Geometry-Driven Nonlinear Equation with an Accelerating Coupled Scheme for Image Enhancement*

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Abstract. In this paper, a geometry-driven nonlinear shock-diffusion equation is presented for image denoising and edge sharpening. An image is divided into three-type different regions according to image features: edges, textures and details, and flat areas. For edges, a shock-type backward diffusion is performed in the gradient direction to the isophote line (edge), incorporating a forward diffusion in the isophote line direction; while for textures and details, a soft backward diffusion is done to enhance image features preserving a natural transition. Moreover, an isotropic diffusion is used to smooth flat areas simultaneously. Finally, a shock capturing scheme with a special limiter function is developed to speed the process with numerical stability. Experiments on real images show that this method produces better visual results of the enhanced images than some related equations.

1 Introduction

Main information of an image resides in such features as its edges, local details and textures. Image features are not only very important to the visual quality of the image, but also are significant to image post processing tasks, for example, image segmentation, image recognition and image comprehension, etc. Among image features, edges are the most general and important, which partition different objectives in an image. Because of some limitations of imaging process, however, edges may not be sharp in images. In addition to noise, both small intensity difference across edge and big edge width would result in a weak and blurry edge [1].

In the past decades there has been a growing amount of research concerning partial differential equations (PDEs) in image enhancement, such as anisotropic diffusion filters [2-5] for edge preserving noise removal, and shock filters [6-9] for edge sharpening. A great deal of successful applications of nonlinear evolving PDEs in “low level” image

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processing can mainly be attributed to their two basic characteristics: “to be local” and “to be iterative”. The word “differential” means that an algorithm is of local processing, while the word “evolving” means that it is of iterative one when numerically implemented.

One of most influential work in using partial differential equations (PDEs) in image processing is the anisotropic diffusion (AD) filter, which was proposed by P. Perona and J. Malik [3] for image denoising, enhancement, sharpening, etc. The scalar diffusivity is chosen as a non-increasing function to govern the behaviour of the diffusion process. Different from the nonlinear parabolic diffusion process, L. Alvarez and L. Mazorra [7] proposed an anisotropic diffusion with shock filter (ADSF) equation by adding a hyperbolic equation, called shock Filter which was introduced by S.J. Osher and L.I. Rudin [6], for noise elimination and edge sharpening.

In image enhancement and sharpening, it is crucial to preserve and even enhance image features when one removes image noise and sharpens edges at the same time. Therefore, image enhancement is composed of two steps: features detection and the processings by corresponding tactic according to different features.

In this paper, incorporating anisotropic diffusion with shock filter, we present a geometry-driven nonlinear shock-diffusion equation to remove image noise, and to sharpen edges by reducing their width simultaneously.

An image comprises regions with different features. Utilizing the techniques of differential geometry, we partition local structures and features of image into flat areas, edges, details such as corners, junctions and fine lines, and textures. These structures should be treated differently to obtain a better result in an image processing task. In our algorithm, for edges between different objects, a shock-type backward diffusion is performed in the gradient direction to the isophote line (edge), incorporating a forward diffusion in the isophote line direction. For textures and details, shock filters with the sign function enhance image features in a binary decision process, which produce unfortunately a false piecewise constant result. To overcome this drawback, we use a hyperbolic tangent function to control softly changes of gray levels of the image. As a result, a soft shock-type backward diffusion is introduced to enhance these features while preserving a natural transition in these areas. Finally, an isotropic diffusion is used to smooth flat areas simultaneously.

In order to solve effectively the nonlinear equation to obtain discontinuous solution with numerical stability, after we have discussed the difficulty of the numerical implementation to this type equation, a shock capturing scheme is developed with a special limiter function to speed the process.

This paper is organized as follows. In section 2, some related equations are introduced for enhancing images: anisotropic diffusions and shock filters. Then, we propose a geometry-driven shock-diffusion equation. In section 3, we implement the proposed method and test it on real images. Conclusions are presented in section 4

2 Geometry-Driven Shock-Diffusion Equation

2.1 Differentials of a Typical Ramp Edge and Edge Sharpening

We first analyze differential properties of a typical ramp edge. In Fig.1 (a), a denotes the profile of a ramp edge, whose center point is o , and, b and c denote its first and

second differential curves respectively. It is evident that b increases in value from 0 gradually, reaches its maximum at o , and then decreases to 0; while c changes its sign at o , from positive to negative in value. Here we control changes of gray level beside the edge center o . More precisely, we reduce gray levels of pixels on the left of o (whose second derivatives are positive), while increase those on the right of o (whose second derivatives are negative), by which the edge can be sharpened by reducing its width (see Fig.1 (b)). Shock-type diffusions later are just based on above analysis.

Fig. 1. Differentials of a typical ramp edge and edge Sharpening. (a) Differentials of 1D typical ramp edge a , with center o , and the first and second differentials b , c respectively; (b) Edge sharpening process (the solid line), compared with original edge (the broken line).

