A Further Characterization on the Sampling Theorem for Wavelet Subspaces^{*}

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Abstract. Sampling theory is one of the most powerful results in signal analysis. The objective of sampling is to reconstruct a signal from its samples. Walter extended the Shannon sampling theorem to wavelet subspaces. In this paper we give a further characterization on some shift-invariant subspaces, especially the closed subspaces on which the sampling theorem holds. For some shift-invariant subspaces with sampling property, the sampling functions are explicitly given.

Keywords: Sampling theorem, Wavelet subspace, Wavelet frame, Shift-invariant subspace.

1 Introduction and Main Results

Sampling theory is one of the most powerful results in signal analysis. The objective of sampling is to reconstruct a signal from its samples. For example, the classical Shannon theorem says that for each

$$f \in B_{\frac{1}{2}} = \{f \in L^2(\mathbf{R}) : \text{supp } \hat{f} \subset [-\frac{1}{2}, \frac{1}{2}]\},$$

$f(x) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)}$, where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$, and the Fourier transform is defined by $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx$. If $\psi(x) = \frac{\sin \pi x}{\pi x}$, then $\{\psi(\cdot - k)\}_{k \in \mathbf{Z}}$ is an orthonormal basis for $B_{\frac{1}{2}}$ and $B_{\frac{1}{2}}$ is a shift-invariant subspace in $L^2(\mathbf{R})$.

The structure of finitely generated shift-invariant subspaces in $L^2(\mathbf{R})$ is studied, e.g., in [1]-[6]. There are fruitful results in wavelet theory in the past 20 years (see [7]-[12]). In [2], Janssen considered the shifted sampling and corresponding aliasing error by means of Zak transform. Walter [1] extended the Shannon sampling theorem to wavelet subspaces. Zhou and Sun [13] characterized the general shifted wavelet subspaces on which the sampling theorem holds:

^{*} This work was supported by the National Natural Science Foundation of China (Grant No. 60572113).

Proposition 1 ([13]). Let V_0 be a closed subspace in $L^2(\mathbf{R})$ and $\{\phi(\cdot - n)\}_{n \in \mathbf{Z}}$ is a frame for V_0 . Then the following two assertions are equivalent:

(i) $\sum_k c_k \phi(x - k)$ converges pointwise to a continuous function for any $\{c_k\} \in l^2$ and there exists a frame $\{S(\cdot - n)\}_{n \in \mathbf{Z}}$ for V_0 such that

$$f(x) = \sum_k f(k)S(x - k) \text{ for any } f \in V_0,$$

where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$.

(ii) $\phi \in C(\mathbf{R})$, $\sup_{x \in \mathbf{R}} \sum_k |\phi(x - k)|^2 < \infty$ and there exist two positive constants A, B such that

$$A\chi_{E_\phi}(\omega) \leq \left| \sum_k \phi(k)e^{-ik\omega} \right| \leq B\chi_{E_\phi}(\omega), \text{ a.e.,}$$

where χ is the characteristic function, $E_\phi = \{\omega \in \mathbf{R} : \sum_k |\widehat{\phi}(\omega + 2k\pi)|^2 > 0\}$.

Moreover, it implies that for S in (i), $f(k) = \langle f, \tilde{S}(\cdot - n) \rangle$ for any $f \in V_0$, where $\tilde{S}$ is defined in the following Proposition 3.

In this paper we characterize some shift-invariant subspaces, especially the closed subspaces in $L^2(\mathbf{R})$ such that the sampling theorem holds.

Notations. Firstly, we discuss functions in $L^2(\mathbf{R})$. Therefore $f = g$ means that $f(\omega) = g(\omega)$ for almost everywhere $\omega \in \mathbf{R}$. $C(\mathbf{R})$ is the space of continuous function. $G_f(\omega) = \sum_k |\widehat{f}(\omega + k)|^2$. It is easy to see that G_f is defined only a.e.. $E_f = \{\omega \in \mathbf{R} : G_f(\omega) > 0\}$ for any $f \in L^2(\mathbf{R})$. χ_E is the characteristic function of the set E . $\widehat{f^*}(\omega) = \sum_n f(n)e^{-i2\pi n\omega}$ for any $f \in L^2(\mathbf{R})$ with $\sum_n |f(n)|^2 < \infty$. $V_f^0 = \{g : g(\cdot) = \sum_n c_n f(\cdot - n), \text{ where the convergence is in } L^2(\mathbf{R}) \text{ and } \{c_n\}_{n \in \mathbf{Z}} \in l^2\}$. $V_f = \overline{\{f(\cdot - n)\}_n}$ which means that any $g \in V_f$ can be approximated arbitrarily well in norm by a finite linear combinations of vectors $f(\cdot - n)$. Moreover, V_f is called a shift-invariant subspace generated by f . Let $\mu(\cdot)$ be the Lebesgue measure on $\mathbf{R}$. For $E, F \subset \mathbf{R}$, $E \stackrel{a.e.}{=} F$ means that $\mu(E \setminus F) = \mu(F \setminus E) = 0$.

