Wavelet Analysis of Bifurcation in a Competition Model

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Abstract. A nonlinear dynamical system which describes two interacting and competing populations (tumor and immune cells) is studied through the analysis of the wavelet coefficients. The wavelet coefficients (also called detail coefficients) are able to reproduce the behaviour of the function, and, being sensible to local changes, are strictly related to the differentiable properties of the function, which cannot be easily derived from the numerical interpolation. So the main features of the dynamical system will be given in terms of detail coefficients that are more adapted to the description of a nonlinear problem.

1 Introduction

In this paper we consider the nonlinear dynamical system

$$\begin{cases} \frac{dx}{d\tau} = \alpha_1 x - [1 - \gamma_0 e^{-2(1+\epsilon)\beta_4\tau}] xy , \\ \frac{dy}{d\tau} = -\beta_1 xy - \beta_2 y + \beta_3 x + \beta_4 - \beta_5 y x^2 , \end{cases} \quad (1)$$

which represents the competition between two cell populations [4,6].

The parameters are such that

$$\begin{cases} 0 < \alpha_1 , \\ |\beta_1| \leq 1 , 0 < \beta_2 , |\beta_3| \leq 1 , 0 \leq \beta_4 \leq 1 , \\ 0 \leq \gamma_0 \leq 1 , |\epsilon| < 1 . \end{cases}$$

In this system, which is a generalization of the Lotka-Volterra model, the unknown quantity $x(t)$ represents the numerical density of tumor cells, while $y(t)$ is the numerical density of lymphocyte population, under conditions $x(t) > 0$ and $y(t) > 0$ [6]. Moreover

1. α_1 is the rate of growth of the tumor population
2. β_1 is the aggressive rate of tumor cells
3. β_2 is the stimulatory effect of the tumor cells on immune cells
4. β_3, β_4 are, in the average, the immune system response
5. β_5 the tumor malignancy
6. ε represents the relative velocity of encounter rates of interacting populations.

Finally γ_0 is a parameter related to the ability of recognition of the competing population by the immune system. In particular ε and γ_0 are coupling the macroscopic with microscopic system [4]. Small values of γ_0 , according to [4], correspond to the maximum learning, i.e. full recognition of the competing population, whereas $\gamma_0 = 1$ correspond to the minimum learning; the competing population is not recognized by the immune system.

We investigate the dynamics of this system through the analysis of the wavelet coefficients which give the possibility to focus on singularities, local high frequencies variation, irregular structure and transient phenomena (see also [2]). A similar analysis was done in some previous papers [2,3] where we studied the Van Der Pol equation with and without damping (pointing out stable and stable solutions), where we observed that if the dynamical system is strongly nonlinear, the detail coefficients show significant jumps. Wavelet coefficients strongly depend on local changes, so that when the dynamical system becomes unstable (or chaotic) many effects appear [8,9]: the amplitude of the detail coefficients grows, the detail coefficients gather around some peaks, showing also some randomness distribution.

The aim of this paper is to investigate these featuring properties of wavelet coefficients for the above system of equations and to focus on what kind of wavelet coefficients give more precise information about the behavior of the studied dynamical system.

The paper is organized as follows: in Sect. 2 some preliminary definitions about Haar wavelets and short Haar wavelet transform [1] are given. The Lotka-Volterra model is introduced in Sect. 3, where we discuss the solutions of the non linear system through a wavelet analysis.

2 Short Haar Wavelet Transform

The *Haar scaling function* $\varphi(t)$ is the characteristic function on $[0, 1]$. By translation and dilation we get the family of functions defined in $[0, 1]$

$$\begin{cases} \varphi_k^n(t) \equiv 2^{n/2} \varphi(2^n t - k) , & (0 \leq n , 0 \leq k \leq 2^n - 1) , \\ \varphi(2^n t - k) = \begin{cases} 1 , t \in \Omega_k^n , \\ 0 , t \notin \Omega_k^n . \end{cases} & \Omega_k^n \equiv \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) , \end{cases} \quad (2)$$

The *Haar wavelet* family $\{\psi_k^n(t)\}$ is the orthonormal basis for the $L^2([0, 1])$ functions [5]:

$$\left\{ \begin{array}{l} \psi_k^n(t) \equiv 2^{n/2} \psi(2^n t - k) , \quad ||\psi_k^n(t)||_{L^2} = 1 , \\ \psi(2^n t - k) \equiv \begin{cases} -1 , & t \in \left[\frac{k}{2^n}, \frac{k+1/2}{2^n} \right) , \\ 1 , & t \in \left[\frac{k+1/2}{2^n}, \frac{k+1}{2^n} \right) , \\ 0 , & \text{elsewhere} . \end{cases} \quad (0 \leq n , 0 \leq k \leq 2^n - 1) , \end{array} \right. \quad (3)$$

Without loss of generality, we restrict ourselves to $0 \leq n , 0 \leq k \leq 2^n - 1 \implies \Omega_k^n \subseteq [0, 1]$. Let $\mathbf{Y} \equiv \{Y_i\}$, ($i = 0, \dots, 2^M - 1$, $2^M = N < \infty$, $M \in \mathbb{N}$), be a real and square summable time-series $\mathbf{Y} \in \mathbb{K}^N \subset \ell^2$ (where $\mathbb{K}$ is a real field); $t_i = i/(2^M - 1)$, is the regular equispaced grid of *dyadic points* on the interval restricted, for convenience and without restriction, to $\Omega = [0, 1]$.

