A New Computing Method for Greeks Using Stochastic Sensitivity Analysis

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Abstract. In a risk management of derivative securities, Greeks, i.e. sensitivity coefficients, are important measures of market risk to evaluate the impact of misspecification of some stochastic model on the expected payoff function. We investigate a new computing method for Greeks based on Malliavin calculus without resort to a direct differentiation of the complex payoff functions. As a result, a new relation between Γ and Δ is obtained for the Asian option.

Keywords: Greeks, Malliavin calculus, Monte Carlo simulation.

1 Introduction

We consider a stochastic integral equation in a well-defined Black-Scholes set-up,

$$S_t = S_0 + \int_0^t r S_\tau d\tau + \int_0^t \sigma S_\tau dW_\tau, \quad (1)$$

where S_t is the price of underlying asset with S_0 denoting the present (initial) value, r denotes the risk-free interest rate, σ is the volatility, and $(W_t)_{0 \leq t \leq T}$ is a standard Brownian motion. It should be noted that, in European options, we have a closed solution to (1) as $S_T = S_0 \exp(\mu T + \sigma W_T)$, where $\mu = r - \sigma^2/2$ for a fixed expiration or maturity time T .

We are interested in studying how to evaluate the sensitivity with respect to model parameters, e.g., present price S_0 , volatility σ , etc., of the expected payoff

$$E[e^{-rT} \Phi(S_T)], \quad (2)$$

for an exponentially discounted value of the payoff function $\Phi(S_T)$, where $E[\cdot]$ denotes the expectation operator. The sensitivity of more sophisticated payoff functions including path-dependent Asian-type options like

$$E[e^{-rT} \Phi(\frac{1}{T} \int_0^T S_t dt)], \quad (3)$$

be treated in a similar manner along the lines that are investigated in this study.

In finance, this is the so-called model risk problem. Commonly referred to as Greeks, sensitivities in financial market are typically defined as the partial

derivatives of the expected payoff function with respect to underlying model parameters. In general, finite difference approximations are heavily used to simulate Greeks. However, the approximation soon becomes inefficient particularly when payoff functions are complex and discontinuous.

To overcome this difficulty, Broadie and Glasserman [1] proposed a method to put the differential of the payoff function inside the expectation operator required to evaluate the sensitivity. But this idea is applicable only when the density of the random variable involved is explicitly known. Recently, Fournie et al. [2] suggested the use of Malliavin calculus, by means of integration by parts, to shift the differential operator from the expected payoff to the underlying diffusion (e.g., Gaussian) kernel, introducing a weighting function.

Another examples that are similar to the present study and explored by the present author (e.g., Ref. [3], [7]) but not covered in this paper are models involving a step function and non-smooth objective functions. In these studies, the stochastic sensitivity analysis technique based on the Novikov's identity is used instead of Malliavin calculus.

In this paper, we present a new constructive approach for the computation of Greeks in financial engineering. The present approach enables the simulation of Greeks without resort to direct differentiation of the complex or discontinuous payoff functions.

2 Malliavin Calculus

Let R be the space of random variables of the form $F = f(W_{t_1}, W_{t_2}, \dots, W_{t_n})$, where f is smooth and W_t denotes the Brownian motion [6]. For a smooth random variable $F \in R$, we can define its derivative $DF = D_t F$, where the differential operator D is closable. Since D operates on random variables by differentiating functions in the form of partial derivatives, it shares the familiar chain rule property, $D_t(f(F)) = \nabla f(F) \cdot D_t F = f'(F)D_t F$, and others.

We denote by D^* the Skorohod integral, defined as the adjoint operator of D . If u belongs to $\text{Dom}(D^*)$, then $D^*(u)$ is characterized by the following integration by parts (ibp) formula:

$$E[FD^*(u)] = E[\int_0^T (D_t F)u_t dt]. \quad (4)$$

Note that (4) gives a duality relationship to link operators D and D^* . The adjoint operator D^* behaves like a stochastic integral. In fact, if u_t is an adapted process, then the Skorohod integral coincides with the Ito integral: i.e., $D^*(u) = \int_0^T u_t dW_t$. In general, one has

