Binding Number and Connected ($g, f + 1$)-Factors in Graphs

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Abstract. Let G be a connected graph of order n and let a, b be two integers such that $2 \leq a \leq b$. Let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. A spanning subgraph F of G is called a $(g, f + 1)$ -factor if $g(x) \leq d_F(x) \leq f(x) + 1$ for every $x \in V(F)$. For a subset X of $V(G)$, let $N_G(X) = \bigcup_{x \in X} N_G(x)$. The binding number of G is defined by $bind(G) = \min\{\frac{|N_G(X)|}{|X|} \mid \emptyset \neq X \subset V(G), N_G(X) \neq V(G)\}$. In this paper, it is proved that if $bind(G) > \frac{(a+b)(n-1)}{an}$, $f(V(G))$ is even and $n \geq \frac{(a+b)^2}{a}$, then G has a connected $(g, f + 1)$ -factor.

1 Introduction

In this paper, we consider only finite undirected graphs without loops and multiple edges. The set of vertices of graph G is denoted by $V(G)$; the set of edges by $E(G)$. We use $\nu(G)$ to denote the cardinality of $V(G)$. If H is a subgraph of G and S is a subset of $V(G)$, we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S , and set $|N_H(S)| = d_H(S)$. In particular, when $H = G$ and $S = u$, then let $N_G(u) = N(u)$ and let $d_G(u) = d(u)$. The binding number of G is defined by $bind(G) = \min\{\frac{|N_G(X)|}{|X|} \mid \emptyset \neq X \subset V(G), N_G(X) \neq V(G)\}$. We write $E_G(X, Y) = \{xy \in E(G) \mid x \in X, y \in Y\}$, and $e_G(X, Y) = |E_G(X, Y)|$. As usual, we use δ for the minimum degree of G . For other definitions and notations not defined here, we refer the readers to [1]. Let G be a connected graph of order n and let a, b be two integers such that $2 \leq a \leq b$. Let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$. A spanning subgraph F of G is called a $(g, f + 1)$ -factor if $g(x) \leq d_F(x) \leq f(x) + 1$ for every $x \in V(F)$. If F is connected, we call it a connected $(g, f + 1)$ -factor. The purpose of this paper is to present a sufficient condition which guarantees the existence of connected $(g, f + 1)$ -factors in graph G related to the binding number of graph G . For convenience, we write $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$, $f(S) = \sum_{x \in S} f(x)$ and $f(T) = \sum_{x \in T} f(x)$.

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Many authors have investigated factors, connected factors and factorizations [3,5,7]. In this paper, we study conditions on the binding number and the order of a graph G which guarantee the existence of a connected $(g, f + 1)$ -factor in G . We begin with some known results.

There is a well-known necessary and sufficient condition for a graph G to have a (g, f) -factor which was given by Lovász.

Theorem 1 ([6]). *A graph G has a (g, f) -factor if and only if*

$$\delta(S, T) = f(S) + d_{G-S}(T) - g(T) - h(S, T) \geq 0$$

for any disjoint subsets S and T of $V(G)$, where $h(S, T)$ denotes the number of components C of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e_G(T, V(C)) + \sum_{x \in V(C)} f(x)$ is odd. Furthermore, if $g(x) = f(x)$ for every $x \in V(G)$, then $\delta(S, T) = f(V(G)) \pmod{2}$.

M. Kano gave a result which shows the relation of binding numbers and existence of f -factors.

Theorem 2 [2]. *Let G be a connected graph of order n and let a and b be integers such that $1 \leq a \leq b$ and $2 \leq b$. Let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for every $x \in V(G)$. Suppose that $n \geq \frac{(a+b)^2}{a}$ and $\sum_{x \in V(G)} f(x) \equiv 0 \pmod{2}$. If*

$$\text{bind}(G) > \frac{(a+b-1)(n-1)}{an - (a+b) + 3},$$

and for any nonempty independent subset X of $V(G)$

$$N_G(X) \geq \frac{(b-1)n + |X| - 1}{a+b-1},$$

then G has an f -factor.

The following two results are essential to the proof of our main theorem.

Theorem 3 [4]. *Let G be a graph and let g and f be two positive integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x) \leq d_G(x)$ for all $x \in V(G)$. If G has both a (g, f) -factor and a Hamiltonian path, then G contains a connected $(g, f+1)$ -factor.*

Theorem 4 [8]. *If $\text{bind}(G) \geq \frac{3}{2}$, then G has a Hamiltonian cycle.*

Liu and Zhang proposed the following problem.

Problem [5]. *Find sufficient conditions for graphs to have connected $[a, b]$ -factors related to other parameters in graphs such as binding number, neighbor union and connectivity.*

We now give our theorem which partially solve the above problem.

