Characterization on Irregular Tight Wavelet Frames with Matrix Dilations^{*}

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Abstract. There are many results in one dimensional wavelet frame theory in recent years. However, since there are some essential differences in high dimensional cases, the classical methods for one dimensional regular wavelet frames are unsuitable for the cases. In this paper, under some conditions on matrix-dilated sequences, a characterization formula for irregular tight frames of matrix-dilated wavelets is developed. It is based on the regular formulation by Chui, Czaja, Maggioni, Weiss, and on the recent multivariate results by Yang and Zhou.

Keywords: Irregular frame, Tight wavelet frame, Matrix dilations, Bessel sequences.

1 Characterization on Irregular Tight Wavelet Frames with Matrix-Dilations

Assume $\psi, \tilde{\psi} \in L^2(\mathbf{R}^n)$, $\{A_j\}_{j \in \mathbf{Z}}$ is a real $n \times n$ matrix sequence, B is a real $n \times n$ nonsingular matrix. A^{-*} is the transpose of A^{-1} . Let

$$\Psi = \{\psi_{j,k} : j \in \mathbf{Z}, k \in \mathbf{Z}^n\}, \quad \tilde{\Psi} = \{\tilde{\psi}_{j,k} : j \in \mathbf{Z}, k \in \mathbf{Z}^n\}, \quad (1)$$

where $\psi_{j,k}(x) = |\det A_j|^{-\frac{1}{2}} \psi(A_j^{-*}x - Bk)$. If $\{\psi_{j,k} : j \in \mathbf{Z}, k \in \mathbf{Z}^n\}$ is a tight frame for $L^2(\mathbf{R}^n)$, then ψ is called a tight frame function with respect to dilation sequence $\{A_j\}_{j \in \mathbf{Z}}$. There are some results in high dimensional wavelet frame theory in recent years (see [1] -[7]). Let $P(f, g) = \sum_{j \in \mathbf{Z}, k \in \mathbf{Z}^n} \langle f, \psi_{j,k} \rangle \langle \tilde{\psi}_{j,k}, g \rangle$, $f, g \in L^2(\mathbf{R}^n)$. Now we consider the following subset of $L^2(\mathbf{R}^n)$:

$$\mathcal{B} = \{f \in L^2(\mathbf{R}^n) : \hat{f} \in L^\infty(\mathbf{R}^n), \hat{f} \text{ has compact support.}\}$$

The following lemma comes from [2].

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Lemma 1. *Two families $\{e_\alpha : \alpha \in \mathcal{A}\}$ and $\{\tilde{e}_\alpha : \alpha \in \mathcal{A}\}$ constitute a dual pair if and only if they are Bessel sequences and satisfy*

$$P(f, g) := \sum_{\alpha \in \mathcal{A}} \langle f, e_\alpha \rangle \langle \tilde{e}_\alpha, g \rangle = \langle f, g \rangle,$$

for all f, g in a dense subset of $\mathcal{H}$.

Given $\{A_j\}_{j \in \mathbf{Z}}$ and B , we need the following notations:

$$\wedge = \{\alpha \in \mathbf{R}^n : \alpha = A_j^{-*}m, (j, m) \in \mathbf{Z} \times B^{-*}(\mathbf{Z}^n)\},$$

$$I(\alpha) = \{(j, m) \in \mathbf{Z} \times B^{-*}(\mathbf{Z}^n) : \alpha = A_j^{-*}m\}.$$

Assume that $\{A_j\}_{j \in \mathbf{Z}}$ satisfy the following conditions: there exist $\beta \in (0, 1)$, $q \in \mathbf{Z}_+$ such that

$$\|A_j A_{j+1}^{-1}\| \leq 1, \|A_j A_{j+q}^{-1}\| \leq \beta \quad \text{for any } j \in \mathbf{Z}, \quad (2)$$

$$\|A_{j+1}^{-1} A_j\| \leq 1, \|A_{j+q}^{-1} A_j\| \leq \beta \quad \text{for any } j \in \mathbf{Z}. \quad (3)$$

Then we have

Theorem 2. *Suppose that $\psi, \tilde{\psi} \in L^2(\mathbf{R}^n)$ have the property that the following two functions are locally integrable:*

$$\sum_{j \in \mathbf{Z}} \left| \widehat{\psi}(A_j \omega) \right|^2, \sum_{j \in \mathbf{Z}} \left| \widehat{\tilde{\psi}}(A_j \omega) \right|^2.$$

Then for $f, g \in \mathcal{B}$, $P(f, g)$ converges absolutely. Moreover,

$$\frac{1}{|\det B|} \sum_{(j, m) \in I(\alpha)} \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + A_j^{-1}m))} = \delta_{\alpha, 0},$$

for a.e. $\omega \in \mathbf{R}^n$ and for all $\alpha \in \wedge$, if and only if $P(f, g) = \langle f, g \rangle$ for all $f, g \in \mathcal{B}$.