$$D^*(Fu) = FD^*(u) - \int_0^T (D_t F)u_t dt. \quad (5)$$

A heuristic derivation of (5) is demonstrated here. Let us assume that F and G are any two smooth random variables, and u_t a generic process, then by product rule of D one has

$$\begin{aligned} E[GF D^*(u)] &= E[\int_0^T D_t(GF)u_t dt] = E[\int_0^T G(D_t F)u_t dt] + E[\int_0^T (D_t G)F u_t dt] \\ &= E[G \int_0^T (D_t F)u_t dt] + E[GD^*(Fu)], \end{aligned}$$

which implies that $E[GD^*(Fu)] = E[G(FD^*(u) - \int_0^T (D_t F)u_t dt)]$ for any random variables G . Hence, we must have (5).

In the present study, we frequently use the following formal relationship to remove the derivative from a (smooth) random function f as follows:

$$E[\nabla f(X)Y] = E[f'(X)Y] = E[f(X)H_{XY}], \quad (6)$$

where X , Y , and H_{XY} are random variables. It is noted that (6) can be deduced from the integration by parts formula (4), and we have an explicit expression for H_{XY} as

$$H_{XY} = D^* \left(\frac{Y}{\int_0^T D_t X dt} \right). \quad (7)$$

If higher order derivatives are involved then one has to repeat the procedure (6) iteratively.

3 European Options

In the case of European option whose payoff function is defined by (2), the essence of the present method is that the gradient of the expected (discounted) payoff, $\nabla E[e^{-rT}\Phi(S_T)]$, is evaluated by putting the gradient inside the expectation, i.e., $E[e^{-rT}\nabla\Phi(S_T)]$, which involves computations of $\nabla\Phi(S_T) = \Phi'(S_T)$ and ∇S_T . Further, applying Malliavin calculus techniques, the gradient is rewritten as $E[e^{-rT}\Phi(S_T)H]$ for some random variable H . It should be noted, however, that there is no uniqueness in this representation since we can add to H any random variables that are orthogonal to S_T . In general, H involves Ito or Skorohod integrals.

3.1 Delta

Now we compute Delta, Δ , the first-order partial differential sensitivity coefficient of the expected outcome of the option, i.e., (2), with respect to the initial asset value S_0 :

$$\Delta = \frac{\partial}{\partial S_0} E[e^{-rT}\Phi(S_T)] = e^{-rT} E[\Phi'(S_T) \frac{\partial S_T}{\partial S_0}] = \frac{e^{-rT}}{S_0} E[\Phi'(S_T)S_T].$$

Then, with $X = Y = S_T$ in (7), we perform the integration by parts (ibp) to give

$$\Delta = \frac{e^{-rT}}{S_0} E[\Phi(S_T)H_{XY}] = \frac{e^{-rT}}{S_0} E \left[\Phi(S_T) D^* \left(\frac{S_T}{\int_0^T D_t S_T dt} \right) \right]. \quad (8)$$

Since $\int_0^T D_t S_T dt = \sigma \int_0^T S_T D_t W_T dt = \sigma S_T \int_0^T 1_{\{t \leq T\}} = \sigma T S_T$, we evaluate the stochastic integral

$$H_{XY} = D^* \left(\frac{S_T}{\int_0^T D_t S_T dt} \right) = D^* \left(\frac{1}{\sigma T} \right) = \frac{D^*(1)}{\sigma T} = \frac{W_T}{\sigma T}$$

with the help of (5) applied to $u = 1$ (a constant process which is adapted and Ito integral yields $D^*(1) = W_T$). Then the final expression for Δ reads

$$\Delta = \frac{e^{-rT}}{\sigma T S_0} E[\Phi(S_T) W_T]. \quad (9)$$

3.2 Vega

Next Greek Vega, V , is the sensitivity of (2) with respect to the volatility σ :

$$V = \frac{\partial}{\partial \sigma} E[e^{-rT} \Phi(S_T)] = e^{-rT} E[\Phi'(S_T) \frac{\partial S_T}{\partial \sigma}] = e^{-rT} E[\Phi'(S_T) S_T \{W_T - \sigma T\}].$$