Theorem 5. *Let G be a connected graph of order n , and let a and b be two integers such that $2 \leq a \leq b$. Let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$ and $f(V(G))$ is even. If $n \geq \frac{(a+b)^2}{a}$, $\text{bind}(G) > \frac{(a+b)(n-1)}{an}$, then G has a connected $(g, f + 1)$ -factor.*

2 Proof of the Theorem 5

In this section, we assume that G satisfies the conditions of Theorem 5. Since $\text{bind}(G) > \frac{(a+b)(n-1)}{an} > \frac{3}{2}$, according to Theorem 4, graph G has a Hamiltonian cycle. Hence by Theorem 3, to prove Theorem 5, we need only to prove that graph G has a (g, f) -factor. At first, we need some lemmas.

Lemma 1. *Suppose that G satisfies the conditions of Theorem 5. Then $|N_G(X)| > \frac{bn+|X|-1}{a+b}$ for all nonempty subset X of $V(G)$ with $N_G(X) \neq V(G)$ and $\delta(G) > \frac{bn}{a+b}$.*

Proof. Let $\text{bind}(G) > l$, and $Y = V(G) \setminus N_G(X)$. Since $N_G(Y) \subseteq V(G) \setminus X$, we obtain $(n - |X|) \geq |N_G(Y)| > l|Y| = l(n - |N_G(X)|)$. Therefore we have $|N_G(X)| > ((l-1)n + |X|)/l$, and $\delta(G) > ((l-1)n + 1)/l$.

Then replace l by $\frac{(a+b)(n-1)}{an-b}$, we have

$$\delta(G) > \frac{bn}{a+b}. \quad (1)$$

Since $n \geq \frac{(a+b)^2}{a}$, similarly we obtain

$$\begin{aligned} |N_G(X)| &> \frac{(an + bn - a - b - an)n + an + |X|}{(a+b)(n-1)} \\ &= \frac{bn(n-1) - an + an + |X|}{(a+b)(n-1)} \\ &\geq \frac{bn + |X| - 1}{a+b}. \end{aligned}$$

So

$$|N_G(X)| > \frac{bn + |X| - 1}{a+b}. \quad (2)$$

We begin to prove that graph G has a (g, f) -factor. By contradiction, we suppose that G has no (g, f) -factors. Then according to Theorem 1, there exist disjoint subsets S and T of $V(G)$ satisfying

$$f(S) + d_{G-S}(T) - g(T) - h(S, T) < 0. \quad (3)$$

Let $s = |S|$ and $t = |T|$. Then by Theorem 1 and (3) we have

$$as + d_{G-S}(T) - bt - \omega \leq -1, \quad (4)$$

where ω denotes the number of components of $G - (S \cup T)$. Obviously,

$$\omega \leq n - s - t. \quad (5)$$

Let m denote the minimum order of components of $G - (S \cup T)$. Then we know that

$$m \leq \frac{n - s - t}{\omega}. \quad (6)$$

By the definition of m , it is obvious that

$$\delta(G) \leq m - 1 + s + t. \quad (7)$$

The following Claims hold.

Claim 1. $T \neq \emptyset$.

Proof. Otherwise $T = \emptyset$.

Case 1. $S = \emptyset$.

Since G is connected and $\delta(S, T) < 0$, $h(S, T) = 1$. So obviously $f(x) = g(x)$ for every $x \in V(G)$ by the definition of $h(S, T)$. Hence $\delta(S, T) = -1$. On the other hand, since $f(V(G))$ is even, according to Theorem 1, $\delta(S, T)$ is even. This is a contradiction.

Case 2. $S \neq \emptyset$. Then by (4) and (5), we get

$$as + 1 \leq \omega \leq n - s. \quad (8)$$

So from (1), (6), (7) and (8) we know that

$$\begin{aligned} \frac{bn}{a+b} &< \delta(G) \leq m - 1 + s \leq \frac{n-s}{\omega} - 1 + s \\ &\leq \frac{n-s}{as+1} - 1 + s \\ &\leq \frac{n-1}{a} - \frac{(n-1-as-s)(s-1)}{as+1}. \end{aligned}$$

Since $n - 1 - as - s \geq 0$ by (8), it follows that

$$\frac{n}{2} \leq \frac{n-1}{a}.$$

This is a contradiction since $a \geq 2$. So $T \neq \emptyset$. ■

Since $T \neq \emptyset$ by Claim 1, let $h = \min\{d_{H-S}(x) \mid x \in T\}$.

Then obviously

$$\delta(G) \leq h + s. \quad (9)$$

Claim 2. $h \neq 0$.

Proof. If $h = 0$, let $X = \{x \in T \mid d_{G-S}(x) = 0\}$. Then $|X| \neq 0$. From (2) we get

$$s \geq |N_G(X)| > \frac{bn + |X| - 1}{a + b}. \quad (10)$$

On the other hand, from (4), (5) we get

$$as - b|X| + (1 - b)(t - |X|) - (b - 1)(n - s - t) \leq -1.$$

Thus

$$s \leq \frac{(b - 1)n + |X| - 1}{a + b - 1}.$$

From (10) and the above inequality, we get

$$\frac{bn + |X| - 1}{a + b} < \frac{(b - 1)n + |X| - 1}{a + b - 1},$$

which means $|X| > an + 1$. This is a contradiction. So $h \neq 0$. ■

Claim 3. $h < b$.