By Theorem 2 and Lemma 1, the following two theorems are the immediate consequences.

Theorem 3. *Assume that $\Psi, \tilde{\Psi} \subset L^2(\mathbf{R}^n)$ are two Bessel sequences,*

$$\sum_{j \in \mathbf{Z}} \left| \widehat{\psi}(A_j \omega) \right|^2, \sum_{j \in \mathbf{Z}} \left| \widehat{\tilde{\psi}}(A_j \omega) \right|^2$$

are locally integrable. Then Ψ and $\tilde{\Psi}$ constitute a pair of dual frames for $L^2(\mathbf{R}^n)$ if and only if

$$\frac{1}{|\det B|} \sum_{(j, m) \in I(\alpha)} \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + A_j^{-1}m))} = \delta_{\alpha, 0},$$

for a.e. $\omega \in \mathbf{R}^n$ and for all $\alpha \in \wedge$.

Theorem 4. $\psi \in L^2(\mathbf{R}^n)$ satisfies

$$\frac{1}{|\det B|} \sum_{(j,m) \in I(\alpha)} \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + A_j^{-1}m))} = \delta_{\alpha,0},$$

for a.e. $\omega \in \mathbf{R}^n$ and for all $\alpha \in \Lambda$, if and only if $\{\psi_{j,k}\}_{j,k}$ is a tight frame with constant 1 for $L^2(\mathbf{R}^n)$.

2 Proof of Main Results

In fact, we only give the proof of Theorem 2.

Proof of Theorem 2: We first prove that $P(f, g)$ is absolutely convergent. Now, let

$$G_j := \sum_{k \in \mathbf{Z}^n} \langle f, \psi_{j,k} \rangle \langle \tilde{\psi}_{j,k}, f \rangle, \quad j \in \mathbf{Z}.$$

Using the Parseval identity, it is easy to show

$$G_j = \frac{1}{|\det B|} \int_{\mathbf{R}^n} \overline{\widehat{f}(\omega)} \widehat{\psi}(A_j \omega) \left(\sum_{s \in \mathbf{Z}^n} \widehat{f}(\omega + A_j^{-1} B^{-*} s) \overline{\widehat{\psi}(A_j \omega + B^{-*} s)} \right) d\omega.$$

Thus, we would like to show that $\sum_{j \in \mathbf{Z}} G_j$ is absolutely convergent. To do so, it is enough to show that the following two series are absolutely convergent:

$$I := \sum_{j \in \mathbf{Z}} \int_{\mathbf{R}^n} \overline{\widehat{f}(\omega)} \widehat{\psi}(A_j \omega) \widehat{f}(\omega) \overline{\widehat{\psi}(A_j \omega)} d\omega,$$

and

$$II := \int_{\mathbf{R}^n} \overline{\widehat{f}(\omega)} \widehat{\psi}(A_j \omega) \left(\sum_{s \in \mathbf{Z}^n \setminus \{0\}} \widehat{f}(\omega + A_j^{-1} B^{-*} s) \overline{\widehat{\psi}(A_j \omega + B^{-*} s)} \right) d\omega.$$

Since

$$\left| \widehat{\psi}(A_j \omega) \widehat{\psi}(A_j \omega + B^{-*} s) \right| \leq \frac{1}{2} \left(\left| \widehat{\psi}(A_j \omega) \right|^2 + \left| \widehat{\psi}(A_j \omega + B^{-*} s) \right|^2 \right), \quad (4)$$

It follows from (4) and the conditions on $\psi, \tilde{\psi}, f$ that I is absolutely convergent.