Then, utilizing (6) and (7) again with $X = S_T$ and $Y = S_T(W_T - \sigma T)$, we apply the ibp to give

$$V = e^{-rT} E[\Phi(S_T) H_{XY}] = e^{-rT} E \left[\Phi(S_T) D^* \left(\frac{S_T(W_T - \sigma T)}{\int_0^T D_t S_T dt} \right) \right] = e^{-rT} E \left[\Phi(S_T) D^* \left(\frac{W_T}{\sigma T} - 1 \right) \right].$$

So, we evaluate the stochastic integral as

$$H_{XY} = D^* \left(\frac{W_T}{\sigma T} - 1 \right) = \frac{1}{\sigma T} D^*(W_T) - D^*(1) = \frac{1}{\sigma T} D^*(W_T) - W_T.$$

With the help of (4) applied to $u = 1$ (adapted process) and $F = W_T$, we have

$$D^*(W_T) = W_T^2 - \int_0^T D_t W_T dt = W_T^2 - \int_0^T dt = W_T^2 - T.$$

If we bring together the partial results obtained above, we derive the final expression

$$V = e^{-rT} E \left[\Phi(S_T) \left\{ \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right\} \right]. \quad (10)$$

3.3 Gamma

The last Greek Gamma, Γ , involves a second-order derivative,

$$\Gamma = \frac{\partial^2}{\partial S_0^2} E[e^{-rT} \Phi(S_T)] = \frac{e^{-rT}}{S_0^2} E[\Phi''(S_T) S_T^2].$$

Utilizing (6) and (7) with $X = S_T$ and $Y = S_T^2$, we obtain after a first ibp

$$\Gamma = \frac{e^{-rT}}{S_0^2} E \left[\Phi'(S_T) D^* \left(\frac{S_T^2}{\int_0^T D_t S_T dt} \right) \right] = \frac{e^{-rT}}{S_0^2} E \left[\Phi'(S_T) D^* \left(\frac{S_T}{\sigma T} \right) \right].$$

With the help of (5) applied to $u = 1/\sigma T$ (constant adapted process) and $F = S_T$, we have

$$D^* \left(\frac{S_T}{\sigma T} \right) = \frac{S_T}{\sigma T} D^*(1) - \frac{1}{\sigma T} \int_0^T D_t S_T dt = S_T \left(\frac{W_T}{\sigma T} - 1 \right).$$

Then, repeated application of (6) and (7) with $X = S_T$ and $Y = S_T(W_T/\sigma T - 1)$, the second ibp yields

$$\Gamma = \frac{e^{-rT}}{S_0^2} E \left[\Phi'(S_T) S_T \left(\frac{W_T}{\sigma T} - 1 \right) \right] = \frac{e^{-rT}}{S_0^2} E \left[\Phi(S_T) D^* \left(\frac{S_T}{\int_0^T D_t S_T dt} \left\{ \frac{W_T}{\sigma T} - 1 \right\} \right) \right].$$

With the help of (5) as before, we can evaluate the stochastic integral as

$$D^* \left(\frac{S_T}{\int_0^T D_t S_T dt} \left\{ \frac{W_T}{\sigma T} - 1 \right\} \right) = \frac{1}{\sigma T} D^* \left(\frac{W_T}{\sigma T} - 1 \right) = \frac{1}{\sigma T} \left\{ \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right\}.$$

If we combine the results obtained above, the final expression becomes

$$\Gamma = \frac{e^{-rT}}{\sigma T S_0^2} E \left[\Phi(S_T) \left\{ \frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma} \right\} \right]. \quad (11)$$

Comparing (11) with (10), we find the following relationship between V and Γ :

$$\Gamma = \frac{V}{\sigma T S_0^2}. \quad (12)$$

Since we have closed solutions for all the Greeks in European options, we can easily check the correctness of the above results.

4 Asian Options

In the case of Asian option whose payoff functional is defined by (3), the essence of the present approach is again that the gradient of the expected (discounted) payoff is rewritten as $E[e^{-rT} \nabla \Phi(\frac{1}{T} \int_0^T S_t dt)] = e^{-rT} E[\Phi(\frac{1}{T} \int_0^T S_t dt) H]$, for some random variable H . Different from the European options, however, we do not have a known closed solution in this case.

