Linear Polynomial-Time Algorithms To Construct 4-Connected 4-Regular Locally Connected Claw-Free Graphs*

MingChu Li¹, Liming Xiong², and Hong Liu³

¹ College of Science, Chongqing Technology and Business University,
Chongqing, P.R. China

School of Software, Dalian University of Technology
Dalian 116620, China

² Department of Mathematics, Beijing Institute of Technology
Beijing 100081, P.R. China

³ School of Software, Dalian University of Technology
Dalian 116620, China

Abstract. A vertex of a graph is locally connected if its neighborhood is connected. A graph G is locally connected if every vertex of G is locally connected. A graph is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. In this paper, we provide a constructive characterization of 4-connected 4-regular locally connected claw-free graphs. From its proof, we can give a linear polynomial-time algorithm to construct a 4-connected 4-regular locally connected claw-free graph.

Keywords: constructive characterization, linear polynomial-time, claw-free graph.

1 Terminology and Notation

We will consider the class of undirected finite graphs without loops or multiple edges, and use [1] for terminology and notation not defined here. Let $G = (V, E)$ be a graph. For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G - H$ and $G[S]$ the induced subgraphs of G by $V(G) - V(H)$ and S , respectively. A clique of G is a complete subgraph of G . A clique with k vertices is called a k -clique. We say that a vertex is adjacent to a clique if it is adjacent to at least one vertex of the clique. A subgraph K and a subgraph B are disjoint if the vertices of K and the vertices of B are disjoint. For any vertex v , $N(v)$ denotes the neighborhood of v . For S in V , let $N(S) = \cup_{x \in S} N(x)$, and $\bar{N}(S) = N(S) - S = \{x \in N(S) \text{ but } x \notin S\}$. A vertex v of G is locally connected if its neighborhood $N(v)$ is connected otherwise v is locally disconnected. A graph G is locally connected if every vertex of G is locally connected. A graph is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. Whenever

* Supported by Nature Science Foundation of China under grant numbers: 60673046 (M. Li) and 10671014 (L. Xiong).

$\{v, v_1, v_2, v_3\}$ denotes an induced claw, then the leftmost vertex v is meant to be the center of the claw.

2 Results

There are a lot of results on algorithms and their complexity analysis (see [2], [3], [4], [5]). In this paper, we will provide a constructive characterization of 4-connected, 4-regular, locally connected, claw-free graphs. From its proof, we can give a linear polynomial-time algorithm to construct a 4-connected 4-regular locally connected claw-free graph. In order to explore our main results, we first prove the following result.

Lemma 1. *Let G be a 4-connected 4-regular claw-free graph on n vertices. If G contains a 4-clique, then every vertex of G is contained in a 4-clique, and all 4-cliques are disjoint if $n \geq 6$.*

Proof. Let K be a clique in G . If there is a vertex in G such that it is not contained in any 4-clique, then there is at least one vertex w in G adjacent to some clique such that w is not contained in any 4-clique. Without loss of generality assume that K is such a 4-clique adjacent to w . Let $V(K) = \{v_1, v_2, v_3, v_4\}$ and $wv_1 \in E(G)$. Then $N(v_1) = \{v_2, v_3, v_4, w\}$. Let w' be a neighbor of w and $w' \neq v_1$. If $w' \in \{v_2, v_3, v_4\}$ (say $w' = v_2$), then $N(v_2) = \{v_1, v_3, v_4, w'\}$. If $n = 5$, then G is K_5 and so we are done. So $n \geq 6$. It follows that $G - \{v_3, v_4, w\}$ is not connected. Thus G is not 4-connected. This contradiction shows that $w' \notin \{v_2, v_3, v_4\}$ and $(N(w) - \{v_1\}) \cap \{v_2, v_3, v_4\} = \emptyset$. For any two vertices w_1, w_2 in $N(w) - \{v_1\}$, we have $w_1w_2 \in E(G)$ since $G[w, v_1, w_1, w_2] \neq K_{1,3}$ and $d(v_1) = 4$. It follows that $G[(N(w) \cup \{w\}) - \{v_1\}]$ is a 4-clique. This contradiction shows that every vertex of G is contained in a 4-clique. Since G is 4-regular, all 4-cliques are disjoint. So the proof of Lemma 1 is completed.

Lemma 2. *Let G be a 4-connected 4-regular claw-free graph on $n \geq 7$ vertices and v a vertex of G . If v is locally connected, then $G[N(v)]$ is an induced hamiltonian path in $G[N(v)]$.*

Proof. Let v be a vertex in G and $N(v) = \{v_1, v_2, v_3, v_4\}$ such that $G[N(v)]$ is connected. If G contains a 4-clique, then, by Lemma 1, every vertex of G is contained in a 4-clique K . Assume that $v \in V(K)$ and $G[(N(v) \cup \{v\}) - \{v_4\}]$ is a 4-clique, then v_4 is adjacent to at least one vertex of $\{v_1, v_2, v_3\}$ since $G[N(v)]$ is connected. Let $v_4v_1 \in E(G)$. Then $N(v_1) = \{v, v_2, v_3, v_4\}$ and $G - \{v_4, v_3, v_2\}$ is not connected. Thus G is not 4-connected. This contradiction shows that G does not contain a 4-clique. It follows that $G[N(v) - \{v_i\}]$ is not a clique for $i = 1, 2, 3, 4$. Assume that $v_1v_2 \notin E(G)$. Since $G[N(v)]$ is connected, there is a path P with three or four vertices connecting v_1 and v_2 in $G[N(v)]$. Thus we consider two cases to prove that $G[N(v)]$ is an induced path on 4 vertices with internal vertices of degree 2 and two end-vertices of degree 1 in $G[N(v)]$.