On the other hand, for $h \in \{\psi, \tilde{\psi}\}$,

$$\begin{aligned} & \sum_{j \in \mathbf{Z}, s \in \mathbf{Z}^n \setminus \{0\}} \int_{\mathbf{R}^n} \left| \widehat{f}(\omega) \widehat{f}(\omega + A_j^{-1} B^{-*} s) \right| \left| \widehat{h}(A_j \omega) \right|^2 d\omega \\ &= \int_{\mathbf{R}^n} \left(\sum_{j \in \mathbf{Z}, s \in \mathbf{Z}^n \setminus \{0\}} \frac{1}{|\det A_j|} \left| \widehat{f}(A_j^{-1} \omega) \widehat{f}(A_j^{-1}(\omega + B^{-*} s)) \right| \right) \left| \widehat{h}(\omega) \right|^2 d\omega. \end{aligned} \quad (5)$$

Since $f \in \mathcal{B}$, it follows from (2) and (3) that there exists constant $C_0 > 0$ such that for each $j \in \mathbf{Z}, \omega \in \mathbf{R}^n$, the number of $s \in \mathbf{Z}^n \setminus \{0\}$ satisfying

$$\widehat{f}(A_j^{-1}\omega)\widehat{f}(A_j^{-1}\omega + A_j^{-1}B^{-*}s) \neq 0$$

is less than $C_0 |\det A_j|$. Then

$$\sum_{j \in \mathbf{Z}, s \in \mathbf{Z}^n \setminus \{0\}} \frac{1}{|\det A_j|} \left| \widehat{f}(A_j^{-1}\omega) \widehat{f}(A_j^{-1}(\omega + B^{-*}s)) \right| \leq C \sum_{j \in \mathbf{Z}} \chi_F(A_j^{-1}\omega), \quad (6)$$

where $C = C_0 \left\| \widehat{f} \right\|_{\infty}^2$ and F is compact in $\mathbf{R}^n \setminus \{0\}$. By (2) and (3), $\{A_j\}_{j \in \mathbf{Z}}$ is an MFS (see [4]). Therefore there exists constant $K > 0$ such that

$$\sum_{j \in \mathbf{Z}} \chi_F(A_j^{-1}\omega) \leq K \quad \text{for any } \omega \in \mathbf{R}^n \setminus \{0\}. \quad (7)$$

Now, it follows from (5), (6) and (7) that the series II is convergent. Hence, we can rearrange the series for $P(f, f)$, to obtain

$$P(f, f) = \sum_{\alpha \in \Lambda} \int_{\mathbf{R}^n} \overline{\widehat{f}(\omega)} \widehat{f}(\omega + \alpha) \left(\frac{1}{|\det B|} \sum_{(j,m) \in I(\alpha)} \widehat{\psi}(A_j\omega) \overline{\widehat{\psi}(A_j(\omega + \alpha))} \right) d\omega.$$

Then using the polarization identity for the form P , we get the sufficiency of the theorem.

Next, we prove the necessary condition. From the above discussion, we have, let

$$P(f, g) = M(f, g) + R(f, g),$$

where

$$M(f, g) := \frac{1}{|\det B|} \int_{\mathbf{R}^n} \overline{\widehat{g}(\omega)} \widehat{f}(\omega) \left(\sum_{j \in \mathbf{Z}} \widehat{\psi}(A_j\omega) \overline{\widehat{\psi}(A_j\omega)} \right) d\omega,$$

and

$$R(f, g) := \frac{1}{|\det B|} \sum_{\alpha \in \Lambda \setminus \{0\}} \int_{\mathbf{R}^n} \overline{\widehat{g}(\omega)} \widehat{f}(\omega + \alpha) \sum_{(j,m) \in I(\alpha)} \widehat{\psi}(A_j\omega) \overline{\widehat{\psi}(A_j(\omega + \alpha))} d\omega.$$

Now, fix $\omega_0 \in \mathbf{R}^n \setminus \{0\}$ and let

$$\widehat{f}_1(\omega) = \frac{1}{\mu(H_k)^{1/2}} \chi_{\omega_0 + H_k}(\omega),$$

where $\mu(\cdot)$ is the Lebesgue measure on $\mathbf{R}^n$ and

$$H_k = A_k^{-1} \Delta, \quad \Delta = \{\xi \in \mathbf{R}^n : |\xi| = 1\}.$$

Then

$$M(f_1, f_1) = \frac{1}{|\det B| \mu(H_k)} \int_{\omega_0 + H_k} \sum_{j \in \mathbf{Z}} \widehat{\psi}(A_j\omega) \overline{\widehat{\psi}(A_j\omega)} d\omega$$

