The Roman Domination Problem in Unit Disk Graphs*

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Abstract. Unit disk graphs are the intersection graphs of equal sized disks in the plane, they are widely used as a mathematical model for wireless ad-hoc networks and some problems in computational geometry. In this paper we first show that the Roman domination problem in unit disk graphs is NP-hard, and then present a simple linear time approximation algorithm and a polynomial-time approximation scheme for this problem, respectively.

Keywords: Domination, approximation algorithm, unit disk graph.

1 Introduction

Let $G = (V, E)$ be a simple and undirected graph. A set $D \subseteq V$ is a *dominating set* if every vertex in $V \setminus D$ is adjacent to at least one vertex in D . The *domination number*, denoted $\gamma(G)$, is the minimal cardinality of a dominating set in G . The *minimum dominating set problem* is to compute a dominating set of minimal cardinality for any given graph G . A *Roman dominating function* [1] of a graph G is defined as a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of a Roman dominating function is $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of Roman dominating functions in G . The minimum Roman domination problem is to compute a minimum weight dominating function for any given graph G .

Minimum Roman domination problem is introduced in [1] as one of variants of classical dominating set problem. It comes with a nice story: The Roman Empire needs to defence itself by positioning some legions on the various parts of the Empire in such a way that either (1) a specific region v is also the location of at least one legion or (2) one region u neighboring v has two legions, so that u can afford sending off one army to the region v (in case of an attack) without losing self-defence capabilities. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending

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the Roman Empire. Roman domination problem has not only historical but also mathematical interests, particularly in the field of server placements [3].

In reality, it is desired that dispatching a legion from one location to another location should not take too much time; In particular, it is desirable that a legion from one location could be dispatched to another location if the Euclidean distance between the two corresponding points is no more than a specified bound. Unit disk graphs satisfy such a property. A undirected graph $G = (V, E)$ is a *unit disk graph* (UDG) if its vertices can be put in one-to-one correspondence with disks of equal radius in the plane in such a way that two vertices are joined by an edge if and only if the corresponding disks intersect. In other words, G is a UDG if there exists a map $f : V \rightarrow R^2$ such that $(u, v) \in E$ if and only if $\|f(u) - f(v)\| \leq 1$, where $\|\cdot\|$ denotes the Euclidean norm. *Grid graphs* compose a subclass of unit disk graphs under such an intersection model that all the disks have centers with integer coordinates and radius $1/2$.

It has been shown in [7] that many graph related problems in unit disks graphs are NP-hard, including the independent set problem, the dominating set and connected dominating set problems; In fact, the dominating set problem is NP-hard even for grid graphs. Polynomial-Time Approximation Schemes (PTASs) for these problems in unit disk graphs are presented in [10, 11]. Roman domination problem in general graphs is proved to be NP-hard in [2], $(2+2\ln(n))$ -approximation algorithms and a PTAS are proposed in [3] for general graphs and for minimum planar Roman domination problem, respectively.

In this paper, we consider the Roman domination problem in unit disk graphs. We will first give some preliminaries about the problem in Section 2, and then we will prove that the Roman domination problem in unit disk graphs is NP-hard in Section 3, after that we will propose a simple linear-time approximation algorithm and a polynomial-time approximation scheme for the problem in Section 4. In Section 5 we conclude the paper.

2 Preliminaries

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For any vertex $v \in V$, the *closed neighborhood* of v is denoted and defined by $N(v) \equiv \{u \in V(G) : uv \in E(G)\} \cup \{v\}$. For a subset S of V , let $N(S) = \bigcup_{v \in S} N(v)$, and for a natural number r , define the r -th neighborhood of $v \in V$ as $N^r(v) \equiv N(N^{r-1}(v))$, where $N^1(v) = N(v)$.

A subset $U \subseteq V$ is an *independent set* (IS) of G if all vertices in U are pairwise non-adjacent, and a *maximal independent set* (MIS) U of G is such an IS that each vertex $V \setminus U$ is adjacent to at least one vertex in U . Clearly, an MIS $I(G)$ of G is also a dominating set of G . Thus any minimum dominating set $D(G)$ of G has size no more than that of an MIS $I(G)$. In addition, a subset S of $V(G)$ is called a *2-packing* if $N(u) \cap N(v) = \emptyset$ for every pair of vertices $u, v \in S$.