Let the set $\mathbf{Y} = \{Y_i\}$ of N data be segmented into σ segments (in general) of different length. Each segment $\mathbf{Y}^s$, $s = 0, \dots, \sigma - 1$ is made of $p_s = 2^{m_s}$, ($\sum_s p_s = N$), data:

$$\mathbf{Y} = \{Y_i\}_{i=0, \dots, N-1} = \bigoplus_{s=0}^{\sigma-1} \{\mathbf{Y}^s\} , \quad \mathbf{Y}^s \equiv \{Y_{sp_s}, Y_{sp_s+1}, \dots, Y_{sp_s+p_s-1}\} ,$$

being, in general, $p_s \neq p_r$. The short discrete Haar wavelet transform of $\mathbf{Y}$ is (see [1]) $\mathbb{W}^{p_s, \sigma} \mathbf{Y}$, being explicitly ($2^{m_s} = p_s$, $\sum_{s=0}^{\sigma-1} p_s = N$)

$$\left\{ \begin{array}{l} \mathbb{W}^{p_s, \sigma} \equiv \bigoplus_{s=0}^{\sigma-1} \mathbb{W}_s^{p_s} , \quad \mathbf{Y} = \bigoplus_{s=0}^{\sigma-1} \mathbf{Y}^s , \\ \mathbb{W}^{p_s, \sigma} \mathbf{Y} = \left(\bigoplus_{s=0}^{\sigma-1} \mathbb{W}_s^{p_s} \right) \mathbf{Y} = \left(\bigoplus_{s=0}^{\sigma-1} \mathbb{W}_s^{p_s} \mathbf{Y}^s \right) , \\ \mathbb{W}^{2^{m_s}} \mathbf{Y}^s = \left\{ \alpha_0^{0(s)}, \beta_0^{0(s)}, \beta_0^{1(s)}, \beta_1^{1(s)}, \dots, \beta_{2^{m_s}-1}^{1(s)} \right\} . \end{array} \right.$$

Where the *discrete Haar wavelet transform* is the operator $\mathbb{W}^N : \mathbb{K}^N \subset \ell^2 \rightarrow \mathbb{K}^N \subset \ell^2$ which maps the vector $\mathbf{Y}$ into the vector of the *wavelet coefficients* $\{\alpha , \beta_k^n\}$:

$$\mathbb{W}^N \mathbf{Y} = \{\alpha, \beta_0^0, \dots, \beta_{2^M-1}^{M-1}\} , \quad \mathbf{Y} = \{Y_0, Y_1, \dots, Y_{N-1}\} . \quad (4)$$

There follows that, the matrix of the wavelet transform is expressed as a direct sum of lower order matrices so that the short transform is a sparse matrix [1]. We want to emphasize that when the short wavelet transform maps short interval values into a few set of wavelet coefficients, it can be considered as a first order approximation. Thus giving information about the linear behavior. However,

since the wavelet transform maps the original signal into uncorrelated sequences [7], the short wavelet transform describes for each sequence of detail coefficients its local behavior. When $p_s = p = N$, $\sigma = 1$, the above coincides with the ordinary wavelet transform. We assume, in the following, $p_s = p = N/\sigma$, $s = 0, \dots, \sigma - 1$, ($\sigma > 1$).

3 System of Competition

Let us consider the competition model (1); by fixing some parameters

$$\begin{cases} \frac{dx}{d\tau} = x - [1 - e^{-2(1+0.1)0.2\tau}] xy, \\ \frac{dy}{d\tau} = -0.1y + x + 0.2 - \beta_5 yx^2, \end{cases} \quad (5)$$

we obtain the numerical solution as in Fig. 1, where we take, as initial conditions, an initial high numerical density of tumor cells $x(0) = 5$ and we neglect the initial number of lymphocytes $y(0) = 0$. The other parameters are taken as $\gamma_0 = 1$, $\varepsilon = 0.1$ and, according to (5), we simulate a competition where the rate of growth of the tumor population is little ($\alpha_1 = 1$), the aggressive rate of tumor cells is neglectable ($\beta_1 = 0$), the stimulatory effect of the tumor cells on immune cells is weak ($\beta_2 = 0.1$), the immune system response is in the average $\beta_3 = 1, \beta_4 = 0.2$. Of course with higher values of β_4 we will have a stronger immune system (weaker for smaller values of β_4).