and

$$\begin{aligned}
 & |R(f_1, f_1)| \\
 & \leq \frac{1}{|\det B| \mu(H_k)} \sum_{\alpha \in \Lambda \setminus \{0\}} \sum_{(j, m) \in I(\alpha)} \int_{(\omega_0 + H_k) \cap (\alpha + \omega_0 + H_k)} \left| \widehat{\psi}(A_j \omega) \widehat{\psi}(A_j(\omega + \alpha)) \right| d\omega \\
 & \leq \frac{1}{|\det B| \mu(H_k)} \left(\sum_{\alpha \in \Lambda \setminus \{0\}} \sum_{(j, m) \in I(\alpha)} \int_{(\omega_0 + H_k) \cap (\alpha + \omega_0 + H_k)} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \right)^{1/2} \\
 & \times \left(\sum_{\alpha \in \Lambda \setminus \{0\}} \sum_{(j, m) \in I(\alpha)} \int_{(\omega_0 + H_k) \cap (\alpha + \omega_0 + H_k)} \left| \widehat{\psi}(A_j(\omega + \alpha)) \right|^2 d\omega \right)^{1/2}.
 \end{aligned}$$

If $(\omega_0 + H_k) \cap (\omega_0 + \alpha + H_k) \neq \emptyset$, then $\alpha \in A_k^{-1}(\Delta - \Delta)$. Thus for $(j, m) \in I(\alpha)$,

$$m \in (A_j A_k^{-1}(\Delta - \Delta)) \cap B^{-*}(\mathbf{Z}^n) \triangleq Q_{j, k}.$$

Using (2) and (3), there exists a constant $c > 0$ such that $j \geq k + c$. However,

$$\begin{aligned}
 & |R(f_1, f_1)| \\
 & \leq \frac{1}{|\det B| \mu(H_k)} \left(\sum_{j \geq k+c} \sum_{m \in Q_{j, k} \setminus \{0\}} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \right)^{1/2} \\
 & \times \left(\sum_{j \geq k+c} \sum_{m \in Q_{j, k} \setminus \{0\}} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j(\omega)) \right|^2 d\omega \right)^{1/2}. \quad (8)
 \end{aligned}$$

For the first factor,

$$\begin{aligned}
 & \frac{1}{\mu(H_k)} \sum_{j \geq k+c} \sum_{m \in Q_{j, k} \setminus \{0\}} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \\
 & \leq |\det A_k| \sum_{j \geq k+c} C |\det(A_j A_k^{-1})| |\det A_j|^{-1} \int_{A_j(\omega_0 + H_k)} \left| \widehat{\psi}(\omega) \right|^2 d\omega \\
 & = C \sum_{j \geq k+c} \int_{A_j(\omega_0 + H_k)} \left| \widehat{\psi}(\omega) \right|^2 d\omega, \quad (9)
 \end{aligned}$$

Here, we have used the fact $\#(Q_{j, k} \setminus \{0\}) \leq C \mu(A_j A_k^{-1}(\Delta - \Delta))$, which can be obtained by (2), (3).

Similarly, we can estimate the second factor. Now, using (2) and (3), it is easy to prove that:

(P_1) there exist $k_1, k_2 \in \mathbf{Z}$ such that the intersection of any k_2 sets in $\{A_j(\omega_0 + H_k)\}_{k \geq k_1, j \geq k+c}$ is empty.

(P_2) there exist constants $k_3 \in \mathbf{Z}, \lambda > 1$ such that for any $k \geq k_3, j \geq k + c$,

$$A_j(\omega_0 + H_k) \subset \{\omega : |\omega| \geq k_3 \lambda^j |\omega_0|\}.$$

Using (8), (9), (P_1) and (P_2) , we can get $|R(f_1, f_1)| \rightarrow 0$ when $k \rightarrow \infty$. Then, by Lebesgue theorem, we have

$$\begin{aligned} 1 &= \lim_{k \rightarrow \infty} \frac{1}{|\det B| \mu(H_k)} \int_{\omega_0 + H_k} \sum_{j \in \mathbf{Z}} \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j \omega)} d\omega \\ &= \frac{1}{|\det B|} \sum_{j \in \mathbf{Z}} \widehat{\psi}(A_j \omega_0) \overline{\widehat{\psi}(A_j \omega_0)}, \end{aligned}$$

which proves our claim for $\alpha = 0$. This also shows that

$$M(f, g) = \langle f, g \rangle,$$

for any $f, g \in \mathcal{B}$.

