Some Results on List Total Colorings of Planar Graphs*

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Abstract. Let G be a planar graph with maximum degree Δ . In this paper, it is proved that if $\Delta \geq 9$, then G is total- $(\Delta+2)$ -choosable. Some results on list total coloring of G without cycles of specific lengths are given.

Keywords: planar graph, total coloring, list total coloring, cycle.

1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [2]. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G , respectively. Let $d(v)$ denote the degree of vertex v .

A *total k -coloring* of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest integer k such that G has a total k -coloring. Total colorings were introduced by Vizing [16] and Behzad [1]. They both, independently, conjectured that $\chi''(G) \leq \Delta(G) + \mu(G) + 1$ holds for multigraphs, where $\mu(G)$ denotes the (edge) multiplicity of G . This conjecture became known as the Total Coloring Conjecture. The conjecture has been verified for multigraphs of sufficiently small maximum degree. Rosenfeld [13] and Vijayaditya [15] independently proved that the total chromatic number for multigraphs of maximum degree 3 is at most 5. Kostochka in [10,11] proved that the total chromatic number of multigraphs of maximum degree 4 (respectively 5) is at most 6 (respectively 7). It is also easy to see that for a bipartite multigraph G , we have $\chi''(G) \leq \Delta(G) + 2$. For planar graphs, the conjecture only remains open for $\Delta(G) = 6$ [3, 20, 14].

The mapping L is said to be a *total assignment* for the graph G if it assigns a list $L(x)$ of possible colors to each element $x \in V \cup E$. If G has a total coloring ϕ such that $\phi(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that G is *total- L -colorable*. Let $f : V \cup E \rightarrow \mathbb{N}$ be a function into the positive integers. We say that G is *total- f -choosable* if it is total- L -colorable for every total

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assignment L satisfying $|L(x)| = f(x)$ for all elements $x \in V(G) \cup E(G)$. The *list total chromatic number* $\chi_l''(G)$ of G is the smallest integer k such that G is totally- f -choosable when $f(x) = k$ for each $x \in V(G) \cup E(G)$. The *list chromatic number* $\chi_l(G)$ of G and the *list edge chromatic number* (or *list chromatic index*) $\chi_l'(G)$ of G are defined similarly in terms of coloring vertices alone, or edges alone, respectively; and so are the concepts of *vertex- f -choosability* and *edge- f -choosability*.

List colorings were introduced by Vizing [17] and independently by Erdős, Rubin, and Taylor [6]. Probably the most well-know conjecture about list colorings is the List Coloring Conjecture which states that every (multi)graph G is edge- $\chi'(G)$ -choosable, where $\chi'(G)$ is the usual chromatic index of G . This conjecture has been proved for special cases, such as bipartite multigraphs [7], complete graphs of odd order [8], multicircuits [19], outerplanar graphs [18], and graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of nonnegative characteristic [5].

As far as list total colorings are concerned, Juvan, Mohar and Škrekovski [9] posed the following conjecture which is the generalization of the Total Coloring Conjecture:

Conjecture 1.1. $\chi_l''(G) \leq \Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the multiplicity of G .

In [9], Juvan et al. proved that the conjecture is true for bipartite multigraphs and multigraphs of maximum degree 3. They also proved that the total-choosability of a graph of maximum degree 2 is equal to its total chromatic number.

In Section 2, we show that Conjecture 1.1 is true for planar graph G with $\Delta(G) \geq 9$. In Section 3, we consider planar graphs without cycles of specific lengths and get some related results on Conjecture 1.1. In Section 4, open problems for future research are given.

