Heavy Cycles in 2-Connected Weighted Graphs with Large Weighted Degree Sums

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Abstract. In this paper, we prove that a 2-connected weighted graph G contains either a Hamilton cycle or a cycle of weight at least $2m/3$ if it satisfies the following conditions: (1) $\sum_{i=1}^3 d^w(v_i) \geq m$, where v_1, v_2 and v_3 are three pairwise nonadjacent vertices of G , and two of them are nonadjacent vertices of an induced claw or an induced modified claw; (2) In each induced claw and each induced modified claw of G , all edges have the same weight. This extends several previous results on the existence of heavy cycles in weighted graphs.

Keywords: Hamilton cycle, weighted graph, induced claw (modified claw.)

1 Terminology and Notation

We use Bondy and Murty [3] for terminology and notation not defined here and consider finite simple graphs only. Let G be a graph. G is called a *weighted graph* if each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e . For a subgraph H of G , $V(H)$ and $E(H)$ denote the sets of vertices and edges of H , respectively. The *weight* of H is defined by $w(H) = \sum_{e \in E(H)} w(e)$. For a vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v . We define the *weighted degree* of v in H by $d_H^w(v) = \sum_{h \in N_H(v)} w(vh)$. When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively.

An unweighted graph can be regarded as a weighted graph in which each edge is assigned weight 1. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v , and the weight of a subgraph is simply the number of its edges.

The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. If G is noncomplete, then for a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k pairwise nonadjacent vertices. If $k > \alpha(G)$, then both $\sigma_k(G)$ and $\sigma_k^w(G)$ are defined as ∞ .

An (x, y) -*path* is a path connecting the two vertices x and y . A y -*path* is a path which has y as one of its end-vertices. The *distance* between two vertices

x and y , denoted by $d(x, y)$, is the length of a shortest (x, y) -path. We call the graph $K_{1,3}$ a *claw*, and the graph $K_{1,3} + e$ (e is an edge between two nonadjacent vertices in the claw) a *modified claw*. For a graph G , if there exist three pairwise nonadjacent vertices in which two of them are nonadjacent in an induced claw or an induced modified claw, then by $\bar{\sigma}_3^w(G)$ we denote the minimum value of the weighted degree sum of such three pairwise nonadjacent vertices. Otherwise, $\bar{\sigma}_3^w(G)$ is defined as ∞ . Clearly we have $\bar{\sigma}_3^w(G) \geq \sigma_3^w(G)$.

2 Results

In [8], Pósa gave a degree sum condition for the existence of long cycles in graphs. This result was generalized by the following two theorems along different lines.

Theorem A (Fan [5]). *Let G be a 2-connected graph such that $\max\{d(x), d(y) \mid d(x, y) = 2\} \geq c/2$. Then G contains either a Hamilton cycle or a cycle of length at least c .*

Theorem B (Fournier & Fraïsse [6]). *Let G be a k -connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.*

Bondy *et al.* [2] generalized the result of Pósa to weighted graph. In [10], it was showed that if one wants to give a generalization of Theorem A to weighted graphs, some extra conditions cannot be avoided. By adding two extra conditions, the authors gave a weighted generalization of Theorem A. Motivated by this result, by adding two same extra conditions, Zhang *et al.* [9] gave a weighted generalization of Theorem B in the case $k = 2$.

Theorem 1 (Zhang *et al.* [9]). *Let G be a 2-connected weighted graph which satisfies: (1) $\sigma_3^w(G) \geq m$; (2) $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$; (3) In every triangle T of G , either all edges of T have different weights or all edges of T have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.*

Theorem A was further extended by the following result.

Theorem C (Bedrossian *et al.* [1]). *Let G be a 2-connected graph. If $\max\{d(x), d(y)\} \geq c/2$ for each pair of nonadjacent vertices x and y , which are vertices of an induced claw or an induced modified claw of G , then G contains either a Hamilton cycle or a cycle of length at least c .*

By adding one extra condition, Fujisawa [7] gave a weighted generalization of Theorem C.