To complete the proof of our theorem, choose $\alpha_0 \in \wedge \setminus \{0\}$, and write

$$R(f, g) = R_1(f, g) + R_2(f, g),$$

where

$$R_1(f, g) = \frac{1}{|\det B|} \int_{\mathbf{R}^n} \overline{\widehat{g}(\omega)} \widehat{f}(\omega + \alpha_0) \sum_{(j, m) \in I(\alpha_0)} \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + \alpha_0))} d\omega.$$

$$R_2(f, g) = \frac{1}{|\det B|} \sum_{\alpha \in \wedge \setminus \{0, \alpha_0\}} \int_{\mathbf{R}^n} \overline{\widehat{g}(\omega)} \widehat{f}(\omega + \alpha) \sum_{(j, m) \in I(\alpha)} \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + \alpha))} d\omega.$$

Next, let $\omega_0 \in \mathbf{R}^n \setminus \{0\}$ be any Lebesgue point of functions $\sum_{j \in \mathbf{Z}} \left| \widehat{\psi}(A_j \omega) \right|^2$ and $\sum_{j \in \mathbf{Z}} \left| \widehat{\psi}(A_j \omega) \right|^2$. For fixed $k \in \mathbf{Z}$, we define f_2, g_2 as follows:

$$\widehat{f}_2(\omega + \alpha_0) = \frac{1}{\mu(H_k)^{1/2}} \chi_{\omega_0 + H_k}(\omega), \quad \widehat{g}_2(\omega) = \frac{1}{\mu(H_k)^{1/2}} \chi_{\omega_0 + H_k}(\omega).$$

Then, using Lebesgue Theorem, we have

$$\lim_{k \rightarrow \infty} R_1(f_2, g_2) = \frac{1}{|\det B|} \sum_{(j, m) \in I(\alpha_0)} \widehat{\psi}(A_j \omega_0) \overline{\widehat{\psi}(A_j(\omega_0 + \alpha_0))}.$$

To estimate $R_2(f_2, g_2)$, we note that if $\widehat{g}_2(\omega) \widehat{f}_2(\omega + \alpha) \neq 0$, then

$$\alpha \in \alpha_0 + A_k^{-1}(\Delta - \Delta).$$

Since $\alpha = A_j^{-1}m \in \wedge \setminus \{0, \alpha_0\}$, it follows from (2), (3) that there exist $J_0 \in \mathbf{Z}$ such that for any $j \leq J_0, m \in B^{-*}(\mathbf{Z}^n) \setminus \{0\}$, $A_j^{-1}m \notin \alpha_0 + D_k$, where

$$D_k = A_k^{-1}(\Delta - \Delta).$$

Hence $R_2(f_2, g_2)$ can be rearranged as

$$\begin{aligned} & R_2(f_2, g_2) \\ &= \frac{1}{|\det B|} \sum_{j \geq J_1} \sum_{m \in (A_j \alpha_0 + A_j D_k) \setminus \{0\}} \int_{\mathbf{R}^n} \overline{\widehat{g}_2(\omega)} \widehat{f}_2(\omega + \alpha) \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + \alpha))} d\omega \\ &+ \frac{1}{|\det B|} \sum_{j=J_0}^{J_1} \sum_{m \in (A_j \alpha_0 + A_j D_k) \setminus \{0\}} \int_{\mathbf{R}^n} \overline{\widehat{g}_2(\omega)} \widehat{f}_2(\omega + \alpha) \widehat{\psi}(A_j \omega) \overline{\widehat{\psi}(A_j(\omega + \alpha))} d\omega \\ &= R_{2,1}(f_2, g_2) + R_{2,2}(f_2, g_2), \end{aligned}$$

where $J_1 \in \mathbf{Z}$. Using (2) and (3), when k is large enough, for each j ($J_0 \leq j \leq J_1$) the number of m satisfying $m \in (A_j \alpha_0 + A_j D_k)$ is a constant which is not related on k . Thus, by Lebesgue theorem, we have $\lim_{k \rightarrow \infty} R_{2,2}(f_2, g_2) = 0$.

To estimate $R_{2,1}(f_2, g_2)$, we would like to prove that for each $\varepsilon > 0$ and k which is large enough, there exists $J_1 \in \mathbf{Z}$ such that $R_{2,1}(f_2, g_2) \leq \varepsilon$.

