Truncation Error Estimate on Random Signals by Local Average

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Abstract. Since signals are often of random characters, random signals play an important role in signal processing. We show that the bandlimited wide sense stationary stochastic process can be approximated by Shannon sampling theorem on local averages. Explicit truncation error bounds are given.

Keywords: stochastic process, random Signals, local averages, truncation error, Shannon sampling theorem.

1 Introduction and the Main Result

The Shannon sampling theorem plays an important role in signal analysis as it provides a foundation for digital signal processing. It says that any bandlimited function f , having its frequencies bounded by $\pi\mathcal{W}$, can be recovered from its sampled values taken at instances $k/\mathcal{W}$, i.e.

$$f(t) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{\mathcal{W}}\right) \text{sinc}(\mathcal{W}t - k), \quad (1)$$

where $\text{sinc}(t) = \sin\pi t/(\pi t)$, $t \neq 0$, and $\text{sinc}(0) = 1$.

This equation requires values of a signal f that are measured on a discrete set. However, due to its physical limitation, say the inertia, a measuring apparatus may not be able to obtain exact values of f at epoch t_k for $k = 0, 1, 2, \dots$. Instead, what a measuring apparatus often gives us is a local averages of f near t_k for each k . The sampled values defined as local averages may be formulated by the following equation

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$$\langle f, u_k \rangle = \int f(x) u_k(x) dx \quad (2)$$

for some collection of averaging functions $u_k(x), k \in \mathbb{Z}$, which satisfy the following properties,

$$\text{supp } u_k \subset [x_k - \frac{\sigma}{2}, x_k + \frac{\sigma}{2}], \quad u_k(x) \geq 0, \quad \text{and} \quad \int u_k(x) dx = 1. \quad (3)$$

Where u_k for each $k \in \mathbb{Z}$ is a weight function characterizing the inertia of measuring apparatus. Particularly, in an ideal case, the function is given by Dirac δ -function, $u_k = \delta(\cdot - t_k)$, because $\langle f, u_k \rangle = f(t_k)$ is the exact value of t_k .

The local averaging method in sampling was studied by a number of papers [1]- [6] from 1994 to 2006.

The associated truncation error of (1) is defined by

$$R_N^f = f(t) - \sum_{k=-N}^{+N} f\left(\frac{k}{\mathcal{W}}\right) \text{sinc}(\mathcal{W}t - k) = \frac{\sin \pi \mathcal{W}t}{\pi} \sum_{|k| > N} (-1)^k \frac{f(k/\mathcal{W})}{\mathcal{W}t - k}. \quad (4)$$

But on the one hand, we can not finish a infinite number of terms in practise, we only approximated signal functions by a finite number of terms. Which is called truncation error deal with bounds by a number of papers [7]- [15].

On the other hand, since signals are often of random characters, random signals play an important role in signal processing, especially in the study of sampling theorems. For example, a signal of speech, where the random portion of the function may be white noise or some other distortion in the transmission channel, perhaps given via a probability distribution. So there are a lots of papers on this topic too. Such as [16]-[24]. Now we give truncation error bounds random signals by local averages.

Before stating the results, let us introduce some notations. $L^p(\mathbb{R})$ is the space of all measurable functions on $\mathbb{R}$ for which $\|f\|_p < +\infty$, where

$$\|f\|_p := \left(\int_{-\infty}^{+\infty} |f(u)|^p du \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \text{ess sup}_{u \in \mathbb{R}} |f(u)|, \quad p = \infty.$$

$B_{\pi\mathcal{W},p}$ is the set of all entire functions f of exponential type with type at most $\pi\mathcal{W}$ that belong to $L^2(\mathbb{R})$ when restricted to the real line [25]. By the Paley-Wiener Theorem, a square integrable function f is band-limited to $[-\pi\mathcal{W}, \pi\mathcal{W}]$ if and only if $f \in B_{\pi\mathcal{W},2}$.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ [26], a real-valued stochastic process $X(t) := X(t, \omega)$ defined on $\mathbb{R} \times \Omega$ is said to be stationary in weak sense if $E[X(t)^2] < \infty, \forall t \in \mathbb{R}$, and the autocorrelation function

$$R_X(t, t + \tau) := \int_{\Omega} X(t, \omega) X(t + \tau, \omega) d\mathbb{P}(\omega)$$

is independent of $t \in \mathbb{R}$, i.e., $R_X(t, t + \tau) = R_X(\tau)$.

A weak sense stationary process $X(t)$ is said to be bandlimited to an interval $[-\pi\mathcal{W}, \pi\mathcal{W}]$ if R_X belongs to $B_{\pi\mathcal{W},p}$ for some $1 \leq p \leq \infty$.

Now we assume that u_k which are given by (3) satisfy the following properties.

