Componentwise Complementary Cycles in Almost Regular 3-Partite Tournaments

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Abstract. The vertex set of a digraph D is denoted by $V(D)$. A c -partite tournament is an orientation of a complete c -partite graph. Let $V_1, V_2, \dots, V_c$ be the partite sets of D . If there exist two vertex disjoint cycles C and C' in D such that $V_i \cap (V(C) \cup V(C')) \neq \emptyset$ for all $i = 1, 2, \dots, c$, then D is cycle componentwise complementary. The global irregularity of D is defined by $i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) | x, y \in V(D)\}$ over all vertices x and y of D ($x = y$ is admissible), where $d^+(x)$ and $d^-(x)$ are the outdegree and indegree of x , respectively. If $i_g(D) \leq 1$, then D is almost regular. In this paper, we consider a special kind of multipartite tournaments which are almost regular 3-partite tournaments, and we show that each almost regular 3-partite tournament D is cycle componentwise complementary, unless D is isomorphic to $D_{3,2}$.

Keywords: almost regular, componentwise complementary cycles, tournament.

1 Terminology and Preliminary Results

We shall assume that the reader is familiar with the standard terminology on directed graphs and refer the reader to [1]. In this paper all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. For a vertex set X of D , we define $D[X]$ as the subdigraph induced by X . If xy is an arc of a digraph D , then we write $x \rightarrow y$ and say x *dominates* y .

A c -partite or multipartite tournament is an orientation of a complete c -partite graph. A tournament is a c -partite tournament with exactly c vertices. A semicomplete multipartite digraph is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs with the same end vertices. Let $V_1, V_2, \dots, V_c$ be the partite sets of a c -partite tournament or semicomplete c -partite digraph D and $|V_1| \leq |V_2| \leq \dots \leq |V_c|$. If

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the vertex x of D belongs to the partite set V_i , then we define $V(x) = V_i$ and define $\gamma(D) = V_1$.

If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y , then we say that X *dominates* Y , denoted by $X \rightarrow Y$. If there is no arc from Y to X , but $x \rightarrow y$ for all $x \in X$ and $y \in Y$ which are in different partite sets, then we denote this by $X \Rightarrow Y$ which denotes the property that there is no arc from Y to X .

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x , and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x . The number $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x , respectively. The *minimum outdegree* and the *minimum indegree* of D are denoted by $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$, and $\delta(D) = \delta = \min\{\delta^+, \delta^-\}$. The *irregularity* $I(D)$ of a digraph is $\max|d^+(x) - d^-(y)|$ over all vertices x and y of D ($x = y$ is admissible). In addition, the *local irregularity* is defined by $i_\ell(D) = \max|d^+(x) - d^-(x)|$ over all vertices x of D and the *global irregularity* of a digraph D is defined by $i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) | x, y \in V(D)\}$ over all vertices x and y of D ($x = y$ is admissible). Clearly, $i_\ell(D) \leq I(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is *almost regular*.

A cycle of length m is an *m-cycle*. A cycle or a path in a digraph D is *Hamiltonian* if it includes all the vertices of D . A set $X \subseteq V(D)$ of vertices is *independent* if the induced subdigraph $D[X]$ has no arcs. The *independence number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D .

A digraph D is *strongly connected* or *strong* if, for each pair of vertices u and v , there is a path from u to v in D . A digraph D with at least $k + 1$ vertices is *k-connected* if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D , denoted by $k(D)$, is then defined to be the largest value of k such that D is k -connected. A set S of vertices of a digraph D is a *separating set* if $D - S$ is not strong.

If we replace every arc xy of D by yx , then we call the resulting digraph, denoted by D^{-1} , the *converse digraph* of D .

A digraph D is *cycle complementary* if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$.

The following results play important roles in the proof of our main result.

