Connectivity of Connected Bipartite Graphs with Two Orbits

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Abstract. Let $X = (V, E)$ be a simple connected graph, the connectivity $\kappa(X)$ of X is the minimum cardinality of all vertex cut sets in X . Let $x \in V(X)$, the set $\{x^g : g \in \text{Aut}(X)\}$ is called an orbit of $\text{Aut}(X)$, the automorphism group of X . In this note, we prove that the connectivity of a connected bipartite graph with two orbits attains its minimum degree.

Keywords: Connectivity, Bipartite graph, Orbit.

1 Introduction

Let $X = (V, E)$ be a simple connected graph, with $V(X)$ the set of vertices and $E(X)$ the set of edges. A *vertex cut* of X is a subset U of V such that the subgraph $X \setminus U$ induced by $V \setminus U$ is either trivial or not connected. The *connectivity* $\kappa(X)$ of a nontrivial connected graph X is the minimum cardinality of all vertex cut sets of X . If we denote by $\delta(X)$ the minimum degree of X , then $\kappa(X) \leq \delta(X)$.

We denote by $\text{Aut}(X)$ the automorphism group of X . The graph X is said to be *vertex transitive* if for any two distinct vertices of X there is an automorphism mapping one to the other. Let $x \in V(X)$, we call the set $\{x^g : g \in \text{Aut}(X)\}$ an *orbit* of $\text{Aut}(X)$. Clearly, $\text{Aut}(X)$ acts transitively on each orbit of $\text{Aut}(X)$.

In [3], Mader proved that the connectivity of a connected vertex transitive graph, which is K_4 -free, attains its minimum degree. It arouses us to consider the relation between the connectivity and the number of orbits. In this note, we prove that the connectivity of a connected bipartite graph with two orbits attains its minimum degree.

In the next, we introduce some terms and theorems which will be used in proving our main results.

Let $X = (V, E)$ be a connected graph and F be a non-empty subset of $V(X)$. Set $N(F) = \{x \in V(X) \setminus F : \exists y \in F, \text{ st. } xy \in E(X)\}$, $C(F) = F \cup N(F)$ and $R(F) = V(X) \setminus C(F)$. Clearly, $N(F)$ is a vertex cut set if $R(F) \neq \emptyset$. A subset $F \subset V(X)$ is said to be a *fragment* if $|N(F)| = \kappa(X)$ and $R(F) \neq \emptyset$. A fragment of minimum cardinality is called an *atom* of X . An *imprimitive block* for a group

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U of permutations on a set T is a proper, nontrivial subset A of T such that if $\sigma \in U$ then either $\sigma(A) = A$ or $\sigma(A) \cap A = \emptyset$. A subset A of $V(X)$ is called an imprimitive block for X if it is an imprimitive block for $\text{Aut}(X)$ on $V(X)$.

Theorem 1. [8] *Let $X=(V, E)$ be a nontrivial connected graph which is not a complete graph.*

- (i) $\kappa(X) = \delta(X)$ if and only if every atom of X has cardinality 1;
- (ii) if $\kappa(X) < \delta(X)$, then each atom has cardinality at most $\lceil (|V| - \kappa(X))/2 \rceil$ and induces a connected subgraph of X .

Theorem 2. [5] *If $X=(V, E)$ is a nontrivial connected graph which is not a complete graph, then distinct atoms of X are disjoint. Thus if $\kappa(X) < \delta(X)$, the atoms of X are imprimitive blocks of X .*

Theorem 3. [5] *Let $X=(V, E)$ be a nontrivial connected graph. If W is a minimum vertex cut set and A an atom of X , then $A \cap W = \emptyset$ or $A \subseteq W$.*

2 Main Results

Let $X=(V, E)$ be a connected bipartite graph with two orbits. Clearly, it is semi-regular. In this section, we use X_0 and X_1 to denote the two orbits of $\text{Aut}(X)$. Without loss of generality, we may assume that $m = d(u)$ for any $u \in X_0$, $n = d(v)$ for any $v \in X_1$, and $m \leq n$. So we have $\delta(X) = m$. Let A be an atom of X . Set $A_0 = A \cap X_0$ and $A_1 = A \cap X_1$, then $A = A_0 \cup A_1$.

Lemma 1. *Let $X=(V, E)$ be a connected bipartite graph with two orbits, and A be an atom of X . If $\kappa(X) < \delta(X)$, then $A_i = A \cap X_i$ ($i = 0, 1$) are nontrivial.*

Proof. By Theorem 1, the induced subgraph $Y = X[A]$ is a nontrivial connected subgraph of X , which is a bipartite graph, thus $A_i = A \cap X_i \neq \emptyset$ ($i = 0, 1$). Suppose that one of these two vertex subsets is trivial.

Case 1: $|A_0|=1$. Thus, $|A_1| \leq m$ since Y is connected. Set $F = N(A)$.

Subcase 1.1: $|A_1|=m$. So we have $|F| \geq n - 1$. If $|F|=n - 1$, then the induced subgraph $Y' = X[A \cup F]$ is a connected component of X , which is impossible since X is connected. Thus $|F| > n - 1$, namely $|F| \geq n \geq m = \delta(X)$, a contradiction.

Subcase 1.2: $|A_1|=p \leq m - 1$. Thus $|N(A_0) \setminus A_1| = m - p$. Let $q = |N(A_1) \setminus A_0|$, we have $q \geq n - 1$. Since

$$|F| = |N(A)| = |N(A_0) \setminus A_1| + |N(A_1) \setminus A_0| = m - p + q \geq m + n - p - 1$$

and $|F| = \kappa(X) < m$, we have $n < p + 1 \leq m$, a contradiction.

Case 2: $|A_1|=1$. Thus, $|A_0| \leq n$ since Y is connected. Set $F = N(A)$.