Some useful facts on Roman dominating functions were proved in [1], which will be used in our following discussions.

Proposition 1. *For any graph G , $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Proposition 2. *Let f be a minimum Roman dominating function of a graph G without isolated vertices. Let V_i be the sets of vertices v with $f(v) = i$ for $i = 0, 1, 2$. Let $f = (V_0, V_1, V_2)$ be such a function that $|V_1|$ is minimal. Then V_1 is a 2-packing and there is no edge between V_1 and V_2 .*

Proposition 3. *For any non-trivial connected graph G , $\gamma_R(G) = \min\{2\gamma(G \setminus S) + |S| : S \text{ is a 2-packing}\}$.*

Clearly, a function $f = (V_0, V_1, V_2)$ is a *Roman dominating function* if the set V_2 dominates the set V_0 , and the weight of f is $f(V) = \sum_{v \in V} f(v) = 2|V_2| + |V_1|$. Moreover, a Roman dominating function $f = (V_0, V_1, V_2)$ defines a Roman dominating set $D_R = (V_1, V_2)$ since, for each vertex $v \in V$, we have either $v \in V_1 \cup V_2$ or v is dominated by some node in V_2 . The weight of a Roman dominating set D_R is $W(D_R) = 2|V_2| + |V_1|$. If D_R is a minimum Roman dominating set (MRDS) of graph G , then $W(D_R) = \gamma_R(G)$.

3 Complexity Study of Roman Domination Problem

Let $V = \{v_1, v_2, \dots, v_n\}$ be the set of points located in the Euclidean plane. Denote the coordinates of v_i by (x_i, y_i) for $i = 1, 2, \dots, n$. For each vertex $v_i \in V$, there is a vertex u_i with coordinates $(x_i + \frac{1}{10}, y_i)$, and let $U = \{u_1, u_2, \dots, u_n\}$ denote the set associated with V . Now construct a graph $G' = (V', E')$ with vertex-set $V' = V \cup U$ and edge-set $E' = \{(u, v) : u, v \in V' \text{ and } \|uv\| \leq 1 + \frac{1}{10}\}$. It is clear that G' is a unit disk graph, and $N(v_i) = N(u_i)$ in graph G' . Moreover, the following lemmas give some other properties of graph G' .

Lemma 1. *Let $f' = (V'_0, V'_1, V'_2)$ be a minimal Roman dominating function of $G' = (V', E')$. Then $|\{u_i, v_i\} \cap V'_2| \leq 1$, for $i = 1, 2, \dots, n$.*

Proof. Suppose, by contradiction, that there exists a j such that $\{u_j, v_j\} \subseteq V'_2$. Let $V''_0 = V'_0 \cup \{u_j\}$, $V''_1 = V'_1$ and $V''_2 = V'_2 \setminus \{u_j\}$. It is easy to verify that function $f'' = (V''_0, V''_1, V''_2)$ is a Roman dominating function satisfying $f''(V') < f'(V')$, a contradicting that f' is minimal. $\square$

Lemma 2. *Let $f' = (V'_0, V'_1, V'_2)$ be a minimal Roman dominating function of $G' = (V', E')$. Then there exists a minimal Roman dominating function $f'' = (V''_0, V''_1, V''_2)$ satisfying $f''(V') = f'(V')$ such that $V''_1 = \emptyset$ and $V''_2 \cap \{u_1, u_2, \dots, u_n\} = \emptyset$.*

Proof. Suppose that $V'_1 \neq \emptyset$. Then let $v_j \in V'_1$ for some j . Thus we have $N(v_j) \cap V'_2 = \emptyset$ and $N(u_j) \cap V'_2 = \emptyset$ since $N(v_j) = N(u_j)$, which implies $u_j \in V'_1$. Hence we have either $|\{u_i, v_i\} \cap V'_1| = 0$ or 2 for each i . Let $V'_1 = \{v_{i_1}, u_{i_1}, v_{i_2}, u_{i_2}, \dots, v_{i_k}, u_{i_k}\}$ for some $1 \leq k \leq n$, $V''_0 = V'_0 \cup \{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$, $V''_1 = \emptyset$, and $V''_2 = V'_2 \cup \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. Clearly, $f'' \equiv (V''_0, V''_1, V''_2)$ is a Roman dominating function satisfying $f''(V') = f'(V')$.

