Semicomplete Multipartite Digraphs Whose Every Arc Is Contained in a Hamiltonian Path

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Abstract. A semicomplete multipartite digraph is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. In 2002, L.Volkman [5] raised a problem: determine some sufficient conditions for semicomplete multipartite digraphs such that every arc is contained in a Hamiltonian path. In this paper, we give a new sufficient condition and describe some semicomplete multipartite digraphs demonstrating that this sufficient condition is independent and in some sense, it is best possible.

Keywords: Semicomplete multipartite digraph, Hamiltonian paths, Semi-partition complete digraphs.

1 Terminology and Introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and the arc set of a digraph are denoted by $V(D)$ and $A(D)$, respectively. The number of vertices in D is called the *order* of D . A digraph obtained by replacing each edge of a complete n -partite graph by an arc or a pair of mutually opposite arcs is called a *semicomplete n -partite* or *multipartite digraph*. Obviously, a tournament is a semicomplete multipartite digraph without cycles of length 2 where each partite set contains exactly one vertex.

If xy is an arc of a digraph D , then we say that x *dominates* y and write $x \rightarrow y$. More generally, for disjoint subsets A and B of $V(D)$ or subdigraphs of D , if every vertex of A dominates every vertex of B , then we say that A *dominates* B , denoted by $A \rightarrow B$. In the case when there is no arc from B to A , we write $A \Rightarrow B$. If both of $A \rightarrow B$ and $A \Rightarrow B$ hold, then we say that A *strictly dominates* B , denoted by $A \mapsto B$. Furthermore, if for all $a \in A$ and

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$b \in B$, either a, b are not adjacent or $a \rightarrow b$, then we denote this relationship between A and B by $A \hookrightarrow B$. For $S \subseteq V(D)$ and for any vertex $x \in V(D)$, we define $d^+(x, S)$ ($d^-(x, S)$) to be the number of out-neighbours (in-neighbours) of x in S . If $S = V(D)$, we also write $d^+(x, D) = d^+(x)$ ($d^-(x, D) = d^-(x)$). Furthermore, $N^+(S)$ ($N^-(S)$) is the set of out-neighbours (in-neighbours) of S .

By a cycle (path) we mean a directed cycle (directed path). A path $P = x_1x_2\dots x_k$ starts with the arc x_1x_2 , or x_1x_2 is the *initial arc* of P . A cycle of length m is an m -cycle. A cycle in a digraph D is *Hamiltonian* if it includes all the vertices of D . A digraph D is *Hamiltonian* if it contains a Hamiltonian cycle. If x is a vertex on a cycle, then x^+ (x^-) denotes the successor (predecessor) of x on C . If a cycle (path) Q contains a subpath from x to y , then we denote it by $Q[x, y]$. A subdigraph F of a digraph D which contains all the vertices of D and which consists of a set of vertex-disjoint cycles is called a *1-regular factor*. F is said to be *minimal* if there is no 1-regular factor F' of D consisting of less cycles than F .

For a vertex set X of digraph D , we define $D[X]$ as the subdigraph induced by X . A set $X \subseteq V(D)$ is *independent* if the induced subdigraph $D[X]$ has no arcs. The *independent number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D . A digraph D is *strong connected* or *strong* if, for each pair of vertices u and v , there is a path in D from u to v . A digraph D with at least $k + 1$ vertices is *k-connected* if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D , denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k -connected.

Some first observation on semicomplete multipartite digraphs can be found in Moon's book [3]. In the following decades graph theorists began to study multipartite semicomplete digraphs more extensively. Excellent sources for more information in this area are found in recent book by Bang-Jensen and Gutin [1]. In 2002, L.Volkman [5] discussed the longest paths through an arc in strong semicomplete multipartite digraphs and gave a sufficient condition such that all arcs are contained in a Hamiltonian path.