Theorem 2 (Fujisawa [7]). *Let G be a 2-connected weighted graph which satisfies: (1) For each induced claw and each induced modified claw of G , all its nonadjacent pair of vertices x and y satisfy $\max\{d^w(x), d^w(y)\} \geq s/2$; (2) For each induced claw and each induced modified claw of G , all of its edges have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least s .*

Clearly, Conditions (2) and (3) of Theorem 1 imply Condition (2) of Theorem 2. In a previous paper [4], the authors proved that Conditions (2) and (3) of Theorem 1 can be replaced by Condition (2) of Theorem 2. Here we will further show that Condition (1) of Theorem 1 can also be replaced by a weaker one.

Theorem 3. *Let G be a 2-connected weighted graph which satisfies the following conditions: (1) $\overline{\sigma}_3^w(G) \geq m$; (2) For each induced claw and each induced modified claw of G , all of its edges have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.*

3 Proof of Theorem 3

We call a path P a *heaviest longest path* of a graph G if P has the following properties

- P is a longest path of G , and
- $w(P)$ is maximum among all longest paths in G .

Proposition 1. *Let G be a non-hamiltonian 2-connected weighted graph which satisfies conditions (1) and (2) of Theorem 3. Then G contains a heaviest longest path $P = v_1v_2 \cdots v_p$, such that $d^w(v_1) + d^w(v_p) \geq 2m/3$.*

We prove this proposition in the next section. Theorem 3 can be proved by combining Proposition 1 and the following lemma. The proof of Lemma 1 is implicit in [2].

Lemma 1 (Bondy et al. [2]). *Let G be a non-hamiltonian 2-connected weighted graph and $P = v_1v_2 \cdots v_p$ be a heaviest longest path in G . Then there is a cycle C in G with $w(C) \geq d^w(v_1) + d^w(v_p)$.*

Proof of Theorem 3. Suppose that G does not contain a Hamilton cycle. Then by Proposition 1 and Lemma 1, we can find a cycle of weight at least $2m/3$.

4 Proof of Proposition 1

In the proof of Proposition 1, we need the following two lemmas.

Lemma 2 (Fujisawa [7]). *Let G be a weighted graph satisfying Condition (2) of Theorem 3. If x_1yx_2 is an induced path with $w(x_1y) \neq w(x_2y)$ in G , then each vertex $x \in N(y) \setminus \{x_1, x_2\}$ is adjacent to both x_1 and x_2 .*

Lemma 3 (Fujisawa [7]). *Let G be a weighted graph satisfying Condition (2) of Theorem 3. Suppose x_1yx_2 is an induced path such that $w_1 = w(x_1y)$ and $w_2 = w(x_2y)$ with $w_1 \neq w_2$, and yz_1z_2 is a path such that $\{z_1, z_2\} \cap \{x_1, x_2\} = \emptyset$ and $x_2z_2 \notin E(G)$. Then the following (i) and (ii) hold:*

(i) $\{z_1x_1, z_1x_2, z_2x_1\} \subseteq E(G)$, and $yz_2 \notin E(G)$. Moreover, all edges in the subgraph induced by $\{x_1, y, x_2, z_1, z_2\}$, other than x_1y , have the same weight w_2 .

(ii) Let Y be the component of $G - \{x_2, z_1, z_2\}$ with $y \in V(Y)$. For each vertex $v \in V(Y) \setminus \{x_1, y\}$, v is adjacent to all of x_1, x_2, y and z_2 . Furthermore, $w(vx_1) = w(vx_2) = w(vy) = w(vz_2) = w_2$.

Proof of Proposition 1. Choose a heaviest longest path $P = v_1 v_2 \cdots v_p$ in G such that $d^w(v_1) + d^w(v_p)$ is as large as possible. It is clear that $N(v_1) \cup N(v_p) \subseteq V(P)$. And it is not difficult to prove that there exists no cycle of length p .

Suppose $d^w(v_1) + d^w(v_p) < 2m/3$. Without loss of generality, we may assume $d^w(v_1) < m/3$.

Claim 1. Let P_1 and P_2 be two heaviest longest paths such that P_1 has v' and v_p as its end-vertices, and P_2 has v'' and v_p as its end-vertices. Then v' and v'' cannot be nonadjacent vertices of an induced claw or an induced modified claw.