Fig. 1. Numerical solution of system (1) with parameters $\alpha_1 = 1$, $\beta_1 = 0$, $\beta_2 = 0.1$, $\beta_3 = 1$, $\beta_4 = 0.2$, $\gamma_0 = 1$, $\varepsilon = 0.1$, and initial conditions $x(0) = 5, y(0) = 0$, in correspondence of different values of β_5

Fig. 2. Numerical solution (plain $x(t)$, dashed $y(t)$) of system (1) with parameters $\alpha_1 = 1$, $\beta_1 = 0$, $\beta_2 = 0.1$, $\beta_3 = 1$, $\beta_4 = 0.2$, $\gamma_0 = 1$, $\varepsilon = 0.1$, and initial conditions $x(0) = 5$, $y(0) = 0$, in correspondence of different values of β_5

It can be observed (see Fig. 1 and Fig. 2) that with small values of the tumor malignancy β_5 , the lymphocyte population $y(t)$ grows, whereas tumor cells $x(t)$ decrease. When $\beta_5 > 0.05$ the number of lymphocytes reach a maximum value and then goes to zero while the tumor cell population grows. It should be noticed that $\beta_5 = 0.05$ represents a bifurcation point for the model with loss of uniqueness of the differential system, because the curve is not simply connected (for the presence of a knot). When the number of tumor cells tends to zero the number of lymphocytes decreases to a particular value, called the sentinel value. The dynamics of equation (5) has been simulated by using the Runge Kutta 4-th order method, with the accuracy 10^{-6} . We obtain as a numerical solution ($0 < t \leq 6$), in correspondence of the values of the parameter $\alpha_1 = 1$, $\beta_1 = 0$, $\beta_2 = 0.1$, $\beta_3 = 1$, $\beta_4 = 0.2$, $\gamma_0 = 1$, $\varepsilon = 0.1$, four sequences (in correspondence of $\beta_5 = 0.01$, $\beta_5 = 0.04$, $\beta_5 = 0.05$, $\beta_5 = 0.06$) of $2^9 = 512$ values $\mathbf{Y} = \{Y_0, Y_1, \dots, Y_{N-1}\}$, with $N = 512$ and $M = 9$. Moreover, using the short Haar wavelet transform, with $p_s = p = 4$, we compare the wavelet coefficients of the two time-series, near the bifurcation value of β_5 , i.e. $\beta_5 = 0.05$ and $\beta_5 = 0.06$ computed numerically (Fig. 3, 4).

4 Critical Analysis

The importance of the wavelet analysis is mainly based on the information content of the detail coefficients. It can be seen (from Fig. 3 and 4) that only a small set of detail coefficients, namely β_0^0 , β_0^1 , β_1^1 , is already enough to have a good information about the dynamical system, in any case better than the numerical evaluation (Fig. 3,4 on top). In fact, the detail coefficients show some local

Fig. 3. Numerical solution and wavelet coefficient of 4-parameters of short Haar transform of the numerical solution $x(t)$ (left) and $y(t)$ (right) of system (1) with parameters $\alpha_1 = 1$, $\beta_1 = 0$, $\beta_2 = 0.1$, $\beta_3 = 1$, $\beta_4 = 0.2$, $\gamma_0 = 1$, $\varepsilon = 0.1$, and initial conditions $x(0) = 5$, $y(0) = 0$, in correspondence of $\beta_5 = 0.05$

Fig. 4. Numerical solution and wavelet coefficient of 4-parameters of short Haar transform of the numerical solution $x(t)$ (left) and $y(t)$ (right) of system (1) with parameters $\alpha_1 = 1$, $\beta_1 = 0$, $\beta_2 = 0.1$, $\beta_3 = 1$, $\beta_4 = 0.2$, $\gamma_0 = 1$, $\varepsilon = 0.1$, and initial conditions $x(0) = 5$, $y(0) = 0$, in correspondence of $\beta_5 = 0.06$

maxima and changes which are hidden in the continuous interpolation of the numerical integration. Each detail coefficient is able, at each scale but mostly at the lower scale $n = 0$, β_0^0 , to reproduce the behaviour of the function, but they are very sensible to local changes and therefore they can easily describe the intervals where the function is monotonic. Moreover, being sensible to local changes they

are strictly related to the differentiable properties of the function, which cannot be easily derived from the numerical interpolation of the function. In particular, we can see that they can focus exactly on local maxima of the function, and on a number of changes higher than the numerical functions $x(t)$ e $y(t)$. Therefore the detail coefficients are more adapted to the description of a nonlinear problem but also show a inflexion in the initial growth which is invisible in the numerical solution. These time spots, where the the detail coefficients are zero, or where the detail coefficients have some local maxima (minima) are reported in Fig. 3,4, and tell us the inversion (inflexion) in the population growth. The positive values of the detail coefficients describe the local growth, the negative values the decreasing of the function. Local maxima (minima) of the detail coefficients define some inflexion which enable us to predict if the phenomenon will increase in time or decrease.

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