Theorem 1 [5]. Let D be a semicomplete multipartite digraphs with $\kappa(D) > \alpha(D)$, then every arc is contained in a Hamiltonian path.

In addition, L.Volkman [5] posed the following problem:

Problem 1 [5]. Determine some sufficient conditions for semicomplete multipartite digraphs such that every arc is contained in a Hamiltonian path.

In this paper, we give a new sufficient condition in term of this problem. Furthermore, through examples we illustrate the independence of this sufficient condition and the best possibility in some sense. Moreover, a new class of semicomplete multipartite digraphs are introduced and we present a sufficient condition for these digraphs to be Hamiltonian.

2 Preliminaries

The following results play an important role in our investigation. First the definition of semi-partition complete digraphs which was first introduced by Guo [2] and some properties of these digraphs are given.

Definition 1. Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \dots, V_k$. If $d^+(x_i, V_j), d^-(x_i, V_j) \geq |V_j|/2$ for every vertex $x_i \in V_i$ and for every $1 \leq i, j \leq k, j \neq i$, then D is called a semi-partition complete digraph.

Theorem 2 [2]. Let D be a semi-partition complete digraph of order n with $\alpha(D) \leq n/2$. Then D is Hamiltonian or there exists two vertex-disjoint cycles C_1 and C_2 in D such that $V(C_1) \cup V(C_2) = V(D)$ and $|V(C_1) \cap V_j| = |V(C_2) \cap V_j| = |V_j|/2$ for all $1 \leq j \leq k$, where $V_1, V_2, \dots, V_k$ are the partite sets of D .

Now we state a shorten version of Yeo's [6] result about the structure of semicomplete multipartite digraphs with a 1-regular factor.

Theorem 3 [6]. Let D be a semicomplete multipartite digraph containing a 1-regular factor. Then there exists a 1-regular factor F in D whose cycles can be labeled, $C_1, C_2, \dots, C_t$, such that the following holds:

There exists a partite set V^* in D such that whenever $x_j \rightarrow x_1$ where $x_j \in V(C_j)$, $x_1 \in V(C_1)$ and $1 < j \leq t$, we have $\{x_j^+, x_1^-\} \subseteq V^*$.

In particular, if F is minimal, then it has the properties described above.

3 Main Results

Theorem 4. Let D be a semicomplete multipartite digraph with partite sets $V_1, V_2, \dots, V_k$. If $d^+(x_i, V_j), d^-(x_i, V_j) \geq (|V_j| + 1)/2$ for every vertex $x_i \in V_i$, for each $1 \leq i, j \leq k, j \neq i$, and $\alpha(D) \leq (n - 1)/2$, then every arc is contained in a Hamiltonian path.

Proof. Let xy be any arc of D . From the assumption we see that $D - x$ is a semi-partition complete digraph which satisfies $\alpha(D - x) \leq \alpha(D) \leq (n - 1)/2 = |V(D - x)|/2$. According to Theorem 2, $D - x$ is either Hamiltonian or there exists two vertex-disjoint cycles C_1 and C_2 in $D - x$ such that $V(C_1) \cup V(C_2) = V(D - x)$ and C_i contain exactly half of the vertices of every partite set of $D - x$, for $i = 1, 2$.

If $D - x$ has a Hamiltonian cycle, denoted by $yy_2y_3 \dots y_{n-1}y$, then xy lies on such a Hamiltonian path of D $xyy_2y_3 \dots y_{n-1}$. Then we are done.