For $\phi \in L^2(\mathbf{R})$, if $\{\phi(\cdot - n)\}_n$ is a frame (Riesz basis) for V_ϕ then ϕ is called a frame (Riesz) function.

Definition 1. A closed subspace V in $L^2(\mathbf{R})$ is called a sampling space, if there exists a frame $\{\psi(\cdot - k)\}_{k \in \mathbf{Z}}$ for V such that $\sum_k c_k \psi(x - k)$ converges pointwisely to a continuous function for any $\{c_k\} \in l^2$ and

$$f(x) = \sum_k f(k)\psi(x - k) \text{ for any } f \in V,$$

where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$.

In this case, ψ is called a sampling function on V .

From the definition, we know that if V is a sampling space then for any $f \in V$ there exists a function $g \in C(\mathbf{R})$ such that $f(x) = g(x)$, *a.e.* $x \in \mathbf{R}$. Therefore, in what follows we assume that all the functions in a sampling space are continuous.

Next we denote the following three sets to characterize the functions with sampling property.

$\mathcal{P}_1$ is the subset in $L^2(\mathbf{R}) \cap C(\mathbf{R})$ in which each function ϕ satisfies that $\sum_k c_k \phi(x - k)$ converges pointwisely to a continuous function for any $\{c_k\} \in l^2$.

$\mathcal{P}_2$ is the set of functions in which each function f satisfies the following three conditions:

- (i) $f \neq 0, f \in L^2(\mathbf{R}) \cap C(\mathbf{R}), \sum_k |f(k)|^2 < \infty$ and f is a bounded function;
- (ii) $\mu(\{\omega : \widehat{f^*}(\omega) = 0, G_f(\omega) \neq 0\}) = 0$;
- (iii) There exist two positive constants A, B such that

$$A \leq \frac{G_f(\omega)}{|\widehat{f^*}(\omega)|^2} \leq B, \text{ a.e. } \omega \in E_f.$$

$$\mathcal{P}_3 = \{\phi \in \mathcal{P}_2 : \sup_{x \in \mathbf{R}} \sum_n |\phi(x - n)|^2 < \infty\}.$$

For any $f \in \mathcal{P}_2$, it is easy to see that $\mu(E_f) > 0$ and $\widehat{f^*}(\omega), G_f(\omega)$ are well defined almost everywhere.

Next we list the following propositions to be used in the proof of our results.

Proposition 2 ([3],[5]). Let $\phi \in L^2(\mathbf{R})$. ϕ is a frame function if and only if there are constants $A, B > 0$ such that $A\chi_{E_\phi} \leq G_\phi \leq B\chi_{E_\phi}$, *a.e.*

Epecially, if ϕ is a Riesz function, then $E_\phi = \mathbf{R}$.

Proposition 3 ([3],[13]). Assume that $f \in L^2(\mathbf{R})$ and f is a frame function. Let

$$\widehat{\widetilde{f}}(\omega) = \begin{cases} \frac{\widehat{f}(\omega)}{G_f(\omega)}, & \text{if } \omega \in E_f \\ 0, & \text{if } \omega \notin E_f \end{cases}$$

then $\{\widehat{\widetilde{f}}(\cdot - n)\}_n$ is a dual frame of $\{f(\cdot - n)\}_n$ in V_f .

Proposition 4 ([13]). Let $\phi \in L^2(\mathbf{R})$. Then $\phi \in \mathcal{P}$ if and only if the following holds

- (i) $\phi \in C(\mathbf{R})$,
- (ii) $\sum_{k \in \mathbf{Z}} |\phi(x - k)|^2 \leq M$ for some constant $M > 0$.