Proof. Suppose v' and v'' are nonadjacent vertices of an induced claw or an induced modified claw. Since P_1 and P_2 are heaviest longest paths, $v'v_p \notin E(G)$ and $v''v_p \notin E(G)$. By the choice of the path P in (b), $d^w(v') \leq d^w(v_1)$ and $d^w(v'') + d^w(v_p) < 2m/3$. So we have $d^w(v') + d^w(v'') + d^w(v_p) < m$, contradicting Condition (1) of Theorem 3. $\square$

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2 . Choose $v_k \in N(v_1)$ such that k is as large as possible. It is clear that $3 \leq k \leq p-1$.

Case 1. $N(v_1) = \{v_2, \dots, v_k\}$.

Since P is longest and $N(v_1) = \{v_2, \dots, v_k\}$, $N(v_i) \subseteq V(P)$ for every i with $i = 2, \dots, k-1$; Since G is a non-hamiltonian 2-connected graph, $k+2 \leq p$; Since $G - v_k$ is connected, there must be an edge $v_j v_s \in E(G)$ with $j < k < s$.

We assume that such an edge $v_j v_s$ was chosen so that: (i) s is as large as possible; (ii) j is as large as possible, subject to (i). Clearly we have $s \leq p-1$.

Claim 2. $w(v_1 v_{j+1}) = w(v_j v_{j+1})$.

Proof. Suppose $j \leq k-2$. By the choices of v_k, v_j and v_s , we have $v_1 v_s \notin E(G)$ and $v_{j+1} v_s \notin E(G)$. So $\{v_j, v_1, v_{j+1}, v_s\}$ induces a modified claw, which implies that $w(v_1 v_{j+1}) = w(v_j v_{j+1})$.

Suppose $j = k-1$ and $s \geq k+2$. By the choices of v_k and v_s , we have $v_1 v_s \notin E(G)$, $v_1 v_{s+1} \notin E(G)$ and $v_j v_{s+1} \notin E(G)$. If $v_k v_{s+1} \in E(G)$, then $\{v_k, v_1, v_{k-1}, v_{s+1}\}$ induces a modified claw. So $w(v_1 v_k) = w(v_{k-1} v_k)$. If $v_k v_s \notin E(G)$, then $\{v_{k-1}, v_1, v_k, v_s\}$ induces a modified claw. So $w(v_1 v_k) = w(v_{k-1} v_k)$. If $v_k v_{s+1} \notin E(G)$ and $v_k v_s \in E(G)$, then $\{v_k, v_1, v_{k+1}, v_s\}$ induces a claw or a modified claw. So $w(v_1 v_k) = w(v_k v_s)$; On the other hand, $\{v_s, v_{k-1}, v_k, v_{s+1}\}$ induces a modified claw. So $w(v_k v_s) = w(v_{k-1} v_k)$; Then $w(v_1 v_k) = w(v_{k-1} v_k)$.

Suppose $j = k-1$ and $s = k+1$. Since P is longest, $N(v_k) \subset V(P)$; Since $G - v_{k+1}$ is connected, there exists an edge $v_k v_t \in E(G)$ with $t \geq k+2$. Choose $v_k v_t$ such that t is as small as possible. So $\{v_k, v_1, v_{k-1}, v_t\}$ induces a modified claw. Then $w(v_1 v_k) = w(v_{k-1} v_k)$. $\square$

Claim 3. $s = k + 1$.

Proof. Suppose $s \geq k + 2$. Let $P' = v_{s-1}v_{s-2} \cdots v_{j+1}v_1 v_2 \cdots v_j v_s v_{s+1} \cdots v_p$. If $v_j v_{s-1} \in E(G)$, then $\{v_j, v_1, v_{s-1}, v_s\}$ induces a modified claw, so $w(v_{s-1}v_s) = w(v_j v_s)$. By Claim 2, now P' is a heaviest longest v_p -path different from P , contradicting Claim 1. If $v_j v_{s-1} \notin E(G)$, then $\{v_s, v_j, v_{s-1}, v_{s+1}\}$ induces a claw or a modified claw. Thus $w(v_{s-1}v_s) = w(v_j v_s)$. By Claim 2, both $v_j v_{j-1} \cdots v_1 v_{j+1} v_{j+2} \cdots v_p$ and P' are heaviest longest v_p -paths, contradicting Claim 1. $\square$

As in the proof of Claim 2, since $s = k + 1$ and $G - v_{k+1}$ is connected, there exists an edge $v_k v_t \in E(G)$ with $k + 2 \leq t \leq p - 1$. Choose $v_k v_t$ such that t is as small as possible.