2 Planar Graphs with Maximum Degree at Least 9

First, let us introduce some notations and definitions. Let $G = (V, E, F)$ be a planar graph. A vertex v is called a k -vertex or k^+ -vertex if $d(v) = k$ or $d(v) \geq k$, respectively. For $f \in F$, we use $b(f)$ to denote the closed boundary walk of f and write $f = [u_1 u_2 \dots u_n]$ if $u_1, u_2, \dots, u_n$ are the vertices on the boundary walk in the clockwise order, with repeated occurrences of vertices allowed. The degree of a face f , denoted by $d(f)$, is the number of edges-steps in $b(f)$. Note that each cut-edge is counted twice. A k -face or a k^+ -face is a face of degree k or of degree at least k , respectively. Let $\delta(f)$ denote the minimum degree of vertices incident with f . When v is a k -vertex, we say that there are k faces incident to v . However, these faces are not required to be distinct, i.e., v may have repeated occurrences on the boundary walk of an indent face. Let $n_k(v)$ or $n_{k^+}(v)$ denote the number of k -faces or k^+ -faces incident with vertex v with repeated occurrence of faces allowed, respectively.

Following theorem is our main result in this section.

Theorem 2.1. *Let G be a planar graph with maximum degree $\Delta \geq 9$. Then G is total- $(\Delta + 2)$ -choosable.*

Proof. Let $G = (V, E, F)$ be a minimal counterexample to the theorem. Then G has the following properties.

(a) G is connected.

(b) G contains no edge uv with $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta+1}{2} \rfloor$ and $d(u) + d(v) \leq \Delta + 2$.

(a) is obvious. We will show (b). Suppose it does contain such edge $e = uv$. Without loss of generality, let $d(u) = \min\{d(u), d(v)\}$. We can list total color $G - e$ by the minimality of G and then erase the color on u . Thus e touches at most $(\Delta + 1)$ colors and can be colored from its list. Now u is adjacent or incident to at most $2d(u) \leq \Delta + 1$ colors and it too can be colored. This contradicts the choice of G as a counterexample and shows that (b) holds.

It is easy to verify $\delta(G) \geq 3$ by (b). Let G_3 be the subgraph induced by the edges incident with the 3-vertices of G . Clearly, G_3 does not contain odd cycles by (b). Thus G_3 is a bipartite graph with partite sets V_1, V_2 , so that $V(G_3) = V_1 \cup V_2$ and for any vertex $v \in V_1$, $d_G(v) = 3$, for any vertex $v \in V_2$, $d_G(v) = \Delta$. Since G is a minimal counterexample, we have

(c) G_3 contains a bipartite subgraph $G' = (V'_1, V'_2, E(G'))$, such that $V'_1 = V_1$ and for any vertex $v \in V'_1$, $d_{G'}(v) = 2$; for any vertex $v \in V'_2$, $d_{G'}(v) = 1$. If $uv \in E(G')$ and $d_G(u) = 3$, then v is called the 3-master of u and u is called the dependent of v .

Before prove (c), we first show that G contains no even cycle $v_1v_2...v_{2t}v_1$ such that $d(v_1) = d(v_3) = ... = d(v_{2t-1}) = 3$. Suppose it does contain such an even cycle C . Total color all elements of $G - E(C)$ from their lists, which is possible by the minimality of G . Then erase the colors on $v_1, v_3, ..., v_{2t-1}$. If e is an edge of C , then e now effectively has a list $L'(e)$ with $|L'(e)| \geq \Delta + 2 - (2 + \Delta - 1) = 2$. Thus the edges of C can be colored from their lists. Finally, if we are coloring vertices $v_1, v_3, ..., v_{2t-1}$, then each vertex $v_{2k-1} \in C$, $1 \leq k \leq t$, is now adjacent or incident to at most 6 colors and so there is at least one color in its list that we can give to v_{2k-1} . Thus we obtain the required contradiction.