- i) $\text{supp } u_k \subset [\frac{k}{W} - \sigma'_k, \frac{k}{W} + \sigma''_k]$, where $\sigma/4 \leq \sigma'_k, \sigma''_k \leq \sigma/2$, σ are positive constants;
- ii) $u_k(t) \geq 0$, $\int u_k(t)dt = 1$;
- iii) $m = \inf_{k \in \mathbb{Z}} \{m_k\}$, where $m_k := \int_{k/W - \sigma/4}^{k/W + \sigma/4} u_k(t)dt$.

In this cases, The associated truncation error of random signals $X(t, \omega)$ is defined dy

$$R_N^X = X(t) - \sum_{k=-N}^{+N} \langle X, u_k \rangle \text{sinc}(Wt - k). \quad (6)$$

where the autocorrelation function of the weak sense stationary stochastic process $X(t, \omega)$ belongs to $B_{\pi\mathcal{W},2}$, and $W > \mathcal{W} > 0$.

The following results is proved by Belyaev and Splettstösser in 1959 and 1981, respectively.

Proposition A. [16, Theorem 5]) If the autocorrelation function of the weak sense stationary stochastic process $X(t, \omega)$ belongs to $B_{\pi\mathcal{W},2}$, for $W > \mathcal{W} > 0$, we have

$$\begin{aligned} E [|R_N^{X*}|^2] &= E \left(\left| X(t, \omega) - \sum_{k=-N}^N X\left(\frac{k}{W}, \omega\right) \text{sinc}(Wt - k) \right|^2 \right) \\ &\leq \frac{16R_X(0)(2 + |t|W)^2}{\pi^2(1 - \mathcal{W}/W)^2 N^2}. \end{aligned} \quad (7)$$

Proposition B. [17, Theorem 2.2]) If the autocorrelation function of the weak sense stationary stochastic process $X(t, \omega)$ belongs to $B_{\pi\mathcal{W},p}$ for some $1 \leq p \leq 2$ and $\Omega > 0$, then

$$\lim_{N \rightarrow \infty} E \left(\left| X(t, \omega) - \sum_{k=-N}^N X\left(\frac{k}{W}, \omega\right) \text{sinc}(Wt - k) \right|^2 \right) = 0. \quad (8)$$

For this case, we have the following result.

Theorem C. If the autocorrelation function R_X of a weak sense stationary stochastic process $X(t, \omega)$ belongs to $B_{\mathcal{W},2}$, for $W > \mathcal{W} > 0$ and $2/\delta \geq N \geq 100$, we have

$$E [|R_N^X|^2] \leq \left(14.80 \|R''\|_\infty + \frac{32R_X(0)(2 + |t|W)^2}{\pi^2(1 - \mathcal{W}/W)^2} \right) \left(\frac{\ln N}{N} \right)^2, \quad (9)$$

where $\{u_k(t)\}$ is a sequence of continuous functions defined by (5).

Or, in other words

$$E [|R_N^X|^2] = O \left(\frac{\ln N}{N} \right)^2, \quad N \rightarrow \infty. \quad (10)$$

2 Proof of the Main Result

Let us introduce some preliminary results first.

Lemma D. [27] One has for $q' > 1$, $1/p' + 1/q' = 1$, and $W > 0$,

$$\sum_{k=-\infty}^{\infty} |\operatorname{sinc}(Wt - k)|^{q'} \leq 1 + \left(\frac{2}{\pi} \right)^{q'} \frac{q'}{q' - 1} < p'. \quad (2.1)$$

Lemma E. [26] If a stationary stochastic process $X(t, \omega)$, $t \in [a, b]$ is continuous in mean square, $f(t), g(t)$ are continuous function on $[a, b]$, then

$$E \left[\int_a^b f(s)X(s)ds \cdot \int_a^b g(t)X(t)dt \right] = \int_a^b \int_a^b f(s)g(t)R(s-t)dsdt. \quad (2.2)$$

Lemma F. Suppose that the autocorrelation function R_X of the weak sense stationary stochastic process $X(t, \omega)$ belongs to $B_{\pi W, 2}$ and $W > 0$, and satisfies $R_X''(t) \in C(\mathbb{R})$. Let

$$\begin{aligned} & D \left(\frac{j}{W}; \frac{\delta}{2} \right) \\ &:= \sup_{\substack{|\delta^*| \leq \frac{\delta}{2} \\ |\delta^{**}| \leq \frac{\delta}{2}}} \left| R_X \left(\frac{j}{W} \right) - R_X \left(\frac{j}{W} - \delta^{**} \right) - R_X \left(\frac{j}{W} + \delta^* \right) + R_X \left(\frac{j}{W} + \delta^* - \delta^{**} \right) \right| \\ &= \sup_{\substack{|\delta^*| \leq \frac{\delta}{2} \\ |\delta^{**}| \leq \frac{\delta}{2}}} \left| \int_{-\delta^{**}}^0 \int_{\delta^*}^0 R_X'' \left(\frac{j}{W} + u + v \right) dudv \right|. \end{aligned}$$