2.2 Local Differential Structure of Image

Consider image as a real function $u(x, y)$ on a 2D rectangular domain Ω , and image edge is isophote line (level set line), along which the image intensity is constant. Image gradient is a vector, $u_N = \nabla u = (u_x, u_y)$. If $\nabla u \neq 0$, then a local coordinates system can be defined at a point o :

$$\tilde{N} = \nabla u / |\nabla u|, \quad \tilde{T} = \nabla u^\perp / |\nabla u^\perp|$$

where $\nabla u^\perp = (-u_y, u_x)$. The Hessian matrix of image function $u(x, y)$ is :

$$Hu = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}$$

For two vectors X and Y , we define $Hu(X, Y) = X^T Hu Y$. Thus, we have the second directional derivatives in the directions $\tilde{N}$ and $\tilde{T}$:

$$u_{NN} = Hu(\tilde{N}, \tilde{N}) = (u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}) / |\nabla u|^2$$

$$u_{TT} = Hu(\tilde{T}, \tilde{T}) = (u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}) / |\nabla u|^2$$

By calculating, the curvature of isophote line at a point o is :

$$k = \text{div}(\tilde{N}) = u_{TT} / |\nabla u| = (u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}) / |\nabla u|^3$$

where div is a divergence operator.

2.3 The Geometry-Driven Shock-Diffusion Equation

An image comprises regions with different features, such as edges, textures and details, and flat areas, which should be treated differently to obtain a better result in an image

processing task. We divide an image into three-type regions by its smoothed gradient magnitude.

For edges between different objects, a shock-type backward diffusion is performed in the gradient direction, incorporating a forward diffusion in the isophote line. For textures and details, in equations (3) and (4), to enhance an image using the sign function $\text{sign}(\mathbf{x})$ is a binary decision process, which is a hard partition without middle transition. Unfortunately, the obtained result is a false piecewise constant image in some areas producing bad visual quality. We notice that the change of texture and detail is gradual in these areas. In order to approach this change, we use a hyperbolic tangent membership function $\text{th}(\mathbf{x})$ to guarantee a natural smooth transition in these areas, by controlling softly changes of gray levels of the image. As a result, a soft shock-type backward diffusion is introduced to enhance these features. Finally, an isotropic diffusion is used to smooth flat areas simultaneously.

Thus, incorporating shock filter with anisotropic diffusion, we develop a nonlinear geometry-driven shock-diffusion equation (GSE) process to reduce noise, and to sharpen edges while preserving image features simultaneously: Let $(x, y) \in \Omega \subset \mathbb{R}^2$, and $t \in [0, +\infty)$, a multi-scale image $u(x, y, t): \Omega \times [0, +\infty) \rightarrow \mathbb{R}$,

$$\begin{cases} u_G = G_\sigma * u \\ \frac{\partial u}{\partial t} = c_N u_{NN} + c_T u_{TT} - w(u_{NN}) \text{sign}((u_G)_{NN}) |u_N| \end{cases} \quad (1)$$

with Neumann boundary condition, where the parameters are chosen as follows according to different image regions:

	c_N	c_T	$w(u_{NN})$
$ (u_G)_N > T_1$	0	$1/(1 + l_1 u_{TT}^2)$	1
$T_2 < (u_G)_N \leq T_1$	0	$1/(1 + l_1 u_{TT}^2)$	$ \text{th}(l_2 u_{NN}) $
else	1	1	0

where G_σ is defined in previous section, c_N and c_T are the normal and tangent flow control coefficients respectively. The tangent flow control coefficient is used to prevent excess smoothness to smaller details; l_2 is a parameter to control the gradient of the membership function $\text{th}(\mathbf{x})$; T_1 and T_2 are two thresholds; l_1 and l_2 are constants. 3. Numerical Implementation and Experimental Results

3 Numerical Implementation and Experimental Results

3.1 A Shock Capturing Scheme

Nonlinear convection-diffusion evolution equation is a very important model in the fluid dynamics, which can be used to depict transmission processes of momentum, energy and mass of fluid. Because of its hyperbolic characteristic, the solution to the convection-diffusion equation often has discontinuity even if its initial condition is very smooth. Mathematically only weak solution can be obtained here. If a weak solution

satisfies the entropic increase principle for an adiabatic irreversible system, then it is called a shock wave.

When one solves numerically a convection-diffusion equation using a difference scheme, he may find some annoying problems in numerical simulation, such as instability, over smoothing, spurious oscillation or wave shift of a scheme. The reason for above is that, despite the original equation are deduced according to some physical conservation laws, its discrete equation may deviate from these laws, which can bring about numerical dissipation, numerical dispersion and group velocity of wave packets effects in numerical solutions specially for the hyperbolic term. Therefore, the hyperbolic term must be discretized carefully so that the flow of small scale and shock waves can be captured accurately.

Besides of satisfying consistence and stability, a good numerical scheme also need to capture shock waves. One method to capture shock waves is to add artificial viscosity term to the difference scheme for controlling and limiting numerical fluctuations near shock waves. But by this method it is inconvenient to adjust free parameters for different tasks, and the resolution of shocks can also be affected. Another method is to try to stop from numerical fluctuations before them appear, which is based on the TVD (Total Variation Diminishing) and nonlinear limiters. Their main idea is to use a limiter function to control the change of the numerical solution by a nonlinear way, and the corresponding schemes satisfy the TVD condition and eliminate above disadvantage effects, which guarantee of capturing shock waves with a high resolution.