Proposition 5 ([5]). $V_f = \{g \in L^2(\mathbf{R}) : \widehat{g} = \tau \widehat{f}, \text{ where } \tau \text{ is a function with period 1 and } \tau \widehat{f} \in L^2(\mathbf{R})\}$.

Proposition 6 ([5]). Let $g \in V_f$. Then $V_g = V_f$ if and only if $\text{supp } \widehat{f} \stackrel{a.e.}{=} \text{supp } \widehat{g}$.

Now we give the following theorem to check whether V_f is the maximum shift-invariant subspace in $L^2(\mathbf{R})$.

Theorem 7. Let $f \in L^2(\mathbf{R})$. There exists $g \in L^2(\mathbf{R})$ such that $f \in V_g$ and $g \notin V_f$ if and only if there exists $E \subseteq [0, 1]$ with $\mu(E) > 0$ such that $\widehat{f}(\omega) = 0$ for any $\omega \in \bigcup_{k \in \mathbf{Z}}(k + E)$.

For a shift-invariant subspace V_f with sampling property, how can we find its sampling functions? For this, we give the following two theorems. Firstly, for any $f \in L^2(\mathbf{R})$ if $\widehat{f^*}$ is well defined, then we define

$$\widehat{f_p}(\omega) = \begin{cases} \frac{\widehat{f}(\omega)}{\widehat{f^*}(\omega)}, & \text{if } \widehat{f^*}(\omega) \neq 0, \\ 0, & \text{if } \widehat{f^*}(\omega) = 0. \end{cases} \quad (1)$$

Theorem 8. If $f \in \mathcal{P}_2$, then $\{f_p(\cdot - n)\}_n$ is a frame for V_f .

Theorem 9. (i) Let $\widehat{f^*}(\omega) \neq 0$, a.e. $\omega \in \mathbf{R}$. If $S \in L^2(\mathbf{R})$ such that

$$f(x) = \sum_k f(k)S(x - k),$$

where the convergence is in $L^2(\mathbf{R})$, then $S = f_p$.

(ii) For any $f \in \mathcal{P}_3$, let

$$\widehat{S}(\omega) = \begin{cases} \frac{\widehat{f}(\omega)}{\widehat{f^*}(\omega)}, & \text{if } \widehat{f^*}(\omega) \neq 0, \\ R(\omega), & \text{if } \widehat{f^*}(\omega) = 0, \end{cases}$$

where $R \in L^2(\mathbf{R})$. Then for any $g \in V_f^0$, $g(x) = \sum_n g(n)S(x - n)$.

Finally, we give the following equivalent characterization on shift-invariant subspaces with sampling property.

Theorem 10. Assume that V is a shift-invariant subspace. Then the following assertions are equivalent:

(i) V is a sampling space.

(ii) For any $\phi \in C(\mathbf{R})$ such that $\{\phi(\cdot - n)\}_n$ is a frame for V , we have

$$\sup_x \sum_n |\phi(x - n)|^2 < \infty, \quad (2)$$

and there exist positive constants A, B such that

$$A\chi_{E_\phi}(\omega) \leq \left| \widehat{\phi^*}(\omega) \right| \leq B\chi_{E_\phi}(\omega), \text{ a.e.} \quad (3)$$

(iii) There exists $\phi \in C(\mathbf{R})$ such that $\{\phi(\cdot - n)\}_n$ is a frame for V and (2), (3) hold.

(iv) There exists $\phi \in C(\mathbf{R})$ such that $\{\phi(\cdot - n)\}_n$ is a frame for V , (2) holds and

$$A\|g\|^2 \leq \sum_n |g(n)|^2 \leq B\|g\|^2, \forall g \in V. \quad (4)$$

for some positive constants A, B .

(v) There exists $\phi \in C(\mathbf{R})$ such that $\{\phi(\cdot - n)\}_n$ is a frame for V , (2) holds and $\{\sum_l \overline{\phi(k - l)}\widehat{\phi}(k - \cdot)\}_k$ is a frame for V .