Then we have for $r, N \geq 1$,

$$\sum_{j=-2N}^{+2N} \left[D \left(\frac{j\pi}{\Omega}; \frac{\delta}{2} \right) \right]^r \leq (4N+1)(\|R_X''(t)\|_{\infty})^r \left(\frac{\delta}{2} \right)^{2r}. \quad (11)$$

Proof. Since R_X is even and $R_X''(t) \in C(\mathbb{R})$, we have

$$\begin{aligned} \sum_{j=-2N}^{+2N} \left[D \left(\frac{j\pi}{\Omega}; \frac{\delta}{2} \right) \right]^r &= \left[D \left(0; \frac{\delta}{2} \right) \right]^r + 2 \sum_{j=1}^{2N} \left[D \left(\frac{j\pi}{\Omega}; \frac{\delta}{2} \right) \right]^r \\ &\leq (4N+1)(\|R_X''(t)\|_{\infty})^r \left(\frac{\delta}{2} \right)^{2r}. \end{aligned}$$

which completes the proof.

Proof of Theorem C. From Proposition A, Proposition B and Lemma E we have

$$\begin{aligned}
E[|R_N^X|^2] &= E \left[\left| X(t, \omega) - \sum_{k=-N}^N \int_{k/W - \delta'_k}^{k/W + \delta''_k} u_k(t) X(t, \omega) dt \operatorname{sinc}(Wt - k) \right|^2 \right] \\
&= E \left[\left| X(t, \omega) - \sum_{k=-N}^N X\left(\frac{k}{W}, \omega\right) \operatorname{sinc}(Wt - k) \right. \right. \\
&\quad \left. \left. + \sum_{k=-N}^N X\left(\frac{k}{W}, \omega\right) \operatorname{sinc}(Wt - k) \right. \right. \\
&\quad \left. \left. - \sum_{k=-N}^N \int_{k/W - \delta'_k}^{k/W + \delta''_k} u_k(t) X(t, \omega) dt \operatorname{sinc}(Wt - k) \right|^2 \right] \\
&= 2E[|R_N^{X*}|^2] \\
&\quad + 2E \left[\left| \sum_{k=-N}^N \left[X\left(\frac{k}{W}, \omega\right) - \int_{k/W - \delta'_k}^{k\pi/\Omega + \delta''_k} u_k(t) X(t, \omega) dt \right] \operatorname{sinc}(Wt - k) \right|^2 \right] \\
&= 2E[|R_N^{X*}|^2] + 2E \left[\left| \sum_{k=-N}^N \left[\int_{-\delta'_k}^{\delta''_k} X\left(\frac{k}{W}, \omega\right) u_k\left(\frac{k}{W} + s\right) ds \right. \right. \right. \\
&\quad \left. \left. - \int_{-\delta'_k}^{\delta''_k} u_k(k/W + t) X(k\pi/\Omega + t, \omega) dt \right] \operatorname{sinc}(Wt - k) \right|^2 \right] \\
&= 2E[|R_N^{X*}|^2] + 2 \sum_{k=-N}^N \sum_{j=-N}^N \\
&\quad \left(\int_{-\delta'_k}^{\delta''_k} \int_{-\delta'_j}^{\delta''_j} E \left[X\left(\frac{k}{W}, \omega\right) X\left(\frac{j}{W}, \omega\right) \right] u_k\left(\frac{k}{W} + u\right) u_j\left(\frac{j}{W} + v\right) dudv \right. \\
&\quad - \int_{-\delta'_k}^{\delta''_k} \int_{-\delta'_j}^{\delta''_j} E \left[X\left(\frac{k}{W}, \omega\right) X\left(\frac{j}{W} + v, \omega\right) \right] u_k\left(\frac{k}{W} + u\right) u_j\left(\frac{j}{W} + v\right) dudv \\
&\quad - \int_{-\delta'_k}^{\delta''_k} \int_{-\delta'_j}^{\delta''_j} E \left[X\left(\frac{k}{W} + u, \omega\right) X\left(\frac{j}{W}, \omega\right) \right] u_k\left(\frac{k}{W} + u\right) u_j\left(\frac{j}{W} + v\right) dudv \\
&\quad \left. + \int_{-\delta'_k}^{\delta''_k} \int_{-\delta'_j}^{\delta''_j} E \left[X\left(\frac{k}{W} + u, \omega\right) X\left(\frac{j}{W} + v, \omega\right) \right] u_k\left(\frac{k}{W} + u\right) u_j\left(\frac{j}{W} + v\right) dudv \right) \\
&\quad \cdot |\operatorname{sinc}(Wt - k)| |\operatorname{sinc}(Wt - j)| \\
&= 2E[|R_N^{X*}|^2] + 2 \sum_{k=-N}^N \sum_{j=-N}^N \int_{-\delta'_k}^{\delta''_k} \int_{-\delta'_j}^{\delta''_j} \left[R_X\left(\frac{(k-j)}{W}\right) - R_X\left(\frac{(k-j)}{W} - v\right) \right. \\
&\quad \left. - R_X\left(\frac{(k-j)}{W} + u\right) + R_X\left(\frac{(k-j)}{W} + u - v\right) \right] u_k\left(\frac{k}{W} + u\right) u_j\left(\frac{j}{W} + v\right) dudv
\end{aligned}$$