Subcase 2.1: $|A_0|=n$. So we have $|F|\geq m-1$. Since $|F|=\kappa(X) < m$, $|F|\leq m-1$. Namely, $|F|=m-1$. Thus we have that the induced subgraph $Y'=X[A\cup F]$ is a connected component of X , a contradiction.

Subcase 1.2: $|A_0|=p\leq n-1$. Thus $|N(A_0)\setminus A_1|=n-p$. Let $q=|N(A_1)\setminus A_0|$, then $q\geq m-1$. Since

$$|F|=|N(A)|=|N(A_0)\setminus A_1|+|N(A_1)\setminus A_0|=n-p+q\geq n+m-p-1$$

and $|F|=\kappa(X) < m$, we have $n < p+1\leq n$, a contradiction.

The result follows. $\square$

Lemma 2. Let $X=(V, E)$ be a connected bipartite graph with two orbits, and A be an atom of X , and $Y=X[A]$. If $\kappa(X) < \delta(X)$, then $\text{Aut}(Y)$ acts transitively on $A_i = A \cap X_i (i=0, 1)$.

Proof. By Lemma 1, $A_i=A\cap X_i (i=0, 1)$ are nontrivial. For any $u, v\in A_0$, there exists an automorphism φ of X with $\varphi(u)=v$ and so $\varphi(A)\cap A\neq\emptyset$. By Theorem 2, $\varphi(A)=A$. Since A_0 and A_1 are contained in different orbits, we have $\varphi(A_0)=A_0$ and $\varphi(A_1)=A_1$. Thus the restriction of φ on A induces an automorphism of Y , and then $\text{Aut}(Y)$ acts transitively on A_0 . Similarly, $\text{Aut}(Y)$ acts transitively on A_1 . The result follows. $\square$

Lemma 3. Let $X=(V, E)$ be a connected bipartite graph with two orbits, and A be an atom of X . If $\kappa(X) < \delta(X)$, then

- (i) Every vertex of X lies in an atom;
- (ii) $|A|\leq\kappa(X)$.

Proof. (i) By Lemma 1, the induced subgraph $Y=X[A]$ is a nontrivial connected subgraph of X , thus at least one vertex of $X_i (i=0, 1)$, respectively, lies in an atom. By the transitivity of X_i , every vertex of X lies in an atom.

(ii) Let $F=N(A)$. By (i), for any $u\in F$, u lies in an atom A' of X . By Theorem 3, $A'\subseteq F$, then $|A|=|A'|\leq|F|=\kappa(X)$. $\square$

Theorem 4. If $X=(V, E)$ is a connected bipartite graph with two orbits, then $\kappa(X)=\delta(X)$.

Proof. Suppose to the contrary that $\kappa(X) < \delta(X)$. By Theorem 1 and Lemma 3, $V(X)$ is a disjoint union of distinct atoms. Let A be an atom of X , then there exist $\sigma_i\in\text{Aut}(X) (i=1, \dots, k)$, such that

$$V(X)=\bigcup_{i=1}^k\sigma_i(A). \quad (1)$$

By Lemma 1, the induced subgraph $Y=X[A]$ is a nontrivial connected subgraph of X , and $A_i=A\cap X_i (i=0, 1)$ are nontrivial. Since $\text{Aut}(X)$ has two orbits X_0, X_1 , for any $1\leq i, j\leq k$ and $i\neq j$, $\sigma_i(A_0)\cap\sigma_j(A_0)=\emptyset$ and $\sigma_i(A_0), \sigma_j(A_0)\subseteq X_0$.

Thus, we have $X_0 = \bigcup_{i=1}^k \sigma_i(A_0)$, $X_1 = \bigcup_{i=1}^k \sigma_i(A_1)$, and $|A_i||X_i| (i = 0, 1)$. Since $|X_0|/|X_1| = n/m$, we have $|A_0|/|A_1| = n/m$.

By Lemma 2, the induced subgraph Y is semi-regular. Let $d = \delta(Y) = \delta_{A_0}$ and $F = N(A)$, we have $\delta_{A_1} = nd/m$. Since every vertex of A_0 has $m - d$ neighbours in F and every vertex of A_1 has $n - nd/m$ neighbours in F , we have $|F| = \kappa(X) \geq m - d + n - nd/m = m + n - (m + n)d/m$. By $\kappa(X) < \delta(X) = m$, we have $d > mn/(m + n)$. Thus, $|A| = |A_0| + |A_1| \geq d + nd/m = (m + n)d/m > n \geq m > \kappa(X)$. By Lemma 3, we deduce a contradiction.

The theorem then follows. $\square$

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References

1. Hamidoune, Y.O.: Sur les atomes d'un graphe orienté. C.R.Acad. Sc. Paris Ser. A 284 (1977) 1253–1256
2. Liang, X., Meng, J.: Connectivity of Bi-Cayley Graphs. accepted by Ars. Comb.
3. Mader, W.: Über den Zusammenhang symmetrischer Graphen. Arch. Math. 21 (1970) 331–336
4. Mader, W.: Minimale n -fach kantenzusammenhängende Graphen. Math. Ann. 191 (1971) 21–28
5. Mader, W.: Ein Eigenschaft der Atome endlicher Graphen. Arch. Math. 22 (1971) 331–336
6. Meng, J.: Connectivity of vertex and edge transitive graphs. Discrete Appl. Math. 127 (2003) 601–613
7. Tindell, R.: Connectivity of Cayley digraphs. In: Du D.Z., Hsu D.F.(Eds.): Combinatorial network theory. Kluwer Academic Publishers, The Netherlands (1996) 41–64
8. Watkins, M.E.: Connectivity of transitive graphs. J.Comb. Theory 8 (1970) 23–29