4.1 Delta

Delta in this case is given by

$$\Delta = \frac{\partial}{\partial S_0} E[e^{-rT} \nabla \Phi(\frac{1}{T} \int_0^T S_t dt)] = \frac{e^{-rT}}{S_0} E[\Phi'(\frac{1}{T} \int_0^T S_t dt) \frac{1}{T} \int_0^T S_t dt].$$

Utilizing (6) and (7) with $X = Y = \int_0^T S_t dt/T$, we may apply the ibp to give

$$\Delta = \frac{e^{-rT}}{S_0} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{Y}{\int_0^T D_t X dt} \right) \right] = \frac{e^{-rT}}{S_0} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{\int_0^T S_t dt}{\sigma \int_0^T \int_0^T S_t dt} \right) \right],$$

where we have used the relationship $\int_0^T \int_0^T D_v S_t dv dt = \sigma \int_0^T t S_t dt$. With the help of (5) applied to $u = 1/\sigma$ (constant adapted process) and $F = \int_0^T S_t dt / \int_0^T t S_t dt$, we may obtain

$$\Delta = \frac{e^{-rT}}{S_0} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) \left(\frac{1}{\langle T \rangle} \left\{ \frac{W_T}{\sigma} + \frac{\langle T^2 \rangle}{\langle T \rangle} \right\} - 1 \right) \right], \quad (13)$$

where we have used the relationship $\int_0^T \int_0^T t D_v S_t dv dt = \sigma \int_0^T t^2 S_t dt$, and where we defined

$$\langle T \rangle = \frac{\int_0^T t S_t dt}{\int_0^T S_t dt} \quad \text{and} \quad \langle T^2 \rangle = \frac{\int_0^T t^2 S_t dt}{\int_0^T S_t dt}.$$

4.2 Vega

Vega in this case becomes

$$\begin{aligned} V &= \frac{\partial}{\partial \sigma} E \left[e^{-rT} \nabla \Phi \left(\frac{1}{T} \int_0^T S_t dt \right) \right] = e^{-rT} E \left[\Phi' \left(\frac{1}{T} \int_0^T S_t dt \right) \frac{1}{T} \int_0^T \frac{\partial S_t}{\partial \sigma} dt \right] \\ &= e^{-rT} E \left[\Phi' \left(\frac{1}{T} \int_0^T S_t dt \right) \frac{1}{T} \int_0^T S_t \{W_t - t\sigma\} dt \right]. \end{aligned}$$

As before, with the help of (6) and (7) applied to $X = \int_0^T S_t dt / T$ and $Y = \int_0^T S_t (W_t - t\sigma) dt / T$, we have

$$V = e^{-rT} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{Y}{\int_0^T D_t X dt} \right) \right] = e^{-rT} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{\int_0^T S_t W_t dt}{\sigma \int_0^T t S_t dt} - 1 \right) \right],$$

which, with the help of (5), yields the following expression

$$V = e^{-rT} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) \left\{ \frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T t S_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{(\int_0^T t S_t dt)^2} - W_T \right\} \right]. \quad (14)$$

4.3 Gamma

Gamma involves a second-order derivative,

$$\Gamma = \frac{\partial^2}{\partial S_0^2} E[e^{-rT} \nabla \Phi(\frac{1}{T} \int_0^T S_t dt)] = \frac{e^{-rT}}{S_0^2} E[\Phi''(\frac{1}{T} \int_0^T S_t dt) (\int_0^T S_t dt / T)^2].$$