2 Proof of Main Results

Proof of Theorem 7. *[necessary]* Assume that $g \in L^2(\mathbf{R})$, $f \in V_g$ and $g \notin V_f$. By Proposition 5 and Proposition 6, $\hat{f} = \tau \hat{g}$ where τ is a function with period 1 and $\mu(\text{supp} \hat{g} \setminus \text{supp} \hat{f}) > 0$. Then there exists $E^* \in \mathbf{R}$ with $\mu(E^*) > 0$ such that $\hat{g}(\omega) \neq 0, \hat{f}(\omega) = 0$ for any $\omega \in E^*$. Thus $\tau(\omega) = 0$ for any $\omega \in E^*$. Let

$$T(E^*) = \{\omega : \omega \in [0, 1], \text{ there exists } k \in \mathbf{Z} \text{ such that } \omega + k \in E^*\}.$$

Then $\tau(\omega) = 0$ for any $\omega \in T(E^*)$. Since $E^* \subseteq \bigcup_{k \in \mathbf{Z}} (k + T(E^*))$ and $\mu(E^*) > 0$, we have $\mu(T(E^*)) > 0$. Therefore, if we take $E = T(E^*)$, then E satisfies the conditions in the theorem.

[sufficiency] Assume that $f \in L^2(\mathbf{R})$ and there exists $E \subseteq [0, 1]$ with $\mu(E) > 0$ such that $\hat{f}(\omega) = 0$ for any $\omega \in \bigcup_{k \in \mathbf{Z}} k + E$. Let

$$\hat{g}(\omega) = \begin{cases} \hat{f}(\omega), & \text{if } \omega \notin E, \\ 1, & \text{if } \omega \in E. \end{cases}$$

Then $\hat{f}(\omega) = \tau(\omega) \hat{g}(\omega)$ for any $\omega \in \mathbf{R}$, where

$$\tau(\omega) = \begin{cases} 1, & \text{if } \omega \notin \bigcup_{k \in \mathbf{Z}} (k + E), \\ 0, & \text{if } \omega \in \bigcup_{k \in \mathbf{Z}} (k + E) \end{cases}$$

is a function with period 1 and $\mu(\text{supp} \hat{g} \setminus \text{supp} \hat{f}) > 0$. Thus we have $f \in V_g$ and $g \notin V_f$.

This completes the proof of Theorem 7.

Proof of Theorem 8. By (1) and $f \in \mathcal{P}_2$, we have $\hat{f}_p(\omega) = \tau(\omega) \hat{f}(\omega)$, where

$$\tau(\omega) = \begin{cases} \frac{1}{\hat{f}^*(\omega)}, & \text{if } \omega \in E_f \\ 0, & \text{if } \omega \notin E_f. \end{cases}$$

Then

$$G_{f_p}(\omega) = \begin{cases} \frac{G_f(\omega)}{|\hat{f}^*(\omega)|^2}, & \text{if } \omega \in E_f \\ 0, & \text{if } \omega \notin E_f. \end{cases}$$

Since $f \in \mathcal{P}_2$, there exist positive constants A, B such that $A\chi_{E_f} \leq G_{f_p} \leq B\chi_{E_f}$. However, τ is a function with period 1. By Proposition 5, $f_p \in V_f$. It is from $\text{supp} f = \text{supp} f_p$ and Proposition 6 that $V_f = V_{f_p}$. Finally, by Proposition 2, f_p is a frame for V_f .

This completes the proof of Theorem 8.

Proof of Theorem 9. *[(i)]* Since $f(x) = \sum_k f(k)S(x - k)$ and $\hat{f}^*(\omega) \neq 0$, we have $\hat{f}(\omega) = \hat{f}^*(\omega) \hat{S}(\omega)$, $\hat{S}(\omega) = \frac{\hat{f}(\omega)}{\hat{f}^*(\omega)}$. Thus $S = f_p$.

[(ii)] For any $g \in V_f^0$, there exist $\{c_k\} \in l^2$ such that $g(\cdot) = \sum_k c_k f(\cdot - k)$, where the convergence is pointwise and $g \in C(\mathbf{R})$. Let $C(\omega) = \sum_k c_k e^{-i2\pi k\omega}$,

$C^*(\omega) = C(\omega)\chi_{E_f}(\omega)$. Then there exists $\{c_k^*\} \in l^2$ such that $C^*(\omega) = \sum_k c_k^* e^{-i2\pi k\omega}$. It follows from

$$\widehat{g}(\omega) = C(\omega)\widehat{f}(\omega) = C^*(\omega)\widehat{f}(\omega)$$

and Proposition 4 that $g(x) = \sum_k c_k^* f(x - k)$ converges pointwisely in $\mathbf{R}$. By Theorem 8, f_p is a frame for V_f . Then using Proposition 3, $\widetilde{f}_p$ is a dual frame of $\{f(\cdot - n)\}_{n \in \mathbf{Z}}$ in V_f . Thus

$$\begin{aligned} (g, \widetilde{f}_p(\cdot - n)) &= \int_{E_f} C^*(\omega) \frac{\widehat{f}^*(\omega)}{G_f(\omega)} \left| \widehat{f}(\omega) \right|^2 e^{i2\pi n\omega} d\omega = \int_{-1/2}^{1/2} C^*(\omega) \widehat{f}^*(\omega) e^{i2\pi n\omega} d\omega \\ &= \sum_k c_k^* f(n - k) = g(n). \end{aligned}$$