Lemma 1 ([2]). *Let D be a c -partite tournament with the partite sets $V_1, V_2, \dots, V_c$. Then $||V_i| - |V_j|| \leq 2I(D) \leq 2i_g(D)$ for $1 \leq i \leq j \leq c$.*

Theorem 1 ([3]). *If D is a multipartite tournament, then*

$$k(D) \geq \frac{|V(D)| - 2i_\ell(D) - \alpha(D)}{3}.$$

Theorem 2 ([14]). *Let D be a regular 3-partite tournament with $|V(D)| \geq 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to $D_{3,2}$. ($D_{3,2}$ is shown in Fig. 1.)*

Fig. 1. The 2-regular 3-partite tournament $D_{3,2}$

Theorem 3 ([4]1994). *Let D be a strong c -partite tournament with $c \geq 3$. Then at least one vertex of every partite set of D is contained in an m -cycle for each $m \in \{3, 4, \dots, c\}$.*

Lemma 2 ([2]). *If D is an almost regular multipartite tournament, then for every vertex x of D we have*

$$\frac{|V(D)| - \alpha(D) - 1}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - \gamma(D) + 1}{2}.$$

Proposition 1 ([2]). *Let D be an almost regular c -partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $r = |V_1| \leq |V_2| \leq \dots \leq |V_c| = r + 2$. Then $|V(D)| - r$ is even.*

2 Main Result

The problem of complementary cycles in tournaments was almost completely solved by Reid [5] in 1985 and by Song [6] in 1993. These authors proved that every 2-connected tournament D on at least 8 vertices has complementary cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \dots, |V(D)| - 3\}$. Later, Guo and Volkmann [7], [8] extended this result to locally semicomplete digraphs. In addition, there are some results on complementary cycles on bipartite tournaments by Z. Song, K. Zhang, J. Wang and Manoussakis. See [9], [11], [10], [12].

Volkmann [13] confirmed that each regular multipartite tournament is cycle complementary, unless D is isomorphic to T_7 , $D_{4,2}$, $D_{4,2}^*$, $D_{3,2}$. But the problem about the existences of complementary cycles in multipartite digraphs which are not tournaments or bipartite tournaments or locally semicomplete digraphs or regular digraphs is still open. It seems that the problem of complementary cycles in semicomplete n -partite digraphs with $n \geq 3$ is difficult. In the following, we give the definition of *componentwise complementary cycles* in multipartite digraphs.

Definition 1. (*componentwise complementary cycles*) Let $V_1, V_2, \dots, V_c$ be the partite sets of D . If there exist two vertex disjoint cycles C and C' in D such that $V_i \cap (V(C) \cup V(C')) \neq \emptyset$ for all $i = 1, 2, \dots, c$, then C and C' are two componentwise complementary cycles of D .

Clearly, a pair of componentwise complementary cycles of D is a pair of complementary cycles of D , if it contains all vertices of D .

If D contains a pair of componentwise complementary cycles, then D is *cycle componentwise complementary*.

Our main result is the following theorem.

Theorem 4. *If D is an almost regular 3-partite tournament with $|V(D)| \geq 6$, then D is cycle componentwise complementary, unless D is isomorphic to $D_{3,2}$.*

Example 1. Let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $V_3 = \{u_1, u_2\}$ be the partite sets of the 2-regular 3-partite tournament $D_{3,2}$ presented in Fig. 1. Then it is a simple matter to verify that $D_{3,2}$ contains no componentwise complementary 3-cycles (and of course no complementary cycles).

3 Proof of Theorem 4

Let V_1, V_2, V_3 be the partite sets of D such that $|V_1| \leq |V_2| \leq |V_3|$. By Theorem 1 and $i_\ell(D) \leq i_g(D)$,

$$k(D) \geq \frac{|V(D)| - 2i_\ell(D) - \alpha(D)}{3} \geq \frac{|V(D)| - 2i_g(D) - \alpha(D)}{3}. \quad (1)$$

If $|V_1| = r$, then because of $i_g(D) \leq 1$ we deduce from Lemma 1 that $|V_3| = r + l$ for some $l \in \{0, 1, 2\}$. Since $|V(D)| \geq 6$, $r \geq 2$.

Clearly, if $l = 0$, then D is a regular 3-partite tournament. By Theorem 2, D contains a pair of complementary cycles, i.e. D contains a pair of componentwise complementary cycles, unless D is isomorphic to $D_{3,2}$. In the following, we consider the case $l \neq 0$. According to (1), we have $k(D) \geq 1$. In view of Theorem 3, there exists a 3-cycle C_3 in D . If we define the 3-partite tournament H by $H = D - V(C_3)$, then $i_\ell(H) \leq i_g(H) \leq 3$. We distinguish five cases.