The mentioned result above implies that G_3 does not contain even cycles. Thus G_3 is a forest. For any component of G_3 , we can select a vertex u with $d_G(u) = 3$ as the root of the tree. We denote edges of distance i from the root to be at level $i + 1$, where $i = 0, 1, ..., d$, and d is the depth of tree. Since G does not contain two adjacent 3-vertices, the distance from any leaf to the root is even. We can select all the edges at even level to form a 3-path $v_1v_2v_3$ such that $d_G(v) = 3$ in this component. Thus we can find a bipartite subgraph $G' = (V'_1, V'_2, E(G'))$, such that $V'_1 = V_1$, for any vertex $v \in V'_1$, $d_{G'}(v) = 2$ and for any vertex $v \in V'_2$, $d_{G'}(v) = 1$. It completes the proof of (c).

Note that each 3-vertex has exactly two 3-masters and each vertex of degree Δ can be the 3-master of at most one 3-vertex.

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0.$$

Now we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(x) = d(x) - 4$ for $x \in V \cup F$. It follows that $\sum_{x \in V \cup F} w(x) < 0$. The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to $\sum_{x \in V \cup F} w(x) \geq 0$. A contradiction follows.

To prove the theorem, we are ready to construct a new charge $w^*(x)$ on G as follows.

R_1 : Each 3-vertex receives $\frac{1}{2}$ from each of its 3-master.

R_2 : Each 9^+ -vertex transfer $\frac{1}{2}$ to each of its incident 3-face.

R_3 : Each k -vertex, where $5 \leq k \leq 8$, transfer $\frac{k-4}{k}$ to each of its incident 3-face.

Let $\gamma(x \rightarrow y)$ denote the amount transferred out of an element x into another element y according to the above rules.

Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 4$. Assume that $f = [v_1 v_2 v_3]$ is a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. If $d(v_1) = 3$, then $d(v_2) = d(v_3) = \Delta$ by (c). So $w^*(f) = w(f) + 2 \times \frac{1}{2} = 0$. If $d(v_1) = 4$, then $d(v_2) \geq 8$ and $d(v_3) \geq 8$. So $\gamma(v_2 \rightarrow f) = \frac{1}{2}$ and $\gamma(v_3 \rightarrow f) = \frac{1}{2}$. Thus $w^*(f) = w(f) + 2 \times \frac{1}{2} = 0$. If $d(v_1) = 5$, then $d(v_2) \geq 7$ and $d(v_3) \geq 7$. So $\gamma(v_1 \rightarrow f) = \frac{1}{5}$, $\gamma(v_2 \rightarrow f) \geq \frac{3}{7}$, and $\gamma(v_3 \rightarrow f) \geq \frac{3}{7}$. Thus $w^*(f) \geq w(f) + \frac{1}{5} + 2 \times \frac{3}{7} > 0$. If $d(v_1) = 6$, then $\gamma(v_i \rightarrow f) \geq \frac{1}{3}$ for $i = 1, 2, 3$. So $w^*(f) \geq w(f) + 3 \times \frac{1}{3} = 0$.

Let v be a k -vertex. If $k = 3$, then there are two vertices, say v_1, v_2 , such that v_i , $i = 1, 2$, is the 3-master of v . So $\gamma(v_i \rightarrow v) = \frac{1}{2}$ for $i = 1, 2$. Thus $w^*(v) \geq w(v) + 2 \times \frac{1}{2} = 0$. If $k = 4$, then $w^*(v) = w(v) = 0$. If $5 \leq k \leq 8$, then $w^*(v) = w(v) - k \times \frac{k-4}{k} = 0$. If $k \geq 9$, then v can be the 3-master of at most one 3-vertex. So $w^*(v) \geq w(v) - \frac{1}{2} - \frac{k}{2} = \frac{k-9}{2} \geq 0$. $\square$

3 Planar Graphs Without Cycles of Specific Lengths

In this section, we consider planar graphs without certain cycles. Note notations and definitions are the same as those in Section 2. Following result is our main theorem in this section.