Case 1. $|V(P)| = 3$.

Assume that $V(P) = \{v_1, v_3, v_2\}$. Since $G[v, v_1, v_2, v_4] \neq K_{1,3}$, $v_4v_1 \in E(G)$ or $v_4v_2 \in E(G)$ (say $v_4v_2 \in E(G)$). Note that $v_3v_4 \notin E(G)$ since otherwise $G[v, v_2, v_3, v_4]$ is a 4-clique. We have that $v_4v_1 \notin E(G)$ since otherwise $G[v_1, v_4, v_3, v'_1] \neq K_{1,3}$, $v'_1v_3 \in E(G)$ or $v'_1v_4 \in E(G)$, where $v'_1 \in N(v_1) - \{v_3, v_4, v\}$. Let $A = \{v'_1, v_2, w'\}$, where $w' = v_3$ if $v_4v'_1 \in E(G)$ and $w' = v_4$ if $v_3v'_1 \in E(G)$. It easy to see that $G - A$ is not connected since $n \geq 7$. Thus G is not 4-connected. This contradiction shows $v_4v_1 \notin E(G)$. Thus $G[N(v)]$ is just an induced hamiltonian path $P = v_1v_3v_2v_4$ with internal vertices of 2 degree and two end-vertices of degree 1 in $G[N(v)]$.

Case 2. $|V(P)| = 4$.

Assume that $P = v_1v_3v_4v_2$. Then $v_1v_4, v_3v_2 \notin E(G)$ since otherwise, say $v_1v_4 \in E(G)$, $G - \{v_1, v_2, v_3\}$ is not connected since $\bar{N}(\{v, v_4\}) = \{v_1, v_2, v_3\}$, a contradiction. Thus $G[N(v)]$ is just an induced hamiltonian path $P = v_1v_3v_4v_2$ with internal vertices of 2 degree and two end-vertices of degree 1 in $G[N(v)]$. It follows that the proof of Lemma 2 is completed.

Fig. 1.

Theorem 3. *Every 4-connected 4-regular locally connected claw-free graph G on at least 7 vertices is isomorphic to a graph of the depiction in Figure 1.*

Proof. Assume that G is not isomorphic to a graph of the form in Figure 1, and v is any vertex of G . Then v is locally connected. By Lemma 2, $N(v)$ is just an induced hamiltonian path in $G[N(v)]$. We construct the graph starting from v . Let $N(v) = \{v_1, v_2, u_1, u_2\}$ and $G[N(v)] = v_2v_1u_1u_2$. Let $W_1 = \{v_1, v_2\}$ and $W_2 = \{u_1, u_2\}$ and let W be the closed neighborhood of v (i.e., $W = W_1 \cup W_2 \cup \{v\}$) and $H = G[W]$. Note that $v_2u_1, v_1u_2, v_2u_2 \notin E(G)$. Then

$N(v_1) \cap N(u_1) = \{v\}$ since otherwise, let $u \in N(v_1) \cap N(u_1) - \{v\}$. Then $\bar{N}(\{v_1, u_1, v\}) = \{u, v_2, u_2\}$. Thus $\{u, v_2, u_2\}$ is a cut set of G with three vertices, a contradiction. Let $N(v_1) = \{v_2, v, u_1, v_3\}$. Then $v_3u_1 \notin E(G)$. Adding v_3 to W_1 . Since $|V(G)| \geq 7$ but $|W| = 6$, $V(G) \neq W$. We have from Lemma 2 that $v_3v_2 \in E(G)$. We further have the following fact.

Claim 1. $v_3 \notin N(W_2)$.

Recall that $v_3u_1 \notin E(G)$. Now we prove $v_3u_2 \notin E(G)$. If $v_3u_2 \in E(G)$, let $N(v_2) = \{v, v_1, v_3, v_4\}$. Then we have from $G[v_2, v_4, v, v_3] \neq K_{1,3}$ that $v_4v_3 \in E(G)$. Since $G[v_3, v_4, v_1, u_2] \neq K_{1,3}$ and $v_1v_4, v_1u_2 \notin E(G)$, $u_2v_4 \in E(G)$. If $V(G) \neq W \cup \{v_4\}$, then $\{v_4, u_1\}$ is a cut-set of two vertices since v_1, v_2, v_3, u_2 are of degree 4 in $G[W \cup \{v_4\}]$, a contradiction. Thus $V(G) = W \cup \{v_4\}$ and add v_4 into W_1 , it follows that $u_1v_4 \in E(G)$ since only v_1, u_1, v_4 are of degree 3 in $G[W]$. So G is isomorphic to a graph of the depiction in Figure 1, a contradiction. Thus Claim 1 is true.