Case 1.1. $t \geq k + 3$.

By the choices of v_s and v_t , $v_j v_{k+2} \notin E(G)$ and $v_k v_{k+2} \notin E(G)$. So $\{v_{k+1}, v_j, v_k, v_{k+2}\}$ induces a claw or a modified claw. Thus $w(v_k v_{k+1}) = w(v_j v_{k+1})$.

Suppose $v_k v_{t+1} \notin E(G)$. By the choice of v_t , $v_k v_{t-1} \notin E(G)$. So $\{v_t, v_k, v_{t-1}, v_{t+1}\}$ induces a claw or a modified claw. Thus $w(v_{t-1}v_t) = w(v_k v_t)$. By Claim 2, both $v_k v_{k-1} \cdots v_{j+1}v_1 \cdots v_j v_{k+1} \cdots v_p$ and $v_{t-1}v_{t-2} \cdots v_{k+1}v_j v_{j-1} \cdots v_1 v_{j+1} \cdots v_k v_t v_{t+1} \cdots v_p$ are heaviest longest v_p -paths, contradicting Claim 1. Suppose $v_k v_{t+1} \in E(G)$. By the choice of v_k , $v_1 v_t \notin E(G)$ and $v_1 v_{t+1} \notin E(G)$. So $\{v_k, v_1, v_t, v_{t+1}\}$ induces a modified claw. Thus $w(v_k v_{t+1}) = w(v_1 v_{t+1})$. By Claim 2, $v_t v_{t-1} \cdots v_{k+1}v_j v_{j-1} \cdots v_1 v_{j+1} \cdots v_k v_{t+1} v_{t+2} \cdots v_p$ is a heaviest longest v_p -path different from P , contradicting Claim 1.

Case 1.2. $t = k + 2$.

By the choice of v_k , now $\{v_k, v_1, v_{k+1}, v_{k+2}\}$ induces a modified claw. So we get $w(v_k v_{k+1}) = w(v_k v_{k+2}) = w(v_{k+1} v_{k+2}) = w(v_1 v_k)$.

Since $G - v_t$ is connected, there exists an edge $v_{k+1} v_{t'} \in E(G)$ with $t' \geq k + 3$. By the choice of v_s , $v_j v_t \notin E(G)$ and $v_j v_{t'} \notin E(G)$. So $\{v_s, v_j, v_t, v_{t'}\}$ induces a claw or a modified claw. In this case $s = k + 1$ and $t = k + 2$, thus we have $w(v_j v_{k+1}) = w(v_{k+1} v_{k+2})$. By Claim 2, now $v_{k+1} v_j v_{j-1} \cdots v_1 v_{j+1} v_{j+2} \cdots v_k v_{k+2} \cdots v_p$ is a heaviest longest v_p -path different from P . At the same time, v_1 and v_{k+1} are nonadjacent vertices in the modified claw induced by $\{v_k, v_1, v_{k+1}, v_{k+2}\}$, contradicting Claim 1.

This completes the proof of Case 1.

Case 2. $N(v_1) \neq \{v_2, \dots, v_k\}$.

Choose $v_r \notin N(v_1)$ with $2 < r < k$ such that r is as large as possible. Then $v_1 v_i \in E(G)$ for every i with $r < i \leq k$. Let j be the smallest index such that $j > r$ and $v_j \notin N(v_1) \cap N(v_r)$. Since $v_{r+1} \in N(v_1) \cap N(v_r)$, we have $j \geq r + 2$. On the other hand, it is obvious that $j \leq k + 1$. By the choice of v_r and v_j , now v_1 and v_r are nonadjacent vertices of a claw or a modified claw induced by $\{v_{j-1}, v_1, v_r, v_j\}$. By Claim 1 and the choice of P , we have the following claim.

Claim 4. *There exists no heaviest longest v_p -path with v_r as it's another end-vertex.*

Claim 5. $w(v_1 v_{r+1}) \neq w(v_r v_{r+1})$.

Proof. If $w(v_1v_{r+1}) = w(v_rv_{r+1})$, then $v_rv_{r-1} \cdots v_1v_{r+1}v_{r+2} \cdots v_p$ is a heaviest longest v_p -path different from P , contradicting Claim 4. $\square$

Claim 6. $v_{r+1}v_j \notin E(G)$.