Proof. Otherwise, $h \geq b$. We consider two cases.

Case 1. $h > b$.

Then by (4),

$$as + (h - b)t - \omega \leq -1. \quad (11)$$

And so

$$\omega \geq as + t + 1 \geq s + t + 1. \quad (12)$$

Since $T \neq \emptyset$ by Claim 1, $\omega \geq 2$.

First suppose that $m \geq 2$. Then from (7) and (12) we get

$$\begin{aligned} \frac{bn}{a + b} &< \delta(G) \leq m - 1 + s + t \leq m + \omega - 2 \\ &\leq m + \omega - 2 + \frac{(m - 2)(\omega - 2)}{2} \\ &= \frac{m\omega}{2} \leq \frac{n}{2}. \end{aligned}$$

This is a contradiction. Hence we may assume $m = 1$. Then from (12) and (5) we have

$$s + t \leq \frac{n-1}{2}.$$

Let C_1 be the least component of $G - (S \cup T)$. Then C_1 contains only one vertex x . Hence

$$\frac{bn}{a+b} < \delta(G) \leq d(x) \leq s + t \leq \frac{n-1}{2}.$$

This also is a contradiction. ■

Case 2. $h = b$.

From (4) we have

$$\omega \geq as + 1. \tag{13}$$

So by (6) and (13)

$$m \leq \frac{n-s-t}{\omega} \leq \frac{n-s-1}{as+1}.$$

From (1) and (9) it is obvious that

$$\frac{bn}{a+b} \leq \delta(G) \leq h + s = b + s.$$

So

$$s > \frac{bn}{a+b} - b = \frac{b(n-a-b)}{a+b}.$$

Let $f(s) = \frac{n-s-1}{as+1}$. Then $f'(s) = \frac{-1-an+a}{(as+1)^2}$. Namely, $f'(s) < 0$. Hence,

$$m < \frac{n - \frac{b(n-a-b)}{a+b} - 1}{a \frac{b(n-a-b)}{a+b} + 1}.$$

Since $n \geq \frac{(a+b)^2}{a}$, it follows that $m < 1$. This is a contradiction. Therefore, from Case 1 and Case 2 we have known that $h < b$. ■

Hence, by Claim 2 and Claim 3 we have $1 \leq h \leq b-1$.

By (4),(5) and $b-h \geq 1$, we obtain

$$as + (h-b)t - (b-h)(n-s-t) \leq -1,$$

which implies

$$s \leq \frac{(b-h)n-1}{a+b-h}. \tag{14}$$

Then from (1),(9) and (14) we obtain

$$\frac{bn}{a+b} \leq \delta(G) \leq s+h \leq \frac{(b-h)n-1}{a+b-h} + h. \quad (15)$$

Let $f(h) = \frac{(b-h)n-1}{a+b-h} + h$. Then by $n \geq \frac{(a+b)^2}{a}$,

$$\begin{aligned} f'(h) &= \frac{-an-1+(a+b-h)^2}{(a+b-h)^2} \\ &< \frac{-1}{(a+b-h)^2} < 0. \end{aligned}$$

Hence the maximum value of $f(h)$ is $\frac{(b-1)n-1}{a+b-1} + 1$.

So by (1) and (15) we have

$$\frac{bn}{a+b} < \frac{(b-1)n-1}{a+b-1} + 1,$$

which implies

$$an < (a+b)^2 - 2(a+b).$$

Thus $n < \frac{(a+b)^2}{a}$. This contradicts the assumption that $n \geq \frac{(a+b)^2}{a}$. Hence we get the desired contradiction and conclude that G has a (g, f) -factor. Since G has both a Hamiltonian path and a (g, f) -factor, according to Theorem 3, graph G has a connected $(g, f + 1)$ -factor. $\blacksquare$

The possible further problem in this topic is the following.

Question. Find the proper binding number and order of G which guarantees the existence of connected k -factors in G .

References

1. Bondy, J. A., Murty, U. S. R.: Graph Theory with Applications. Macmillan, London (1976).
2. Kano, M., Tokushige, N.: Binding number and f -factors of graphs. J. Combin. Theory Ser. B 54(1992) 213-221
3. Kouider, M., Vestergaard, P. D.: Connected factors in graphs - a survey. Graphs and Combinatorics 21 (2005) 1-26
4. Li, G., Xu, Y., Chen, C., Liu, Z.: On connected $(g, f + 1)$ -factors in graphs. Combinatorica 25 (4) (2005) 393-405
5. Liu, G., Zhang, L.: Factors and factorizations of graphs(in Chinese). Advances in Math. 29(2000) 289-296
6. Lovász, L.: Subgraphs with prescribed valencies. J. Combin. Theory 8 (1970) 391-416
7. Plummer, M. D.: Graph factors and factorization. Chapter 5.4 in Handbook on Graph Theory Eds.: J. Gross and R. Yellen CRC Press , New York (2003) 403-430
8. Woodall, D. R.: The binding number of a graph and its Anderson number. J. Combin. Theory Ser. B 15 (1973) 225-255