Now suppose that $V'_2 \cap \{u_1, u_2, \dots, u_n\} \neq \emptyset$. Assume, without loss of generality, that $V'_2 \cap \{u_1, u_2, \dots, u_n\} = \{u_1, u_2, \dots, u_m\}$ for some $1 \leq m \leq n$. Since $|\{u_i, v_i\} \cap V'_2| \leq 1$ for each i , then $\{v_1, v_2, \dots, v_m\} \subseteq V'_0$. Let $V''_0 = (V'_0 \setminus \{v_1, v_2, \dots, v_m\}) \cup \{u_1, u_2, \dots, u_m\}$, $V''_1 = V'_1$ and $V''_2 = (V'_2 \setminus \{u_1, u_2, \dots, u_m\}) \cup \{v_1, v_2, \dots, v_m\}$. It is easy to verify that $f'' \equiv (V''_0, V''_1, V''_2)$ is a Roman dominating function satisfying $f''(V') = f'(V')$ and $V''_2 \cap \{u_1, u_2, \dots, u_n\} = \emptyset$. The lemma is then proved. $\square$

Theorem 1. *The minimum Roman dominating set problem in unit disk graphs is NP-hard.*

Proof. The reduction is from the minimum dominating set problem in grid graphs, which is known to be NP-hard. Given a grid graph $G = (V, E)$, we will construct a unit disk graph $G' = (V', E')$ such that G has a dominating set D with $|D| \leq k$ if and only if G' has a Roman dominating set D_R with $W(D_R) \leq 2k$.

Given any grid graph $G = (V, E)$ with vertex-set $V = \{v_1, v_2, \dots, v_n\}$, where each v_i has integral coordinates (x_i, y_i) and $(v_i, v_j) \in E$ if and only if $\|v_i v_j\| = 1$. For each vertex $v_i \in V$, we add a new vertex u_i with coordinates $(x_i + \frac{1}{10}, y_i)$. Now construct the graph $G' = (V', E')$ with vertex-set $V' = \{v_1, u_1, v_2, u_2, \dots, v_n, u_n\}$, and for any two vertices $u, v \in V'$, $(u, v) \in E'$ if and only if $\|uv\| \leq 1 + 1/10$. It is clear that G' is a unit disk graph, and for each $1 \leq i \leq n$, v_i and u_i have the same neighborhood $N(v_i) = N(u_i)$, and the same degree $d_{G'}(u_i) = d_{G'}(v_i) = 2d_G(v_i) + 1$.

Let D_R be a minimal Roman dominating set of G' . Then by Lemma 2, we can assume, without loss of generality, that $D_R = (\emptyset, V_2)$ and $V_2 \cap \{u_1, u_2, \dots, u_n\} = \emptyset$. Suppose that D is a dominating set of G with $|D| \leq k$. Let $D_R = (\emptyset, D)$, then D_R is a Roman dominating set with $W(D_R) \leq 2k$. Conversely, suppose that G' has a Roman dominating set $D_R = (\emptyset, V_2)$, $V_2 \cap \{u_1, u_2, \dots, u_n\} = \emptyset$ and $W(D_R) \leq 2k$. Then V_2 is also a dominating set of G , $|V_2| = \frac{1}{2}W(D_R) \leq k$. Let $D = V_2$, then G has a dominating set D with $|D| \leq k$. Therefore, G has a dominating set D with $|D| \leq k$ if and only if G' has a Roman dominating set D_R with $W(D_R) \leq 2k$. The proof is then completed. $\square$

4 Approximation Algorithms

It is well known that any maximal independent set D is also a dominating set. Thus $D_R = (\emptyset, D)$ is a Roman dominating set. One method of finding a Roman dominating set in a graph is to find a maximal independent set. A straightforward method of finding a maximal independent set is as follows: Select an arbitrary vertex $v \in V$, add v to the current independent set D , which is initially set to be an empty set, and delete $N(v)$ from the current graph G , and then repeat the process until all vertices in $V(G)$ have been deleted.