Proof. By the choice of v_r , $v_1v_r \notin E(G)$. So, if $v_{r+1}v_j \in E(G)$, then by the choice of v_j , we know that $\{v_{r+1}, v_1, v_r, v_j\}$ induces a claw or a modified claw. Thus $w(v_1v_{r+1}) = w(v_rv_{r+1})$, contradicting Claim 5. $\square$

By Claim 6, we have $r+3 \leq j \leq k+1$. Let n be the largest index such that $r+1 \leq n \leq j-2$ and $v_nv_j \notin E(G)$. Then $v_iv_j \in E(G)$ for every i with $n+1 \leq i \leq j-1$.

Claim 7. If $r+3 \leq j \leq k$, then $w(v_1v_{n+1}) = w(v_nv_{n+1})$.

Proof. By the choice of v_n , $v_{n+1}v_j \in E(G)$. So $\{v_{n+1}, v_1, v_r, v_j\}$ induces a modified claw. Thus $w(v_1v_{n+1}) = w(v_nv_{n+1})$. Since $v_rv_j \notin E(G)$ and $v_nv_j \notin E(G)$, $\{v_r, v_n, v_{n+1}, v_j\}$ induces a modified claw. Thus $w(v_nv_{n+1}) = w(v_{n+1}v_j)$. So $w(v_1v_{n+1}) = w(v_nv_{n+1})$. $\square$

Case 2.1. $r+3 \leq j \leq k-1$.

Suppose $v_{r+1}v_{j+1} \notin E(G)$. By Claim 6, $v_{r+1}v_j \notin E(G)$, so $\{v_1, v_j, v_{j+1}, v_{r+1}\}$ induces a modified claw. Thus $w(v_1v_{j+1}) = w(v_jv_{j+1})$. Suppose $v_{r+1}v_{j+1} \in E(G)$. By Claim 5 and the choice of v_j , applying Lemma 3 (i) to $\{v_1, v_{r+1}, v_r, v_{j+1}, v_j\}$, we get $w(v_1v_{j+1}) = w(v_jv_{j+1})$.

By Claim 7, both $v_nv_{n-1} \cdots v_1v_{n+1}v_{n+2} \cdots v_p$ and $v_jv_{j-1} \cdots v_1v_{j+1}v_{j+2} \cdots v_p$ are heaviest longest v_p -paths. Furthermore, v_n and v_j are nonadjacent vertices in the modified claw induced by $\{v_r, v_n, v_{n+1}, v_j\}$, contradicting Claim 1.

Case 2.2 $j = k$.

Since $\{v_{j-1}, v_1, v_r, v_j\}$ induces a claw or a modified claw and $j = k$, we have

Claim 8. $w(v_1v_{k-1}) = w(v_1v_k) = w(v_{k-1}v_k)$.

Claim 9. $v_{r+2}v_k \notin E(G)$.

Proof. Suppose $v_{r+2}v_k \in E(G)$. Applying Lemma 3 (ii) to $\{v_1, v_{r+1}, v_r, v_{r+2}, v_k\}$ and the vertex v_2 (which is adjacent to v_1), we get $v_2v_{r+1} \in E(G)$ and $w(v_1v_2) = w(v_2v_{r+1}) = w(v_1v_k) = w(v_rv_{r+1})$. By Claim 8, $v_rv_{r-1} \cdots v_2v_{r+1}v_{r+2} \cdots v_{k-1}v_1v_kv_{k+1} \cdots v_p$ is a heaviest longest v_p -path, contradicting Claim 4. $\square$

Now, we have $v_{r+1}v_{k-1} \notin E(G)$. Since otherwise, applying Lemma 3 (ii) to $\{v_1, v_{r+1}, v_r, v_{k-1}, v_k\}$ and the vertex v_{r+2} (which is adjacent to v_{r+1}), we get $v_{r+2}v_k \in E(G)$, contradicting Claim 9. By Claim 6, v_{k-1} and v_{r+1} are nonadjacent vertices in the modified claw induced by $\{v_1, v_{r+1}, v_{k-1}, v_k\}$.