Thus we get $g(x) = \sum_n g(n) f_p(x - n)$. Let $\widehat{h}(\omega) = R(\omega) \chi_{\{\omega: \widehat{f}^*(\omega)=0\}}$. Then

$$\widehat{S}(\omega) = \widehat{f}_p(\omega) + \widehat{h}(\omega), S = f_p + h.$$

Since $g(n) = \sum_k c_k^* f(n - k)$ for $n \in \mathbf{Z}$, we have $\widehat{g}^*(\omega) = \sum_k c_k^* e^{-i2\pi k\omega} \widehat{f}^*(\omega)$. Then

$$\widehat{g}^*(\omega) \widehat{h}(\omega) = 0, \sum_n g(n) h(x - n) = 0.$$

Therefore

$$g(x) = \sum_n g(n) (f_p(x - n) + h(x - n)) = \sum_n g(n) S(x - n).$$

This completes the proof of Theorem 9.

For the proof of Theorem 10, we give the following Lemma.

Lemma 1. Assume that $\phi \in \mathcal{P}_1$ is a frame function. If $f \in V_\phi$ satisfies $\widehat{f}(\omega) = b(\omega) \widehat{\phi}(\omega)$, where $b(\omega)$ is a function with period 1 and bounded on E_ϕ , then $f \in \mathcal{P}_1$.

Specially, for any frame function $\psi \in V_\phi$, we have $\psi \in \mathcal{P}_1$.

Proof. Assume that $f \in V_\phi$ satisfies $\widehat{f}(\omega) = b(\omega) \widehat{\phi}(\omega)$, where $b(\omega)$ is a function with period 1 and bounded on E_ϕ . Let $B(\omega) = b(\omega) \chi_{E_\phi}(\omega)$. Since $B(\omega)$ is bounded on $[-\frac{1}{2}, \frac{1}{2}]$, there exists $\{B_n\} \in l^2$ such that $B(\omega) = \sum_n B_n e^{-i2\pi n\omega}$. Since $\phi \in \mathcal{P}_1$ and

$$\widehat{f}(\omega) = b(\omega) \widehat{\phi}(\omega) = B(\omega) \widehat{\phi}(\omega),$$

we have $f \in C(\mathbf{R})$ and $f(x) = \sum_n B_n \phi(x - n)$, where the convergence is both in $L^2(\mathbf{R})$ and pointwisely.

Now using $\phi \in \mathcal{P}_1$ and Proposition 4, we have $\sup_{x \in \mathbf{R}} \sum_n |\phi(x - n)|^2 < \infty$. Hence

$$\sup_x \sum_k |f(x - k)|^2 = \sup_x \sum_k \left| \sum_n B_n \phi(x - k - n) \right|^2$$

$$\begin{aligned}
 &= \sup_x \int_{-1/2}^{1/2} |B(\omega)|^2 \left| \sum_k \phi(x-k) e^{i2\pi k\omega} \right|^2 d\omega \\
 &\leq \sup_x \|B(\omega)\|_\infty^2 \sum_k |\phi(x-k)|^2 < \infty.
 \end{aligned} \tag{5}$$

Then by $f \in C(\mathbf{R})$, (5) and Proposition 4, we get $f \in \mathcal{P}_1$.

For any frame function $\psi \in V_\phi$, there exists a function τ with period 1 such that $\hat{\psi}(\omega) = \tau(\omega)\hat{\phi}(\omega)$, then $G_\psi = |\tau(\omega)|^2 G_\phi(\omega)$. By Proposition 2, τ is bounded on E_ϕ . Thus $\psi \in \mathcal{P}_1$.

This completes the proof of Lemma 1.