$$\begin{aligned}
& \cdot |sinc(Wt - k)| |sinc(Wt - j)| \\
& \leq 2E [|R_N^{X*}|^2] + 2 \sum_{k=-N}^N \sum_{j=-N}^N \int_{-\delta'_k}^{\delta''_k} \int_{-\delta'_k}^{\delta''_k} D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) \\
& \quad u_k\left(\frac{k}{W} + u\right) u_j\left(\frac{j}{W} + v\right) dudv \cdot |sinc(Wt - k)| |sinc(Wt - j)| \\
& = 2E [|R_N^{X*}|^2] + 2 \sum_{k=-N}^N \sum_{j=-N}^N D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) |sinc(Wt - k)| |sinc(Wt - j)|
\end{aligned}$$

Using Hölder's inequality and Lemma D, we have

$$\begin{aligned}
& \sum_{k=-N}^N \sum_{j=-N}^N D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) |sinc(Wt - k)| \cdot |sinc(Wt - j)| \\
& \leq \left(\sum_{k=-N}^N \left| \sum_{j=-N}^N D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) |sinc(Wt - j)| \right|^{p^*} \right)^{1/p^*} \\
& \quad \cdot \left(\sum_{k=-N}^N |sinc(Wt - k)|^{q^*} \right)^{1/q^*} \\
& \leq (p^*)^{1/q^*} \left(\sum_{k=-N}^N \left| \sum_{j=-N}^N D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) |sinc(Wt - j)| \right|^{p^*} \right)^{1/p^*} \\
& \leq p^* \left(\sum_{k=-N}^N \left| \sum_{j=-N}^N D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) |sinc(Wt - j)| \right|^{p^*} \right)^{1/p^*},
\end{aligned}$$

where $1/p^* + 1/q^* = 1$. By Hausdorff-Young inequality [28, page176] and Lemma F, we have

$$\begin{aligned}
& \left(\sum_{k=-N}^N \left| \sum_{j=-N}^N D\left(\frac{(k-j)}{W}; \frac{\delta}{2}\right) |sinc(Wt - j)| \right|^{p^*} \right)^{1/p^*} \\
& \leq \left(\sum_{j=-2N}^{2N} \left[D\left(\frac{j}{W}; \frac{\delta}{2}\right) \right]^{r^*} \right)^{1/r^*} \left(\sum_{j=-2N}^{2N} |sinc(Wt - j)|^{s^*} \right)^{1/s^*} \\
& \leq (4N+1)^{1/r^*} \|R_X''(t)\|_\infty \left(\frac{\delta}{2}\right)^2 \left(\sum_{j=-2N}^{2N} |sinc(\Omega t - j\pi)|^{s^*} \right)^{1/s^*}
\end{aligned}$$

$$\leq (4N+1)^{1/r^*} \|R_X''(t)\|_\infty \left(\frac{1}{N}\right)^2 \left(\sum_{j=-\infty}^{\infty} |\text{sinc}(\Omega t - j\pi)|^{s^*}\right)^{1/s^*},$$

where $0 \leq 1/s^* + 1/r^* - 1 = 1/p^*$. Let $r^* = \ln N/2$. Notice that $N \geq 100$, we have

$$(4N+1)^{1/r^*} < \left(\frac{401N}{400}\right)^{2/\ln N} \leq 7.40.$$

Let $s^* = 2r^*/(2r^*-1)$ and $s' = 2r^*$. Then $1/s^* + 1/s' = 1$ and $p^* = 2r^* = \ln N$. By Lemma E, we have

$$\left(\sum_{j=-\infty}^{\infty} |\text{sinc}(\Omega t - j\pi)|^{s^*}\right)^{1/s^*} \leq s' = p^* = \ln N.$$

Hence

$$E[|R_N^X|^2] \leq \left(14.80\|R''\|_\infty + \frac{32R_X(0)(2+|t|W)^2}{\pi^2(1-W/W)^2}\right) \left(\frac{\ln N}{N}\right)^2.$$

This completes the proof.

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