Algorithm A. 5-Approximation Algorithm for MRDS**Input:** A unit disk graph $G = (V, E)$ **Output:** A Roman dominating set D_R

1. $S := V, D := \emptyset$
2. **while** $S \neq \emptyset$ **do**
3. Choose $v \in S$
4. $D := D \cup \{v\}, S := S \setminus N(v)$
5. **end-while**
6. **return** $D_R = (\emptyset, D)$

Theorem 2. *Let $G = (V, E)$ be a unit disk graph, Algorithm A returns a Roman dominating set in time of $O(|E|)$ whose size is at most five times that of the minimum Roman dominating set.*

Proof. It is clear that the algorithm returns a Roman dominating set in time of $O(|E|)$. Let $D_R^* = (V_1^*, V_2^*)$ denote an optimal Roman dominating set of G such that V_1^* is a minimal. By Proposition 2 we have V_1^* is a 2-packing and there is no edge between V_1^* and V_2^* . Note that G is a unit disk graph and the returned set D is an independent set. Moreover, no vertex in V_2^* can dominate more than 5 vertices in $D \setminus V_1^*$, and each vertex in V_1^* only can dominate itself. Hence we have

$$|V_2^*| \geq \frac{|D| - |V_1^* \cap D|}{5},$$

and

$$\gamma_R(G) = f(V) = 2|V_2^*| + |V_1^*| \geq \frac{2|D|}{5} = \frac{W(D_R)}{5},$$

and then the theorem follows. $\square$

In the following we will show that there exists an algorithm that can return a better approximation solution at the expenses of increasing the time complexity. For this purpose, we first show the following lemma.

Lemma 3. *If the maximum independent set $I(G)$ of graph $G = (V, E)$ has cardinality $|I(G)|$ no greater than a constant c . Then $\gamma_R(G)$ can be computed in polynomial time.*

Proof. By $|I(G)| \leq c$, we have $\gamma(G) \leq c$ since any maximal independent set is also a dominating set. As $\gamma_R(G) = \min\{2\gamma(G - S) + |S| : S \text{ is a 2-packing}\}$, $\gamma(G \setminus S)$ and $|S|$ both are at most c . Thus to compute $\gamma_R(G)$ we can first choose a 2-packing set S in G , and then find a minimum dominating set in $G \setminus S$, finally choose the set S such that $|S| + 2\gamma(G - S)$ is minimal. Clearly we can compute $\gamma_R(G)$ by a brute-force method in time of $O(|V|^c)$. $\square$

The basic idea of our polynomial time approximation scheme is as follows: First, compute a local Roman dominating set for a neighborhood of a vertex, and

then expand this neighborhood until we have formed some sets that satisfy the desired bound. Secondly, eliminate the current neighborhoods and repeat the same operation for the remaining graph.

Algorithm B PTAS for MRDS

Input: A unit disk graph $G = (V, E)$, $\varepsilon > 0$

Output: $(1 + \varepsilon)$ -approximation MRDS D_R

1. $V := V$, $D_R := \emptyset$
 2. **while** $V \neq \emptyset$ **do**
 3. choose a vertex $v \in V$
 4. **while** $W(D_R(N^{r+2}(v))) > (1 + \varepsilon)W(D_R(N^r(v)))$ **do**
 5. compute $D_R(N^r(v))$
 6. **end-while**
 7. compute $\bar{r} = \min\{r : W(D_R(N^{r+2}(v))) \leq (1 + \varepsilon)W(D_R(N^r(v)))\}$
 8. $V := V \setminus N^{\bar{r}+2}(v)$
 9. $D_R := D_R \cup D_R(N^{\bar{r}+2}(v))$
 10. **end-while**
 11. **Return** D_R
-

Let k be the total number of executed iterations by Algorithm B. Let $v_1, v_2, \dots, v_k$ be the vertices chosen in step 3, and $N_i, i = 1, 2, \dots, k$, denote the corresponding neighborhoods, i.e., $N_i = N^{\bar{r}_i+2}(v_i)$. By the rules of Algorithm B, it is easy to verify that the returned set $D_R = \bigcup_{i=1}^k D_R(N_i)$ is a Roman dominating set of G . Moreover, we have the following lemma.