Now, we assume that there exists two vertex-disjoint cycles C_1 and C_2 in $D - x$ satisfying the condition described as above. Assume without loss of generality that $y \in C_1 \cap V_1$. Since x and y^- are both adjacent to y , x and y^- do not belong to V_1 . Note that $C_2 \cap V_1 \neq \emptyset$, we claim that there exists at least one vertex $z \in C_2 \cap V_1$ such that $y^- \rightarrow z$. In fact, if this is not true, then we have $C_2 \cap V_1 \mapsto y^-$. Hence, $N^+(y^-, V_1) \subseteq C_1 \cap V_1$ holds. Recalling x does not belong to V_1 , we have $|C_1 \cap V_1| = |V_1|/2$. This contradicts the assumption

$d^+(y^-, V_1) \geq (|V_1| + 1)/2$. Now, the arc xy is on such a Hamiltonian path $xyC_1[y^+, y^-]zC_2[z^+, z^-]$. This completes the proof. $\square$

An example will show that conditions in Theorem 4 are independent to Theorem 1.

Example 1. Let $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4, v_5, v_6\}$, and $V_3 = \{v_7\}$ be the partite sets of a semicomplete 3-partite digraphs D . The arcs between v_1, v_2, v_3, v_4 are $v_1 \rightarrow v_4 \rightarrow v_2 \rightarrow v_4 \rightarrow v_3$, and the other arcs between two vertices of different partite sets are two mutually opposite arcs. It is not difficult to check that this digraph D satisfies all conditions in Theorem 4 and hence, every arc is contained in a Hamiltonian path.

Note that the vertices $\{v_5, v_6, v_7\}$ is a separating set of D . So $\kappa(D) \leq 2$, but $\alpha(D) = 3$. Therefore, D does not satisfy the conditions in Theorem 1.

In addition, at the proof of Theorem 4, we actually showed that every arc is not only on a Hamiltonian path, but also the arc is exactly the initial arc of the Hamiltonian path. In this sense, the degree-condition in Theorem 4 is best possible. See the following example.

Example 2. Let $V_1 = \{x_1, x_2, x_3, x_4\}$, $V_2 = \{y_1, y_2, y_3, y_4\}$ and $V_3 = \{z_1, z_2, z_3, z_4\}$ be the partite sets of a semicomplete 3-partite digraphs D . The arcs between any two vertices in $S_1 = \{x_1, y_1, z_1, x_2, y_2, z_2\}$, which belong to different partite sets are defined by two mutually opposite arcs. Similarly, we define the arcs in $S_2 = \{x_3, y_3, z_3, x_4, y_4, z_4\}$. Moreover, $S_1 \hookrightarrow S_2$.

It is easy to see that D is a semicomplete 3-partite digraph with order 12, $\alpha(D) = 4$ and $d^+(x_3, V_2) = 2$. But every arc in S_2 is not on any Hamiltonian path which starts with this arc.

In the following we abstract the conditions in Theorem 4 and define a new class of semicomplete multipartite digraphs. Furthermore, we will present a sufficient condition for these digraphs to be Hamiltonian.

Definition 2. Let D be a semicomplete multipartite digraph with partite sets $V_1, V_2, \dots, V_k$. If $d^+(x_i, V_j), d^-(x_i, V_j) \geq (|V_j| + 1)/2$ for every vertex $x_i \in V_i$ and for every $1 \leq i, j \leq k$, $j \neq i$, then D is called a quasi-semi-partition complete digraph.

Remark 1. According to Definition 2, Theorem 4 can be stated as: Let D be a quasi-semi-partition complete digraph with order n and $\alpha(D) < n/2$, then every arc of D is contained in a Hamiltonian path.

Remark 2. Clearly, quasi-semi-partition complete digraphs are also semi-partition complete digraphs.

Theorem 5. Every quasi-semi-partition complete digraph is strong.

Proof. Let D be a quasi-semi-partition complete digraph with partite sets $V_1, V_2, \dots, V_k$, and x, y any two distinct vertices of D .