By Claims 6 and 9, now $\{v_1, v_{r+1}, v_{r+2}, v_k\}$ induces a modified claw. Then $w(v_1v_{r+2}) = w(v_{r+1}v_{r+2})$. By Claim 8, both $v_{k-1}v_{k-2} \cdots v_1v_kv_{k+1} \cdots v_p$ and $v_{r+1}v_r \cdots v_1v_{r+2} \cdots v_p$ are heaviest longest v_p -paths, contradicting Claim 1.

Case 2.3 $j = k + 1$.

Since $\{v_{j-1}, v_1, v_r, v_j\}$ induces a claw or a modified claw, we have

Claim 10. $w(v_1 v_k) = w(v_k v_{k+1}) = w(v_r v_k)$.

Claim 11. $v_{r+1} v_k \notin E(G)$.

Proof. Suppose $v_{r+1} v_k \in E(G)$. Applying Lemma 3 (ii) to $\{v_1, v_{r+1}, v_r, v_k, v_{k+1}\}$ and the vertex v_{r-1} (which is adjacent to v_r), we get $v_{r-1} v_{k+1} \in E(G)$ and $w(v_{r-1} v_r) = w(v_{r-1} v_{k+1})$. By Claim 10, we get a heaviest longest v_p -path $v_r v_{r+1} \cdots v_k v_1 v_2 \cdots v_{r-1} v_{k+1} v_{k+2} \cdots v_p$, contradicting Claim 4. $\square$

Claim 12. $v_i v_{k+1} \notin E(G)$ for every $v_i \in N(v_1) \cap N(v_r) \setminus \{v_k\}$.

Proof. Suppose there exists a vertex $v_i \in N(v_1) \cap N(v_r) \setminus \{v_k\}$ such that $v_i v_{k+1} \in E(G)$. If $w(v_1 v_{r+1}) \neq w(v_1 v_k)$, then applying Lemma 3 (i) to $\{v_{r+1}, v_1, v_k, v_i, v_{k+1}\}$, we get $w(v_1 v_{r+1}) = w(v_k v_{k+1})$. So $w(v_1 v_k) \neq w(v_k v_{k+1})$, contradicting Claim 10. So we get $w(v_1 v_{r+1}) = w(v_1 v_k)$. Similarly, we can prove that $w(v_r v_{r+1}) = w(v_r v_k)$. By Claim 10, we get $w(v_1 v_{r+1}) = w(v_r v_{r+1})$, contradicting Claim 5. $\square$

Claim 13. $w(v_1 v_{k-1}) = w(v_{k-1} v_k)$.

Proof. By the choice of v_k and Claim 12, we get $v_1 v_{k+1} \notin E(G)$ and $v_{k-1} v_{k+1} \notin E(G)$. So $\{v_k, v_1, v_{k-1}, v_{k+1}\}$ induces a modified claw and $w(v_1 v_{k-1}) = w(v_{k-1} v_k)$. $\square$

Case 2.3.1. $w(v_r v_{r+1}) \neq w(v_r v_k)$.

By Claim 11, $v_{r+1} v_k \notin E(G)$. Furthermore, $v_r v_{k+1} \notin E(G)$, since otherwise it follows from Lemma 2 that $v_{r+1} v_{k+1} \in E(G)$, which contradicts Claim 6.

Applying Lemma 2 to the induced path $v_{r+1} v_r v_k$ and the vertex $v_{r-1} \in N(v_r) \setminus \{v_{r+1}, v_k\}$, we get $v_{r-1} v_{r+1} \in E(G)$ and $v_{r-1} v_k \in E(G)$. By Claim 5, $w(v_1 v_{r+1}) \neq w(v_r v_{r+1})$. Applying Lemma 2 to the induced path $v_1 v_{r+1} v_r$ and the vertex $v_{r-1} \in N(v_{r+1}) \setminus \{v_1, v_r\}$, we get $v_1 v_{r-1} \in E(G)$. This implies that $v_{r-1} \in N(v_1) \cap N(v_r)$. By Claim 12, we have $v_{r-1} v_{k+1} \notin E(G)$. Then $\{v_k, v_{r-1}, v_r, v_{k+1}\}$ induces a modified claw, so $w(v_{r-1} v_r) = w(v_{r-1} v_k)$. Therefore, by Claim 13, $v_r v_{r+1} \cdots v_{k-1} v_1 v_2 \cdots v_{r-1} v_k v_{k+1} \cdots v_p$ is a heaviest longest v_p -path, contradicting Claim 4.