Lemma 4. *Let $G = (V, E)$ be a unit disk graph. Then $W(D_R(N^r(v))) \leq 8(r + 1/2)^2$ for every $v \in V$ and any positive integer r .*

Proof. From the definition of a unit disk graph, we know that for any $w \in N^r(v)$, the Euclidean distance between w and v is at most r . Let $I(N^r(v))$ denote an independent set in $N^r(v)$. Then the sets of unit-diameter disks centered at the nodes in $I(N^r(v))$ are disjoint and their convex hull is contained in the disk of radius $r + 1/2$. Then $|I(N^r(v))| \leq 4(r + 1/2)^2$, and $W(D_R(N^r(v))) \leq 2|D(N^r(v))| \leq 2|I(N^r(v))|$. Thus the lemma follows. $\square$

Lemma 5. *The obtained $\bar{r}$ in Step 7 of Algorithm B is no greater than constant $c = O(\frac{1}{\varepsilon^2} \ln \frac{1}{\varepsilon})$.*

Proof. Note that $W(D_R(N^0(v))) = 1$ and $W(D_R(N^1(v))) = 2$ since vertex v dominates itself and all its neighbors. For an arbitrary value $r < \bar{r}$, consider the following two cases.

Case (1) r is an even number. By the rules of Algorithm B and Lemma 4 we have

$$8(r + \frac{1}{2})^2 \geq W(D_R(N^r(v)))$$

$$\begin{aligned}
&> (1 + \varepsilon)W(D_R(N^{r-2}(v))) \\
&> (1 + \varepsilon)^{\frac{r}{2}}W(D_R(N^0(v))) \\
&= (1 + \varepsilon)^{\frac{r}{2}}.
\end{aligned}$$

Case (2) r is an odd number. In this case we have

$$\begin{aligned}
8(r + \frac{1}{2})^2 &\geq W(D_R(N^r(v))) \\
&> (1 + \varepsilon)W(D_R(N^{r-2}(v))) \\
&> (1 + \varepsilon)^{\frac{r-1}{2}}W(D_R(N^1(v))) \\
&= 2(1 + \varepsilon)^{\frac{r-1}{2}}.
\end{aligned}$$

In both cases the above inequalities will not hold for sufficiently large r , the bound on $\bar{r}$, which is achieved when these inequalities are violated the first time, only depends on ε and not on the order of graph G . By applying a similar argument used in [10], we can show $\bar{r} \leq c$, where $c = O(\frac{1}{\varepsilon^2} \ln \frac{1}{\varepsilon})$. The proof is then proved. $\square$

The number of independent sets in $N^{\bar{r}}(v)$ is polynomially bounded in $\bar{r}$, and $\bar{r}$ is also polynomially bounded. By Lemma 3, a minimum Roman dominating set $D_R(N^{\bar{r}}(v))$ can be computed in polynomial time $O(|V|^{c^2})$, where $c = O(\frac{1}{\varepsilon^2} \ln \frac{1}{\varepsilon})$. The following theorem shows the correctness and approximation guarantee of the algorithm.

Theorem 3. *Algorithm B returns a Roman dominating set $D_R = \bigcup_{i=1}^k D_R(N_i)$ whose weight is no more than $(1 + \varepsilon)\gamma_R(G)$.*

Proof. Let D_R^* denote the optimal Roman dominating set. It is clear that $N(N^{\bar{r}_i}(v_i))$ and $N(N^{\bar{r}_j}(v_j))$ are mutually disjoint in G for $i \neq j$, and $N(N^{\bar{r}_i}(v_i)) \cap D_R^*$ dominates $N^{\bar{r}_i}(v_i)$. Thus by Lemma 5 we have

$$\begin{aligned}
\gamma_R(G) = W(D_R^*) &\geq \sum_{i=1}^k W(D_R^* \cap N(N^{\bar{r}_i}(v_i))) \\
&\geq \sum_{i=1}^k W(D_R(N^{\bar{r}_i}(v_i))) \\
&\geq \sum_{i=1}^k \frac{1}{1 + \varepsilon} W(D_R(N^{\bar{r}_i+2}(v_i))) \\
&\geq \frac{1}{1 + \varepsilon} W(\bigcup_{i=1}^k D_R(N^{\bar{r}_i+2}(v_i))) \\
&= \frac{1}{1 + \varepsilon} W(D_R).
\end{aligned}$$

Hence the theorem holds. $\square$

5 Conclusion

In this paper we first show that the Roman domination problem in unit disk graphs is NP-hard, and then present a 5-approximation algorithm of linear time and a polynomial-time approximation scheme for this problem, respectively.

Grid graphs compose a subclass of unit disk graphs. However, hardness results for the Roman domination problem in grid graphs are still unknown (to the best knowledge of the authors). This raises an interesting problem for future work.

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