1. $|V_1| = r$, $|V_2| = r$, $|V_3| = r + 1$.
2. $|V_1| = r$, $|V_2| = r + 1$, $|V_3| = r + 1$.
3. $|V_1| = r$, $|V_2| = r$, $|V_3| = r + 2$.
4. $|V_1| = r$, $|V_2| = r + 1$, $|V_3| = r + 2$.
5. $|V_1| = r$, $|V_2| = r + 2$, $|V_3| = r + 2$.

According to Proposition 1, we have case 4 is impossible.

Since the proofs of the cases above are similar, we give the proof of case 1 only.

Let $V_1 = \{x_1, x_2, \dots, x_r\}$, $V_2 = \{y_1, y_2, \dots, y_r\}$, $V_3 = \{u_1, u_2, \dots, u_{r+1}\}$, and let $C_3 = x_1 y_1 u_1 x_1$. By Lemma 2, we have $r - 1 \leq d_D^+(x), d_D^-(x) \leq r + 1$, for

$x \in V(D)$. If H contains a cycle C , then we have $V(C_3) \cap V(C) = \emptyset$ and $(V(C_3) \cup V(C)) \cap V_i \neq \emptyset$ for all $i = 1, 2, 3$. Otherwise, let $P = v_1 v_2 \dots v_p$ be a longest path of H . Clearly, $v_1 \Rightarrow V(D - V(C_3))$ and $V(D - V(C_3)) \Rightarrow v_p$. Hence, we have

$$d_D^+(v_1) \geq 3r + 1 - 3 - r = 2r - 2, \quad \text{and}$$

$$d_D^-(v_1) \leq 2.$$

If $r \geq 4$, then $d_D^+(v_1) \geq 2r - 2 > r + 1$. The contradiction to the almost regular of D .

If $r = 3$, then $d_D^+(v_1) \geq 2r - 2$. Since $d_D^-(v_1) \leq 2$, we have $d_D^+(v_1) - d_D^-(v_1) \geq 2r - 4 = 2$, a contradiction to $i_g(D) \leq 1$.

If $r = 2$, then $|V_1| = 2$, $|V_2| = 2$, and $|V_3| = 3$. If $p < 3$, i.e. $p = 2$, then there is one vertex u such that it is adjacent with y for all $y \in V(P)$. Since P is a longest path in H , $v_1 \rightarrow u$ and $u \rightarrow v_2$. Then H has a path $x_1 u x_2$, a contradiction. Therefore, $p = 3$ or $p = 4$. If there exists V_i such that $V_i \cap V(P) = \emptyset$ for some $i \in \{1, 2, 3\}$, then there is a vertex $z \in V_i$ such that it is adjacent with y for all $y \in V(P)$. Since P is a longest path in H , $v_1 \rightarrow z$ and $z \rightarrow v_p$. Then there is an integer $j \in \{1, 2, \dots, p\}$ such that $v_{j-1} \rightarrow z \rightarrow v_j$. Therefore, H has a $(p + 1)$ -path, a contradiction. We consider two cases.

(1) Let $p = 3$. Then $P = v_1 v_2 v_3$.

If $v_2 \in V_3$, then there is one vertex $x \in V_3$ such that $v_1 \rightarrow x \rightarrow v_3$. Let, without loss of generality, $v_1 \in V_1$ and $v_3 \in V_2$. Since $i_g(D) \leq 1$, $\{y_1, u_1\} \rightarrow v_1$, $v_3 \rightarrow \{x_1, u_1\}$. If $y_1 \rightarrow v_2$, then $v_2 \rightarrow x_1$ and we obtain the componentwise complementary cycles $x_1 y_1 v_2 x_1$ and $u_1 v_1 x v_3 u_1$. If $v_2 \rightarrow y_1$, then $x_1 \rightarrow v_2$. Furthermore, if $x_1 \rightarrow x$, then we have the componentwise complementary cycles $x_1 x v_3 u_1 x_1$ and $y_1 v_1 v_2 y_1$; if $x \rightarrow x_1$, then $y_1 \rightarrow x$ and we obtain the componentwise complementary cycles $x_1 y_1 x x_1$ and $u_1 v_1 v_2 v_3 u_1$.