Theorem 3.1. *Let G be a planar graph with maximum degree Δ such that G is free of k -cycles, where $k \geq 3$. Then G is total- $(\Delta + 2)$ -choosable, if*

- (1) $\Delta \geq 5$ and $k = 3$, or
- (2) $\Delta \geq 5$ and $k = 4$, or
- (3) $\Delta \geq 6$ and $k = 5$, or
- (4) $\Delta \geq 6$ and $k = 6$.

Proof. Let $G = (V, E, F)$ be a minimal counterexample to the theorem. Then G has the following proprieties.

(a) G is connected.

(b) G contains no edge uv with $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta+1}{2} \rfloor$ and $d(u) + d(v) \leq \Delta + 2$.

(c) $\delta(G) \geq 3$ and for any 3-vertex $v \in V$, v has exactly two 3-masters. For each vertex v of degree Δ , v can be the 3-master of at most one 3-vertex.

(a) is obvious. The proofs of (b) and (c) are the same as the above in Theorem 2.1, so we omit.

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0.$$

Now we define the initial charge function $w(x) = d(x) - 4$ for each $x \in V \cup F$. It follows that $\sum_{x \in V \cup F} w(x) < 0$.

In order to prove (1), we are ready to construct a new charge $w^*(x)$ on G as follows.

R_1 : Each 3-vertex receives $\frac{1}{2}$ from each of its 3-master.

Clearly, $w^*(f) = w(f) \geq 0$ for any face $f \in F$. If v is a 3-vertex by (c), then v has exactly two 3-masters. So $w^*(v) = w(v) + 2 \times \frac{1}{2} = 0$. If v is a 4-vertex, then $w^*(v) = w(v) = 0$. If v is a 5^+ -vertex, then v can be the 3-master of at most one 3-vertex. So $w^*(v) \geq w(v) - \frac{1}{2} > 0$.

We will prove the following claim before we proof (2).

Claim 3.2. *If $\Delta \geq 5$, then G contains no 3-face $f = uvw$ such that $d(u) = d(v) = d(w) = 4$.*

Proof. Suppose on the contrary, such 3-face does exist. Thus $G - \{uv, uw, vw\}$ has a list total coloring ϕ by the minimality of G , then erase the colors on u, v and w . For each element x incident with f , we define a reduced total list $L'(x)$ such that $L'(x) = L(x) \setminus \{\phi(x') | x' \text{ is incident or adjacent to } x \text{ and } x' \text{ is not incident with } f\}$. Then $|L'(u)| \geq 3$, $|L'(v)| \geq 3$, $|L'(w)| \geq 3$, $|L'(uv)| \geq 3$, $|L'(uw)| \geq 3$ and $|L'(vw)| \geq 3$. It follows from $\chi'_l(C_3) = \chi''(C_3) = 3$ that uncolor elements incident with f in G can be colored from its total lists L' . This contradicts the choice of G as a counterexample. $\square$

To prove (2), we are ready to construct a new charge $w^*(x)$ on G as follows:

$R_{2.1}$: Each 3-vertex receives $\frac{3}{10}$ from each of its 3-master.

$R_{2.2}$: For a k -face f , where $k \geq 5$, and each occurrence of a vertex $v \in b(f)$, transfer the amount $\frac{k-4}{k}$ from f to v .

$R_{2.3}$: From each 4-vertex to each of its incident 3-face, transfer $\frac{1}{5}$.

$R_{2.4}$: From each 5^+ -vertex v to each of its incident 3-face, transfer $\frac{3}{5}$.

Clearly, $w^*(f) = w(f) - k \times \frac{k-4}{k} = 0$ if $k \geq 5$. Let f be a 3-face. If f is incident with a 3-vertex, then the other vertices f incident with are 5^+ -vertices. So $w^*(f) = w(f) + 2 \times \frac{3}{5} = \frac{1}{5} > 0$. Otherwise, f is incident with at least one 5^+ -vertex by Claim 3.2. So $w^*(f) \geq w(f) + 2 \times \frac{1}{5} + \frac{3}{5} = 0$.