If $x, y \in V_i$ for some $i \in \{1, 2, \dots, k\}$, then for any $j \neq i$, and $1 \leq j \leq k$, let $A_j = \{a \in V_j \mid x \mapsto a\}$, $B_j = \{b \in V_j \mid x \rightarrow b, b \rightarrow x\}$, and $C_j = \{c \in V_j \mid c \mapsto x\}$. Obviously, A_j, B_j, C_j are pairwise disjoint and $A_j \cup B_j \cup C_j = V_j$. Because of $d^+(x, V_j) \geq (|V_j|+1)/2$, we know $|A_j \cup B_j| \geq (|V_j|+1)/2$ and then, $|C_j| < |V_j|/2$. Also note that $d^-(y, V_j) \geq (|V_j|+1)/2$, so there is at least one vertex in $A_j \cup B_j$ which dominates y , say u . According to the definitions of A_j and B_j , xuy is a path in D from x to y . Similarly, we can show that there is also a path from y to x .

Now we suppose that $x \in V_i, y \in V_j$ for some $i, j \in \{1, 2, \dots, k\}$ and $i \neq j$. Since x and y are adjacent, we assume without loss of generality that $x \rightarrow y$. If $y \rightarrow x$ also holds, then there is nothing to prove. Otherwise, we have $x \mapsto y$. Let $A = \{a \in V_i \mid a \mapsto y\}$, $B = \{b \in V_i \mid y \rightarrow b, b \rightarrow y\}$, $C = \{c \in V_i \mid y \mapsto c\}$, $L = \{d \in V_j \mid x \mapsto d\}$, $M = \{e \in V_j \mid x \rightarrow e, e \rightarrow x\}$ and $N = \{f \in V_j \mid f \mapsto x\}$. Clearly, L, M, N are pairwise disjoint, $L \cup M \cup N = V_j$ and $B \neq \emptyset$. Because of $d^-(x, V_j) \geq (|V_j|+1)/2$, we know $|M \cup N| \geq (|V_j|+1)/2$ and then, $|L| < |V_j|/2$. Now we claim that there is at least one arc uv in D from B to $M \cup N$. In fact, if this is not true, then $M \cup N \mapsto B$ and for any vertex b of B , we have $N^+(b, V_j) \subseteq L$. So $d^+(b, V_j) \leq |L| < |V_j|/2$, a contradiction! By the definitions of B, M and N , we see that $y \rightarrow u$ and $v \rightarrow x$ hold. Hence, $yuvx$ is just a path in D from y to x .

In summary, D is strongly connected. $\square$

By using Theorem 5, we obtain a sufficient condition for a quasi-semi-partition complete digraph to be Hamiltonian.

Theorem 6. Let D be a quasi-semi-partition complete digraph with $\alpha(D) \leq n/2$. Then D is Hamiltonian.

Proof. From Remark 2 and Theorem 2, D is either Hamiltonian or there exists two vertex-disjoint cycles C_1 and C_2 in D such that $V(C_1) \cup V(C_2) = V(D)$ and $|V(C_1) \cap V_j| = |V(C_2) \cap V_j| = |V_j|/2$ for all $1 \leq j \leq k$, where $V_1, V_2, \dots, V_k$ are the partite sets of D . We only need to consider the latter case. Obviously, C_1, C_2 is the minimal 1-regular factor. Thus, according to Theorem 3, we may assume that C_1, C_2 and V^* just satisfies Theorem 3. Theorem 5 ensue that D is strong, hence there are vertices $u \in C_1$ and $v \in C_2$ such that $v \rightarrow u$. Furthermore, $u^- \in V^*$ and $u^- \mapsto V(C_2) \cap (V - V^*)$. In fact, u^- is adjacent to any vertex of $V(C_2) \cap (V - V^*)$. If there exists a vertex in $V(C_2) \cap (V - V^*)$ which dominates u^- , then again by Theorem 3, we deduce $u^{--} \in V^*$, but this is clearly not impossible. For any partite set V' different from V^* , we have $d^-(u^-, V') \leq |C_1 \cap V'| = |V'|/2$. This contradicts the definition of quasi-semi-partition complete digraphs. Therefore, D is Hamiltonian. $\square$

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