Proof of Theorem of 10. (i) $\Rightarrow$ (ii) Given any continuous function ϕ such that $\{\phi(\cdot - n)\}_n$ is a frame for sampling space V . Let ψ be a sampling function for V . By Proposition 4, $\sup_{x \in \mathbf{R}} \sum_n |\psi(x-n)|^2 < \infty$.

Let $\hat{\phi}(\omega) = b(\omega)\hat{\psi}(\omega)$, where $b(\omega) = \sum_k b_k e^{-i2\pi k\omega}$ for some $\{b_k\} \in l^2$. Then $G_\phi(\omega) = |b(\omega)|^2 G_\psi(\omega)$. By Proposition 2, $b(\omega)$ is bounded on E_ψ . By Lemma 1, $\phi \in \mathcal{P}_1$. Then by Proposition 1, we get (ii).

(ii) $\Rightarrow$ (iii) It is trivial.

(iii) $\Rightarrow$ (i) By Proposition 1, there exists a frame $\{\psi(\cdot - n)\}_n$ for V such that $f(x) = \sum_n f(n)\psi(x-n)$ for any $f \in V$. By Lemma 1 and Proposition 4, $\psi \in \mathcal{P}_1$. Thus we get (i).

(iv) $\Rightarrow$ (i) For any $n \in \mathbf{Z}$, $Tg = g(n)$ is a bounded linear functional on V . Then there exists $S_n \in V$ such that

$$g(n) = \langle g, S_n \rangle \text{ for any } n \in \mathbf{Z}, g \in V.$$

Let $S := S_0$ and $g_1(x) = g(x+1)$, for any $g \in V, x \in \mathbf{R}$. Then

$$\langle g_1, S \rangle = g_1(0) = g(1) = \langle g, S_1 \rangle.$$

Thus for any $g \in V$, we have

$$\begin{aligned}
 \int g(x) \overline{S(x-1)} dx &= \int g(x+1) \overline{S(x)} dx = \int g_1(x) \overline{S(x)} dx = g_1(0) = g(1) \\
 &= \langle g, S_1 \rangle = \int g(x) \overline{S_1(x)} dx.
 \end{aligned}$$

Thus $S_1(x) = S(x-1)$. Similarly, we have $S_n(x) = S(x-n)$ for any $n \in \mathbf{Z}, x \in \mathbf{R}$. Therefore

$$g(n) = \langle g, S(\cdot - n) \rangle \text{ for any } g \in V.$$

Now by (4), S is a frame for V . Let $\tilde{S}$ be defined as Proposition 3. Then by Lemma 1 and Proposition 4, $\tilde{S}$ is the sampling function for V . Therefore we get (i).

(i) $\Longleftrightarrow$ (iv) Note that in Proposition 1, $f(n) = \langle f, \tilde{S}(\cdot - n) \rangle$ for any $f \in V_0$, where $\{\tilde{S}(\cdot - n)\}_n$ is a frame for V_0 . Thus it is just from Proposition 1.

(iv) $\iff$ (v) Assume that there exists a continuous function ϕ which is a frame function for V such that $\sup_x \sum_n |\phi(x-n)|^2 < \infty$. Then by Lemma 1, we have $\tilde{\phi} \in \mathcal{P}_1$. Therefore $q_x(\cdot) := \sum_l \overline{\phi(x-l)} \tilde{\phi}(\cdot-l)$ convergences both in $L^2(\mathbf{R})$ and uniformly on $\mathbf{R}$.

Now for any $g \in V$,

$$\langle g, q_x \rangle = \sum_k \left\langle g, \tilde{\phi}(\cdot - k) \right\rangle \phi(x - k) \text{ for any } x \in \mathbf{R}. \quad (6)$$

The last series converges both in $L^2(\mathbf{R})$ to g and uniformly to $\langle g, q_x \rangle$. Thus

$$g(x) = \langle g, q_x \rangle \text{ for any } x \in \mathbf{R}. \quad (7)$$

If (iv) holds, then there exist two positive constants A, B such that

$$A \|g\|^2 \leq \sum_k |\langle g, q_k \rangle|^2 \leq B \|g\|^2 \text{ for any } g \in V.$$

Thus $\{q_k(\cdot)\}_k = \{\sum_l \overline{\phi(k-l)} \tilde{\phi}(k-\cdot)\}_k$ is a frame for V . We get (v).

If (v) holds, then $\{q_k(\cdot)\}_k$ is a frame for V . By (6) and (7), we get (iv).

This completes the proof of Theorem 10.

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