Let v be a k -vertex. If $k = 3$, then v has exactly two 3-master by (c) and v is incident with at least two 5^+ -faces since G is C_4 -free. So $w^*(f) \geq w(f) + 2 \times \frac{3}{10} + 2 \times \frac{1}{5} = 0$. If $k = 4$, then v is incident with at most two 3-faces. So $w^*(f) \geq w(f) + 2 \times \frac{1}{5} - 2 \times \frac{1}{5} = 0$. If $k = 5$, then v can be the 3-master of at most one 3-vertex and v is incident with at most two 3-faces. So $w^*(f) \geq w(f) + 3 \times \frac{1}{5} - 2 \times \frac{3}{5} - \frac{3}{10} = \frac{1}{10} > 0$. If $k \geq 6$, then v is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces, since G is C_4 -free. So $w^*(f) \geq w(f) + \frac{1}{5} \times \frac{k}{2} - \frac{k}{2} \times \frac{3}{5} - \frac{3}{10} > 0$.

To prove (3), we are ready to construct a new charge $w^*(x)$ on G as follows.

$R_{3,1}$: Each 3-vertex receives $\frac{1}{2}$ from each of its 3-master.

$R_{3,2}$: For a k -face f , where $k \geq 6$, and each occurrence of a vertex $v \in b(f)$, transfer the amount $\frac{1}{3}$ from f to v .

$R_{3,3}$: From each 4-vertex to each of its incident 3-face, transfer $\frac{1}{3}$.

$R_{3,4}$: From each 5^+ -vertex to each of its incident 3-face, transfer $\frac{1}{2}$.

Clearly, $w^*(f) = w(f) = 0$ if $d(f) = 4$. Assume that $d(f) = 3$. If $\delta(f) = 3$, then f is incident with two 6^+ -vertices by (b). So $w^*(f) = w(f) + 2 \times \frac{1}{2} = 0$. Otherwise, $w^*(f) \geq w(f) + 3 \times \frac{1}{3} = 0$. If f is a 6^+ -face, then $w^*(f) = w(f) - d(f) \times \frac{1}{3} = \frac{2d(f)}{3} - 4 \geq 0$.

Let v be a k -vertex. If $k = 3$, then $w^*(v) \geq w(v) + 2 \times \frac{1}{2} = 0$. Assume that $k = 4$. If v is incident with no 3-faces, then $w^*(v) = w(v) = 0$. Otherwise, v is incident with at least two 6^+ -faces, since G is C_5 -free, so $w^*(v) \geq w(v) + 2 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0$. Assume that $k = 5$. If v is incident with no 3-faces, then $w^*(v) \geq w(v) = 1 > 0$. Otherwise, v is incident with at most three 3-faces and if v is incident with exactly three 3-faces, then the other faces f incident with are 6^+ -faces. So $w^*(v) \geq w(v) + \min\{-2 \times \frac{1}{2}, -3 \times \frac{1}{2} + 2 \times \frac{1}{3}\} = 0$. Let $k = 6$. Then v can be the 3-master of at most one 3-vertex. If v is incident with at most three 3-faces, then $w^*(v) \geq w(v) - \frac{1}{2} - 3 \times \frac{1}{2} = 0$. Otherwise, v is incident with exactly four 3-faces since G is C_5 -free, and the other faces f incident with are 6^+ -faces. So $w^*(v) \geq w(v) - \frac{1}{2} - 4 \times \frac{1}{2} + 2 \times \frac{1}{3} = \frac{1}{6} > 0$. If $k \geq 7$, then v is incident with at most $(k - 2)$ 3-faces since G is C_5 -free. So $w^*(v) \geq w(v) - \frac{1}{2} - (k - 2) \times \frac{1}{2} = \frac{k-7}{2} \geq 0$.

In order to proof (4) in the theorem, we need the following claim.

