G) \rceil + 1$. So suppose that $\text{diam}(G) \geq 4$.

Let u, v be any two vertices of G . Without loss of generality assume $u \in C_{i_j}$ and $v \in C_{i_k}$ for some $j \leq k$. When $k-1 \leq j \leq k$, according to steps 1 and 2 in Case 3, $d_H(u, v) \leq 5$ if $j = k = 1$ or $j = k = d$ and, according steps 3, 4, and 5 in Case 3, $d_H(u, v) \leq 3$ if $j = k \neq 1, d$. When $j = k-1$, $d_H(u, v) \leq 4$ according to steps 1-5 in Case 3.

Suppose now $j < k - 1$. Let $u' \in C_{i_1} - \{z_1, x_1\}$ if $|S_1| = 1$ and $u' \in C_{i_1} - S_1$ if $|S_1| \geq 2$, and let $v' \in C_{i_d} - \{z_{d-1}, x'_d\}$ if $|S_{d-1}| = 1$ and $v' \in C_{i_d} - S_{d-1}$ if $|S_{d-1}| \geq 2$.

Claim: $d_H(u, v) \leq d_H(u', v')$ and $d_H(v, u) \leq d_H(v', u')$.

Proof of Claim: We only prove $d_H(u, v) \leq d_H(u', v')$ as the proof of the other inequality is similar. Denote $P_t'' = C_{i_t} \cup C_{i_{t+1}}$ with $C_{i_t} \in \mathcal{L}$. If u does not lie in P_t'' for any t , then $d_H(u, z_j) \leq 2 \leq d_H(u', z_j)$; If u lies in P_t'' for some t , then $3 \leq t \leq d - 3$ and either $t = j$ or $t = j - 1$, so $d_H(u, z_{t+1}) \leq 3 \leq d_H(u', z_{t+1})$.

If v does not lie in P_t'' for any t , then $d_H(z_{k-1}, v) \leq 2 \leq d_H(z_{k-1}, v')$; If v lies in P_p'' for some p , then $3 \leq p \leq d - 3$ and either $p = k$ or $p = k - 1$, so $d_H(z_{p-1}, v) \leq 3 \leq d_H(z_{p-1}, v')$. So

$$\begin{aligned} d_H(u, v) &\leq d_H(u, z_r) + d_H(z_r, z_m) + d_H(z_m, v) \\ &\leq d_H(u', z_r) + d_H(z_r, z_m) + d_H(z_m, v') = d_H(u', v') \end{aligned}$$

where $r = t + 1$ if u lies in P_t'' for some t , and $r = j$ otherwise; $m = p - 1$ if v lies in P_p'' for some p , and $m = k - 1$ otherwise. $\square$

To estimate $d_H(u', v')$ and $d_H(v', u')$, first we have

$$\begin{aligned} d_H(u', v') &= d_H(u', z_2) + d_H(z_2, z_{d-2}) + d_H(z_{d-2}, v') \\ &\leq 2 + 2\lceil \frac{\eta}{2} \rceil + 3\lceil \frac{\eta}{2} \rceil + \lceil \frac{\ell}{2} \rceil + 2\lfloor \frac{\ell}{2} \rfloor + (d - 4 - 2\eta - \ell) + 3 \\ &= 1 + \lceil \frac{\eta}{2} \rceil + \lfloor \frac{\ell}{2} \rfloor + d. \end{aligned}$$

Since $\ell \leq (d - 4 - 2\eta - \ell) + 1$, we have

$$d_H(u', v') \leq 1 + \lceil \frac{\eta}{2} \rceil + \lfloor \frac{d - 3 - 2\eta}{4} \rfloor + d \leq 1 + \lceil \frac{\eta}{2} \rceil + \lceil \frac{d}{4} \rceil - \lceil \frac{3}{4} + \frac{\eta}{2} \rceil + d = \lceil \frac{5}{4} \text{diam}(G) \rceil.$$

Similarly, $d_H(v', u') \leq 1 + \lfloor \frac{\eta}{2} \rfloor + \lceil \frac{\ell}{2} \rceil + d$ and hence

$$d_H(v', u') \leq 1 + \lceil \frac{\ell}{2} \rceil + \lfloor \frac{d - 3 - 2\ell}{4} \rfloor + d \leq 1 + \lceil \frac{\ell}{2} \rceil + \lceil \frac{d}{4} \rceil - \lceil \frac{3}{4} + \frac{\ell}{2} \rceil + d = \lceil \frac{5}{4} \text{diam}(G) \rceil.$$

$\square$

When $\text{diam}(G) = 1$, the upper bound of $OD(G)$ in Theorem 2 is achieved by a minimum diameter orientation of K_4 (see Figure 1). The following proposition from [3] shows that the upper bound in Theorem 2 is sharp for $\text{diam}(G) \geq 4$.

Proposition 1. [3] For $d \geq 3$, there is a 2-connected proper interval graph G with $\text{diam}(G) \geq d$ and $OD(G) \geq \frac{5}{4}\text{diam}(G)$. $\square$

The sharpness of upper bounds of $OD(G)$ for the cases when $2 \leq \text{diam}(G) \leq 3$ is given by Propositions 2 and 3 below. Indeed, it can be verified that the digraphs in Figures 3 and 4 are the minimum diameter orientations of their underlying graphs. Therefore we have

Fig. 3. A minimum diameter orientation of a 2-connected proper interval graph of diameter two, where unoriented edges may be oriented arbitrarily

Fig. 4. A minimum diameter orientation of 2-connected proper interval graph of diameter three, where unoriented edges may be oriented arbitrarily

Proposition 2. Let G_1 be the underlying graph of the digraph in Figure 3. Then $OD(G_1) \geq 4$. □

Proposition 3. Let G_2 be the underlying graph of the digraph in Figure 4. Then $OD(G_2) \geq 5$. □

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