In fact, similar to $R(f_1, f_1)$, we have

$$\begin{aligned} R_{2,1}(f_2, g_2) &\leq \frac{1}{|\det B| |H_k|} \left(\sum_{j \geq J_1} \sum_{m \in (A_j \alpha_0 + A_j D_k) \setminus \{0\}} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \right)^{1/2} \\ &\times \left(\sum_{j \geq J_1} \sum_{m \in (A_j \alpha_0 + A_j D_k) \setminus \{0\}} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \right)^{1/2}. \end{aligned}$$

Therefore it is enough to estimate just one of these factors.

In fact, fix any k which is large enough. Using the conditions (2) and (3), there exists a constant C such that for any $j \geq J_1$,

$$\# \left((A_j \alpha_0 + A_j D_k) \cap B^{-T}(\mathbf{Z}^n) \right) \leq 1 + C |\det A_j| |\det A_k|^{-1},$$

where $\#(\cdot)$ is the number of elements in a given set. Then

$$\begin{aligned} & \frac{1}{|H_k|} \sum_{j \geq J_1} \sum_{m \in (A_j \alpha_0 + A_j D_k) \setminus \{0\}} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \\ &\leq \frac{1}{|H_k|} \sum_{j \geq J_1} (1 + C |\det A_j| |\det A_k|^{-1}) \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega \\ &= \sum_{j \geq J_1} \frac{1}{|H_k|} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega + \sum_{j \geq J_1} \int_{A_j(\omega_0 + H_k)} \left| \widehat{\psi}(\omega) \right|^2 d\omega. \end{aligned}$$

Let $J_1 \in \mathbf{Z}$ such that $\sum_{j \geq J_1} \left| \widehat{\psi}(A_j \omega_0) \right|^2 < \varepsilon$. Then

$$\lim_{k \rightarrow \infty} \sum_{j \geq J_1} \frac{1}{\mu(H_k)} \int_{\omega_0 + H_k} \left| \widehat{\psi}(A_j \omega) \right|^2 d\omega < \varepsilon/2.$$

By (P_1) , (P_2) and Lebesgue theorem, we have

$$\lim_{k \rightarrow \infty} \sum_{j \geq J_1} \int_{A_j(\omega_0 + H_k)} |\widehat{\psi}(\omega)|^2 d\omega = 0.$$

Thus for any $\varepsilon > 0$, there exists J_1 such that $\lim_{k \rightarrow \infty} |R_{2,1}(f_2, g_2)| \leq \varepsilon$. Finally, we obtain

$$\frac{1}{|\det B|} \sum_{(j,m) \in I(\alpha_0)} \widehat{\psi}(A_j \omega_0) \overline{\widehat{\psi}(A_j(\omega_0 + \alpha_0))} = 0, \text{ for any } \alpha_0 \in \Lambda \setminus \{0\}.$$

We complete the proof of Theorem 2.

3 Remarks

In the proof of Theorem 2, we have used the following result in some places.

Remark 3.1. If (2) and (3) holds, then there exist constants C , $q \in \mathbf{Z}_+$ and $\beta \in (0, 1)$ such that

- (i) for any $p, j \in \mathbf{Z}_+$, $j \geq pq$ and $\xi \in \mathbf{R}^n$, $|A_j \xi| \geq C(\frac{1}{\beta})^p |\xi|$.
- (ii) for any $p, -j \in \mathbf{Z}_+$, $j \leq -pq$ and $\xi \in \mathbf{R}^n$, $|A_j \xi| \leq C\beta^p |\xi|$.
- (iii) for any $j, k \in \mathbf{Z}$, $j < k$ and $\xi \in \mathbf{R}^n$, we have $|A_k \xi| \geq C(\frac{1}{\beta})^{\frac{k-j}{q}} |A_j \xi|$.

Remark 3.2. Assume that A is an expand matrix, $S = \Gamma \setminus \bigcup_{j=-\infty}^{-1} (A^j \Gamma)$, where Γ is a bounded measurable subset in $\mathbf{R}^n$, $\sup_{x,y \in \Gamma} \|x - y\| < 1$ and the origin is an interior point of Γ . Let $\widehat{\psi} = \chi_S$. By Theorem 4, ψ is a tight frame function with respect to dilation sequence $\{A^j\}_{j \in \mathbf{Z}}$. If $D_i (i = 1, \dots, p)$ are nonsingular matrices which is commutative with A , and $A_{sp+i} = A^{s+1} D_i$ for any $s \in \mathbf{Z}$, $i = 1, \dots, p$, then using Theorem 4 again, it is easy to show that ψ is also a tight frame function with respect to dilation sequence $\{A_j\}_{j \in \mathbf{Z}}$.

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