Claim 3.3. *If $\Delta \geq 6$ and G contains a 3-face $f = uvw$ such that $d(u) = d(v) = 4$, then $d(w) \geq 6$.*

Proof. Otherwise, $d(w) \leq 5$. We can list total color $G - \{uv, uw, vw\}$ by the minimality of G , then erase the colors on u, v and w . For each element x incident with f , we define a reduced total list $L'(x)$ similar to Claim 3.2. Then $|L'(w)| \geq 2$, $|L'(u)| \geq 4$, $|L'(v)| \geq 4$, $|L'(uw)| \geq 3$, $|L'(vw)| \geq 3$ and $|L'(uv)| \geq 4$. If there is a color $\alpha \in L'(wu) \setminus L'(u)$, then color wu with color α and color w, uv, v, uv and u successively. So $L'(wu) \subseteq L'(u)$. If there is a color $\beta \in L'(u) \setminus L'(v)$, then color u with β and color w, uw, uv, uv and v successively. So $L'(u) \subseteq L'(v)$. This implies that $L'(wu) \subseteq L'(v)$. Choose a color $\gamma \in L'(wu)$ and color wu

and v with color γ , then color w, wv, uv and u successively. Thus we obtain the required contradiction. $\square$

We are ready to construct a new charge $w^*(x)$ on G as follows.

$R_{4.1}$: Each 3-vertex receives $\frac{1}{2}$ from each of its 3-master.

$R_{4.2}$: For a k -face f , where $k \geq 5$, and each occurrence of a vertex $v \in b(f)$, transfer the amount $\frac{k}{k-4}$ from f to v .

$R_{4.3}$: From each 4-vertex v to each of its incident 3-face f , transfer

0, if $n_3(v) \geq 3$.
 $\frac{3}{5}$, if $n_{7^+}(v) \geq 2$.
 $\frac{1}{5}$, otherwise.

$R_{4.4}$: From each 5-vertex v to each of its incident 3-face, transfer

$\frac{1}{2}$, if $\delta(f) = 4$.
 $\frac{1}{3}$, if $\delta(f) \geq 5$.

$R_{4.5}$: From each 6^+ -vertex v to each of its incident 3-face, transfer

$\frac{1}{2}$, if $\delta(f) = 3$.
 $\frac{3}{5}$, if $\delta(f) = 4$.
 $\frac{1}{3}$, otherwise.

Clearly, $w^*(f) = w(f) = 0$ if $d(f) = 4$ and $w^*(f) = w(f) - k \times \frac{k-4}{k} = 0$ if $k \geq 5$. Let $f = [u_1 u_2 u_3]$ be a 3-face with $d(u_1) \leq d(u_2) \leq d(u_3)$. If $d(u_1) = 3$, then $d(u_2) = d(u_3) = \Delta$. So $w^*(f) \geq w(f) + 2 \times \frac{1}{2} = 0$. If $d(u_1) \geq 5$, then $w^*(f) \geq w(f) + 3 \times \frac{1}{3} = 0$. Assume that $d(u_1) = 4$. If $d(u_2) = 4$, then $d(u_3) \geq 6$ by Claim 3.3. So $\gamma(u_3 \rightarrow f) = \frac{3}{5}$. In this case, without loss of generality, we suppose that $\gamma(u_1 \rightarrow f) \leq \gamma(u_2 \rightarrow f)$. If $\gamma(u_1 \rightarrow f) = 0$, then u_1 is incident with at least three 3-faces. This implies that u_2 is incident with at least two 7^+ -faces. So $\gamma(u_2 \rightarrow f) = \frac{3}{7}$. Thus $w^*(f) \geq w(f) + \frac{3}{7} + \frac{3}{5} > 0$. If $\gamma(u_1 \rightarrow f) = \frac{1}{5}$, then $\gamma(u_2 \rightarrow f) \geq \frac{1}{5}$. So $w^*(f) \geq w(f) + 2 \times \frac{1}{5} + \frac{3}{5} = 0$. If $\gamma(u_1 \rightarrow f) = \frac{3}{7}$, then $w^*(f) \geq w(f) + \frac{3}{7} + \frac{3}{5} > 0$. If $d(u_2) \geq 5$, then $\gamma(u_2 \rightarrow f) \geq \frac{1}{2}$ and $\gamma(u_3 \rightarrow f) \geq \frac{1}{2}$. So $w^*(f) \geq w(f) + 2 \times \frac{1}{2} = 0$.