Note that v_2 is of degree 3 and v_3 is of degree 2 and v_1 is of degree 4 in H . Consider the vertex v_3 of degree 3 in H . By Claim 1, we have $V(G) \neq W$. Let $N(v_2) = \{v, v_1, v_3, v_4\}$ and add v_4 into W_1 . From Lemma 2, $v_i v_{i-1} \in E(G)$, where $i = 4$. We further have the following fact.

Claim 2. If $V(G) \neq W$, then $v_4 \notin N(W_2)$.

If $v_4u_2 \in E(G)$, let $N(v_3) = \{v_1, v_2, v_4, v_5\}$. Then $v_5v_4 \in E(G)$ since $G[v_3, v_5, v_4, v_1] \neq K_{1,3}$. Since $G[v_4, v_2, v_4, u_2] \neq K_{1,3}$ and $v_2v_5, v_2u_2 \notin E(G)$, $v_5u_2 \in E(G)$. Thus $d_H(v_4) = 4$. If $V(G) \neq W \cup \{v_5\}$, then $\{v_5, u_1\}$ is a cut set of two vertices, a contradiction. Thus $V(G) = W \cup \{v_5\}$, and add v_5 into W_1 . Since $d_H(v_5) = d_H(u_1) = 3$, $u_1v_5 \in E(G)$. Thus G is isomorphic to a graph of the depiction in Figure 1, a contradiction. So $v_4u_2 \notin E(G)$. If $v_4u_1 \in E(G)$, then $G[u_1, u_2, v_1, v_4] = K_{1,3}$. This contradiction shows that Claim 2 is true.

From the above, we have established the following claim.

Claim 3. v_i is of degree 2, v_{i-1} is of degree 3, and v_j is of degree 4 in H for $j = 1, 2, \dots, i-2$, where $i = 4$.

Now we return to the proof of Theorem 3. If $V(G) = W$, then we have from $d_H(u_2) = 2$ that $u_2v_4, u_2v_3 \in E(G)$. Thus $u_1v_4 \in E(G)$. It follows that G is isomorphic to Figure 1. We assume that $V(G) \neq W$. Then, by Claim 3, consider the vertex v_3 of degree 3 in H and add the neighbor v_5 of v_4 outside W into W_1 . By Lemma 2, $v_5v_4 \in E(G)$. Then v_5 has similar properties in Claim 2 and Claim 3 ($i = 5$). But G is finite, we must have $V(G) = W$ and $W_1 = \{v_1, v_2, \dots, v_{n-3}\}$. And Claim 3 is true for $i = n-3$. It follows that G is isomorphic to a graph of the depiction in Figure 1. Therefore, the proof of Theorem 3 is completed.

Algorithm 4. *Construction of a 4-connected 4-regular locally connected claw-free graph G on $n \geq 7$ vertices.*

1. Pick up any vertex v .
2. Find 4 neighbors of v . Let $v_2v_1u_1u_2$ is a hamiltonian path in $N(v)$ and $W_1 = \{v_1, v_2\}$ and $W = \{v, u_1, u_2\} \cup W_1$, and set $i = 1$.
3. Find a neighbor v_{i+2} of v_i . Set $W_1 = (W_1 \cup \{v_{i+2}\}) - \{v_i\}$ and $W = W \cup \{v_i\}$
4. If $|W| = |V(G)|$, go to Step 5. Otherwise, set $i = i + 1$ and go to 3.
5. Check whether u_2 is adjacent to all vertices of W_1 and u_1 is only adjacent to v_{n-3} .

We now analyze time in Algorithm 4. Note that n is the number of vertices.

Step 1 needs $O(1)$. Step 2 needs $O(n)$. Step 3 needs $O(n)$ and Step 4 needs $O(1)$. By Theorem 3, such a graph is 4-connected 4-regular locally connected claw-free. Thus we obtain a linear polynomial time algorithm to construct a 4-connected 4-regular locally connected claw-free graph.

References

1. J. A. Bondy and U.S.R. Murty, Graph Theory with its applications, MacMillan, New York, 1976
2. M.R. Garey and D.S. Johnson, Computers and Intractability-A Guide to the theory of NP-completeness, Freeman, New York, 1979.
3. M. R. Garey, D.S. Johnson and R.E. Tarjan, The planar hamiltonian circuit problem is NP-complete, SIAM J. Computing, 5(1976), 704-714.
4. M. Li, D.G. Corneil and E. Mendelsohn, Pancyclicity and NP-completeness in planar graphs, Discrete Applied Math., 98(2000) 219-225.
5. L. G. Valiant, The complexity of computing the permanent, Theor. Computer Sci., 8(1979), 189-201.