Case 2.3.2. $w(v_r v_{r+1}) = w(v_r v_k)$.

By Claim 5 and Claim 10, we can get that $w(v_1 v_{r+1}) \neq w(v_1 v_k)$.

Claim 14. $v_2 v_k, v_2 v_{r+1}$ and $v_2 v_r \in E(G)$.

Proof. By Claim 11, $v_{r+1} v_k \notin E(G)$. Applying Lemma 2 to the induced path $v_{r+1} v_1 v_k$ and the vertex $v_2 \in N(v_1) \setminus \{v_{r+1}, v_k\}$, we get $v_2 v_k \in E(G)$ and $v_2 v_{r+1} \in E(G)$. By Claim 5, $w(v_1 v_{r+1}) \neq w(v_r v_{r+1})$. Applying Lemma 2 to the induced path $v_1 v_{r+1} v_r$ and the vertex $v_2 \in N(v_{r+1}) \setminus \{v_1, v_r\}$, we get $v_2 v_r \in E(G)$. $\square$

Claim 15. $v_r v_{k+1} \notin E(G)$.

Proof. Suppose $v_r v_{k+1} \in E(G)$. By Claim 14, we get $v_2 \in N(v_1) \cap N(v_r)$, $v_2 v_k$ and $v_2 v_{r+1} \in E(G)$. By Claims 6 and 12, we get $v_{r+1} v_{k+1} \notin E(G)$ and $v_2 v_{k+1} \notin E(G)$. Thus both $\{v_r, v_2, v_{r+1}, v_{k+1}\}$ and $\{v_k, v_1, v_2, v_{k+1}\}$ induce modified claws. Then we have $w(v_2 v_{r+1}) = w(v_r v_{r+1}) = w(v_1 v_2) = w(v_1 v_k)$. By Claim 13, we get $w(v_1 v_{k-1}) = w(v_{k-1} v_k)$. So, $v_r v_{r-1} \cdots v_2 v_{r+1} v_{r+2} \cdots v_{k-1} v_1 v_k v_{k+1} \cdots v_p$ is a heaviest longest v_p -path, contradicting Claim 4. $\square$

Suppose $v_1 v_{r-1} \in E(G)$. Then by applying Lemma 2 to the induced path $v_k v_1 v_{r+1}$ and $v_{r-1} \in N(v_1) \setminus \{v_k, v_{r+1}\}$, we get $v_{r-1} v_k \in E(G)$ and $v_{r-1} v_{r+1} \in E(G)$. By Claims 12 and 15, we get $v_{r-1} v_{k+1} \notin E(G)$ and $v_r v_{k+1} \notin E(G)$. Then $\{v_k, v_{r-1}, v_r, v_{k+1}\}$ induces a modified claw. Thus $w(v_{r-1} v_r) = w(v_{r-1} v_k)$. Therefore, by Claim 13, we have $v_r v_{r+1} \cdots v_{k-1} v_1 v_2 \cdots v_{r-1} v_k v_{k+1} \cdots v_p$ is a heaviest longest v_p -path, contradicting Claim 4.

Suppose $v_1 v_{r-1} \notin E(G)$. Then by Claim 5 and Lemma 2, we have $v_{r-1} v_{r+1} \notin E(G)$. Furthermore, $v_2 v_{r-1} \notin E(G)$, since otherwise, applying Lemma 3 (ii) to $\{v_r, v_{r+1}, v_1, v_2, v_{r-1}\}$ and the vertex v_k (which is adjacent to v_r), we get $w(v_1 v_{r+1}) = w(v_1 v_k)$, a contradiction. By Claims 12 and 14, $v_2 v_{k+1} \notin E(G)$. Then, both $\{v_r, v_{r+1}, v_2, v_{r-1}\}$ and $\{v_k, v_1, v_2, v_{k+1}\}$ induce modified claws. Thus $w(v_2 v_{r+1}) = w(v_r v_{r+1})$ and $w(v_1 v_2) = w(v_1 v_k)$. By Claim 13, we know that $v_r v_{r-1} \cdots v_2 v_{r+1} v_{r+2} \cdots v_{k-1} v_1 v_k v_{k+1} \cdots v_p$ is a heaviest longest v_p -path, contradicting Claim 4.

The proof of the theorem is complete.

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