Let v be a k -vertex. If $k = 3$, then $w^*(v) \geq w(v) + 2 \times \frac{1}{2} = 0$. Let $k = 4$. If $n_3(v) = 0$, then $w^*(v) \geq w(v) = 0$. If $n_3(v) \geq 3$, then $w^*(v) = w(v) = 0$. If $n_{7^+}(v) \geq 2$, then v is incident with at most two 3-faces. So $w^*(v) \geq w(v) + 2 \times \frac{3}{7} - 2 \times \frac{3}{7} = 0$. Otherwise, let f be a 3-face incident with v . Then $\gamma(v \rightarrow f) = \frac{1}{5}$. In this case, if $n_3(v) = 1$, then v is incident with at least 5^+ -faces since G is C_6 -free. So $w^*(v) \geq w(v) + \frac{1}{5} - \frac{1}{5} = 0$. If $n_3(v) = 2$, then v is incident with at least one 7^+ -face. So $w^*(v) \geq w(v) + \frac{3}{7} - 2 \times \frac{1}{5} > 0$. Let $k = 5$. Then v is incident with at most three 3-faces. If $n_3(v) \leq 2$, then $w^*(v) \geq w(v) - 2 \times \frac{1}{2} = 0$. If $n_3(v) = 3$, then v is incident with at least one 7^+ -face. So $w^*(v) \geq w(v) - 3 \times \frac{1}{2} + \frac{3}{7} > 0$. Let $k = 6$. Then v is incident with at most four 3-faces. If $n_3(v) \leq 2$, then $w^*(v) \geq w(v) - 2 \times \frac{3}{5} - \frac{1}{2} = \frac{3}{10} > 0$. If $n_3(v) = 3$, then v is incident with at least one 7^+ -face. So $w^*(v) \geq w(v) - 3 \times \frac{3}{5} - \frac{1}{2} + \frac{3}{7} > 0$. If $n_4(v) = 4$, then the other face f incident with v are 7^+ -faces. In this case, if v is not adjacent to a 3-vertex, then $w^*(v) \geq w(v) - 4 \times \frac{3}{5} + 2 \times \frac{3}{7} > 0$. Otherwise, v is adjacent to a 3-vertex. This implies that v is incident with at least one

3-face f with $\delta(f) = 3$. So $w^*(v) \geq w(v) - \frac{1}{2} - \frac{1}{2} - 3 \times \frac{3}{5} + 2 \times \frac{3}{7} > 0$. If $k \geq 7$, then $n_3(v) \leq k - 2$. If $n_3(v) \leq k - 3$, then $w^*(v) \geq w(v) - \frac{1}{2} - (k - 3) \times \frac{3}{5} > 0$. If $n_3(v) = k - 2$, then the face v incident with are 7^+ -faces. So $w^*(v) \geq w(v) - \frac{1}{2} - (k - 2) \times \frac{3}{5} + 2 \times \frac{3}{7} > 0$. $\square$

4 Open Problems for Future Research

Following conjecture was posed by Borodin, Kostochka, and Woodall [5] which is stronger than Conjecture 1.1.

Conjecture 4.1. *If G is a multigraph, then $\chi_l''(G) = \chi''(G)$.*

Problem 4.2. *Consider Conjecture 4.1 for planar graphs. Especially, when planar graphs without cycles of specific lengths.*

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