Application of (6) with $X = \int_0^T S_t dt / T$ and $Y = (\int_0^T S_t dt / T)^2$, and utilizing a close variant of (7) (see Ref. [4], [5]), i.e.,

$$H_{XY} = D^* \left(\frac{S_t Y}{\int_0^T S_v D_v X dv} \right),$$

we may obtain after a first integration by parts

$$\begin{aligned}
\Gamma &= \frac{e^{-rT}}{S_0^2} E \left[\Phi' \left(\frac{1}{T} \int_0^T S_\tau d\tau \right) D^* \left(\frac{S_T Y}{\int_0^T S_v D_v X dv} \right) \right] \\
&= \frac{e^{-rT}}{S_0^2} E \left[\Phi' \left(\frac{1}{T} \int_0^T S_\tau d\tau \right) D^* \left(\frac{2S_T}{\sigma T} \right) \right], \quad (15)
\end{aligned}$$

where the relation $\int_0^T \int_0^T S_v D_v S_t dv dt = \sigma \int_0^T \int_0^t S_t S_v dv dt = \frac{\sigma}{2} (\int_0^T S_t dt)^2$ is used. Further, we obtain

$$\Gamma = \frac{2e^{-rT}}{\sigma T S_0^2} E \left[\Phi' \left(\frac{1}{T} \int_0^T S_\tau d\tau \right) \int_0^T S_t dW_t \right] = \frac{2e^{-rT}}{\sigma^2 T S_0^2} E \left[\Phi' \left(\frac{1}{T} \int_0^T S_t dt \right) (S_T - S_0 - r \int_0^T S_t dt) \right],$$

which involves (1).

Then, repeated application of (6) and (7) with $X = \int_0^T S_t dt/T$ and $Y = S_T - S_0 - r \int_0^T S_t dt$, the second integration by parts yields

$$\begin{aligned}
\Gamma &= \frac{2e^{-rT}}{\sigma^2 S_0^2} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{S_T - S_0 - r \int_0^T S_t dt}{\sigma \int_0^T t S_t dt} \right) \right] \\
&= \frac{2e^{-rT}}{\sigma^2 S_0^2} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{S_T - S_0}{\sigma \int_0^T t S_t dt} \right) \right] - \frac{2re^{-rT}}{\sigma^2 S_0^2} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{\int_0^T S_t dt}{\sigma \int_0^T t S_t dt} \right) \right].
\end{aligned}$$

With the help of (5) applied to $u = 1/\sigma$ and $F = (S_T - S_0)/\int_0^T t S_t dt$, the present approach yields a brand new estimate which gives an explicit relationship between Γ and Δ as follows:

$$\begin{aligned}
\Gamma &= \frac{2e^{-rT}}{\sigma^2 S_0^2} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) D^* \left(\frac{S_T - S_0}{\sigma \int_0^T t S_t dt} \right) \right] - \frac{2r}{\sigma^2 S_0} \Delta \\
&= \frac{2e^{-rT}}{\sigma^2 S_0^2} E \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) \frac{1}{\int_0^T t S_t dt} \left(\left\{ \frac{W_T}{\sigma} + \frac{< T^2 >}{< T >} \right\} (S_T - S_0) - T S_T \right) \right] \\
&\quad - \frac{2r}{\sigma^2 S_0} \Delta. \quad (16)
\end{aligned}$$

5 Monte Carlo Simulations of Asian Option

Here, we present the simulation results with parameters $r = 0.1$, $\sigma = 0.25$, $T = 0.2$ (in years), and $S_0 = K = 100$ (in arbitrary cash units) where K denotes the strike price. We have divided the entire interval of integration into 252 pieces, representing the trading days in a year.

In Fig. 1, we compare the convergence behavior of Δ with the result obtained by Broadie and Glasserman [1]. The result indicates a fairly good convergence to the steady-state value that is attained at 10,000th iteration stage in [1].

In Fig. 2, we compare the simulation result of V with the one that is obtained at 10,000th iteration stage in [1]. This indicates that some noticeable bias may remain in the present Monte Carlo simulation, and further study may be necessary to analyze and reduce the bias involved.

Although we cannot compare the proposed method with others, the present simulations may provide most detailed and extensive results currently available for Greeks of Asian option.

Fig. 1. Estimated Delta

Fig. 2. Estimated Vega

6 Conclusions

We have presented a stochastic sensitivity analysis method, in particular, a constructive approach for computing Greeks in finance using Malliavin calculus. As a result, a new relation between Γ and Δ is obtained for the Asian option. The present approach may be useful when the random variables are smooth in the sense of stochastic derivatives. It is, however, necessary to further investigate and improve Monte Carlo procedures to reduce the bias involved in the simulation of Vega in Asian-type options and other sophisticated options.

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