In a word, when solving numerically a nonlinear convection-diffusion equation like (1) using a difference scheme, the hyperbolic term must be discretized carefully because discontinuity solutions, numerical instability and spurious oscillation may appear. Shock capturing methods with high resolution are effective tools. For more details, we refer the reader to the book [10]. Here, we develop a speeding scheme by using a proper limiter function.

An explicit Euler method with central difference scheme is used to approximate equation (1) except the gradient term $|u_N|$. Below we detail a numerical approach to it. On the image grid, the approximate solution is to satisfy:

$$u_{ij}^n \approx u(ih, jh, n\Delta t), \quad i, j, n \in Z^+ \quad (2)$$

where h and Δt are the spatial and temporal step respectively. Let $h = 1$, $\delta^+ u_{ij}^n$ and $\delta^- u_{ij}^n$ are forward and backward difference schemes of u_{ij}^n respectively. A limiter function MS is used to approximate the gradient term:

$$|u_N| = \sqrt{(MS(\delta_x^+ u_{ij}^n, \delta_x^- u_{ij}^n))^2 + (MS(\delta_y^+ u_{ij}^n, \delta_y^- u_{ij}^n))^2} \quad (3)$$

where

$$MS(x, y) = \begin{cases} x, & |x| < |y| \\ y, & |x| > |y| \\ x, & |x| = |y| \text{ and } xy > 0 \\ 0, & |x| = |y| \text{ and } xy \leq 0 \end{cases} \quad (4)$$

The MS function bears fewer 0 in value than the $minmod$ function does in the x - y plane, which also make the scheme satisfy the numerical stability. Because the gradient term represents the transport speed of the scheme, the MS function makes our scheme evolve faster with a bigger transport speed than those with the $minmod$ function.

In [8], other than above flux limitation technique, a fidelity term $(u - u_0)$ is used to carry out the stabilization task, and they also displayed that the SNRs of results tend towards 0 if $a_f=0$. However, this is not enough to eliminate overshoots, and this term also affect its performance.

3.2 The Coupled Iteration

Based on preceding discussion, when implementing iteratively equation (1), we find that the shock and diffusion forces will cancel mutually in a single formula. We split equation (1) into two formulas and propose the following coupled scheme by iterating with time steps:

$$\begin{cases} v^0 = u^0, & u_G = G_\sigma * u \\ v^{n+1} = u^n + \Delta t(-w(u_{NN}^n)\text{sign}((u_G^n)_{NN})|u_N|) \\ u^{n+1} = v^{n+1} + \Delta t(c_N v_{NN}^{n+1} + c_T v_{TT}^{n+1}) \end{cases} \quad (5)$$

where Δt is the time step, u^0 is an original image. By computing iteratively in the order of $u^0 \rightarrow v^0 \rightarrow v^1 \rightarrow u^1 \rightarrow v^2 \rightarrow u^2 \rightarrow \dots$, we obtain the enhanced image after some steps.

3.3 Experiments

We present results obtained by using our scheme (5), and compare its performance with those of above related methods, where the parameters are selected such that best results are obtained for all methods.

We compare performances of related methods on the blurred Cameraman image (Gaussian blur, $\sigma=2.5$) with added high level noise (SNR=14dB). In this case, weaker features are smeared by big noise in the image, which are difficult to be restored completely. As it can be seen, although the AD method denoises the image well specially in the smoother segments, it produces the blurry image with unsharp edges, whose ability to sharpen edges is limited, because of its poor sharpening process with the improper diffusion coefficient along the gradient direction. Moreover, with the diffusion coefficient in inverse proportion to the image gradient magnitude along the tangent direction, it does not diffuse fully in this direction and presents rough contours.

For the ADSF method, though it sharpens edges very well, in a binary decision process they yield the false piecewise constant images, which look unnatural with a discontinuous transition in the homogenous areas. Further, it cannot reduce noise well only by a single directional diffusion in the smoother regions.

The best visual quality is obtained by enhancing the image using GSE, which enhances most features of the image with a natural transition in the homogenous areas, and produces pleasing sharp edges and smooth contours while denoising the image effectively.

Finally, we discuss the performances of these methods in smoothing image contours on bigger gradients in the tangent direction of edges. As we explain above, image contours obtained by AD are not smooth with blurry edges in the gradient direction. The results obtained using ADSF and GSE respectively all present smooth contours in the tangent direction.

4 Conclusions

This paper deals with image enhancement for noisy blurry images. By reducing the width of edges, a geometry-driven nonlinear shock-diffusion equation is proposed to remove noise and to sharpen edges.

Our model performs a powerful process to noise blurry images, by which we not only can remove noise and sharpen edges effectively, but also can smooth image contours even in the presence of high level noise. Enhancing image features such as edges, textures and details with a natural transition in interior areas, this method produces better visual quality than some relative equations.

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