Let $v_3 \in V_3$. If $v_1 \in V_3$, then we consider D^{-1} . Let, without loss of generality, $v_1 \in V_1$ and $v_2 \in V_2$. Since P is a longest path in H , $|V_1| = 2$, $|V_2| = 2$, and $|V_3| = 3$, there is one vertex $x \in V_3$ such that $\{v_1, v_2\} \rightarrow x$. Since $i_g(D) \leq 1$, $\{v_3, x\} \rightarrow \{x_1, y_1\}$, $\{y_1, u_1\} \rightarrow v_1$, and $x_1 \rightarrow v_2$. If $v_2 \rightarrow u_1$, then we have the componentwise complementary cycles $x_1 v_2 u_1 x_1$ and $y_1 v_1 v_3 y_1$. If $u_1 \rightarrow v_2$, then we have the componentwise complementary cycles $x_1 v_2 x x_1$ and $y_1 u_1 v_1 v_3 y_1$.

(2) Let $p = 4$. Then $P = v_1 v_2 v_3 v_4$.

Note that $|V_1| = 2$, $|V_2| = 2$, and $|V_3| = 3$.

First we assume that $V(v_1) = V(v_4)$, i.e. $V(v_1) = V(v_4) = V_3$. Let, without loss of generality, $v_2 \in V_1$ and $v_3 \in V_2$. Since $i_g(D) \leq 1$, $\{x_1, y_1\} \rightarrow v_1$ and $v_4 \rightarrow \{x_1, y_1\}$.

If $v_3 \rightarrow x_1$ and $u_1 \rightarrow v_2$, then we obtain the componentwise complementary cycles $x_1 v_1 v_3 x_1$ and $y_1 u_1 v_2 y_1$.

If $v_3 \rightarrow x_1$ and $v_2 \rightarrow u_1$, then we have $y_1 \rightarrow v_2$ and $u_1 \rightarrow v_3$. This leads to the componentwise complementary cycles $y_1 v_2 v_4 y_1$ and $x_1 v_1 v_3 x_1$.

If $x_1 \rightarrow v_3$, then we have $v_3 \rightarrow u_1$ and $u_1 \rightarrow v_2$. Furthermore, if $v_2 \rightarrow y_1$, then we obtain the componentwise complementary cycles $y_1 u_1 v_2 y_1$ and $x_1 v_1 v_3 v_4 x_1$;

if $y_1 \rightarrow v_2$, then we obtain the componentwise complementary cycles $y_1v_2v_4y_1$ and $x_1v_1v_3u_1x_1$.

Now we assume that $V(v_1) \neq V(v_4)$. Let, without loss of generality, $\{v_2, v_4\} \subset V_3$. If $\{v_1, v_3\} \subset V_3$, then we consider D^{-1} . Let, without loss of generality, $v_1 \in V_1$ and $v_3 \in V_2$. Since $i_g(D) \leq 1$, $\{y_1, u_1\} \rightarrow v_1$ and $v_4 \rightarrow \{x_1, y_1\}$.

If $y_1 \rightarrow v_2$, then we have $v_2 \rightarrow x_1$, $x_1 \rightarrow v_3$, and $v_3 \rightarrow u_1$. This leads to $x_1v_3u_1x_1$ and the componentwise complementary cycle $y_1v_1v_4y_1$.

If $v_2 \rightarrow y_1$, then $x_1 \rightarrow v_2$. Furthermore, if $v_3 \rightarrow u_1$, then we obtain the componentwise complementary cycles $u_1x_1v_2v_3u_1$ and $y_1v_1v_4y_1$; if $u_1 \rightarrow v_3$, then $v_3 \rightarrow x_1$ and the componentwise complementary cycles $x_1v_2v_3x_1$ and $y_1u_1v_1v_4y_1$.

The proof is complete.

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