k -Colored Point-Set Embeddability of Outerplanar Graphs^{*}

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Abstract. This paper addresses the problem of designing drawing algorithms that receive as input a planar graph G , a partitioning of the vertices of G into k different semantic categories $V_0, \dots, V_{k-1}$, and k disjoint sets $S_0, \dots, S_{k-1}$ of points in the plane with $|V_i| = |S_i|$ ($i \in \{0, \dots, k-1\}$). The desired output is a planar drawing such that the vertices of V_i are mapped onto the points of S_i and such that the curve complexity of the edges (i.e. the number of bends along each edge) is kept small. Particular attention is devoted to outerplanar graphs, for which lower and upper bounds on the number of bends in the drawings are established.

1 Introduction

Semantic constraints for the vertices of a graph G define the placement that these vertices must have in a readable visualization of G [3,9,12]. For example, in the context of data base design some particularly relevant entities of an ER schema may be required to be represented in the center and/or along the boundary of the diagram (see, e.g., [13]). A possible way of modeling semantic constraints for a (sub)set $\{v_1, v_2, \dots, v_h\}$ of the vertices of a graph G is to specify a set $\{p_1, p_2, \dots, p_h\}$ of locations for their placement. Often, it is sufficient for the application that every vertex v_i ($i = 1, \dots, h$) is placed to any location p_j ($1 \leq j \leq h$), that is the mapping of each vertex to a specific location is not part of the input. A key reference in this scenario is the work by Kaufmann and Wiese [10]. Given a planar graph G with n vertices and a set S of n distinct points in the plane, they show how to compute a planar drawing of G such that each vertex is mapped to any point of S and every edge bends at most twice, which is proved to be worst case optimal. It is also known that for specific classes

^{*} Research partially supported by MIUR under Project “ALGO-NEXT(Algorithms for the Next Generation Internet and Web: Methodologies, Design and Experiments)”, and NSERC. Omitted or sketched proofs can be found in [4].

of graphs, such as outerplanar graphs and trees, the number of bends per edge can be reduced to zero (see, e.g., [1,2,7]). The work by Kaufmann and Wiese, however, does not seem to be immediately scalable to applications where the vertices of G are grouped based on their relevance or meaning and for each of these groups a different semantic constraint should be applied (for example a subset of the vertices on the boundary, some others in a central position, and so on). A solution in this case could be to fix in advance the location of each vertex in each semantic category and then route the edges. Halton [8] and, independently, Pach and Wenger [11], showed that a planar graph G always admits a planar drawing such that the location of each vertex is part of the input; however, fixing the vertex positions in advance may give rise to drawings with high visual complexity. Pach and Wenger [11] show that a linear number of bends per edge is asymptotically optimal in the worst case even for graphs as simple as paths.

This paper studies the above mentioned problem without imposing that the position of the vertices is part of the input. The input is a planar graph G , a partitioning of the vertices of G into k different semantic categories $V_0, \dots, V_{k-1}$, and k disjoint sets $S_0, \dots, S_{k-1}$ of points in the plane with $|V_i| = |S_i|$ ($i \in \{0, \dots, k-1\}$). The required output is a planar drawing such that the vertices of V_i are mapped to the points of S_i ; for each vertex $v \in V_i$ the drawing algorithm can choose which point of S_i represents v . We study the visual complexity of these types of drawings, expressed in terms of number of bends per edge. The intuition is that if the number of categories is constant, then a constant number of bends per edge may be sufficient, at least for simple classes of planar graphs. This type of investigation was started in [5,6], where the apparently simple case of $k = 2$ is studied. In [6] and in [5] a constant number of bends per edge is proved to be sufficient for constructing planar poly-line drawings of subclasses of outerplanar graphs, including paths, cycles, caterpillars, and wreaths. In [5] it is also shown that there exists a 2-outerplanar graph G with a vertex partition V_0, V_1 and two disjoint sets S_0, S_1 of points such that any planar drawing of G that maps a vertex $v \in V_i$ to a distinct point of S_i ($i = 0, 1$) has at least one edge with a linear number of bends. The 2-outerplanarity of the counterexample on one hand, and the outerplanarity of the families of graphs for which a constant number of bends per edge is possible, motivated us to further investigate how many bends are required for general outerplanar graphs and then extend the research to cases where $k > 2$.

In this paper, each integer $i \in \{0, \dots, k-1\}$ identifying a partition set of the vertices of G is called a *color*, G is called a *k -colored graph*, and the set of points $S = S_0 \cup \dots \cup S_{k-1}$ such that $|V_i| = |S_i|$ (for each each color $i \in \{0, \dots, k-1\}$) is called a *k -colored set compatible with G* . A planar drawing of G such that each $v \in V_i$ is drawn as a distinct point $p \in S_i$ is a *point-set embedding* of G on S . Graph G is *k -colored point-set embeddable* if it admits a point-set embedding on every k -colored set compatible with G . Our main results are as follows.

- Every outerplanar 2-colored graph is 2-colored point-set embeddable with at most 5 bends per edge. Also, a 2-colored embedding of this type can be computed in $O(n \log n)$ time. (See Section 3).

- For every positive integer $h > 0$, there exists an outerplanar 3-colored graph G , whose number of vertices depends on h , and a set of points S compatible with G such that every point-set embedding of G on S has an edge with more than h bends. (See Section 4).
- For k colors in which one restricts all the points of the point set with the same color to vertical regions separated by vertical lines, at most $4k + 1$ bends per edge are required for outerplanar graphs. The drawings can be computed in $O(n \log n + kn)$ time. (See Section 5).

2 Preliminaries

Let $G = (V, E)$ be a graph. A k -coloring of G is a partition $\{V_0, V_1, \dots, V_{k-1}\}$ of V where the integers $0, 1, \dots, k-1$ are called *colors*. For each vertex $v \in V_i$ ($0 \leq i \leq k-1$) we denote by $col(v)$ the color i of v . For any subset of vertices $U \subseteq V_i$ ($0 \leq i \leq k-1$) we denote by $col(U)$ the color i of all the elements of U . A k -coloring of G is *proper* if for every edge $(u, v) \in E$ we have $col(u) \neq col(v)$. A graph G with a (proper) k -coloring is called a (*proper*) k -colored graph. Let S be a set of distinct points in the plane. For any point $p \in S$ we denote by $x(p)$ and $y(p)$ the x - and y -coordinates of p , respectively. A k -coloring of S is a partition $\{S_0, S_1, \dots, S_{k-1}\}$ of S . A set of points S with a k -coloring is called a k -colored set. For each point $p \in S_i$ ($0 \leq i \leq k-1$), $col(p)$ denotes the color i of p , and for any subset $R \subseteq S_i$ ($0 \leq i \leq k-1$), $col(R)$ denotes the color i of all the elements of R . A k -colored set S is *compatible with a k -colored graph G* if $|V_i| = |S_i|$ for every $0 \leq i \leq k-1$; if G is planar we say that G has a *point-set embedding on S* if there exists a planar drawing of G such that: (i) every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$, (ii) each edge e of G is drawn as a polyline λ ; a point shared by any two consecutive segments of λ is called a *bend* of e . A k -colored sequence σ is a sequence of (possibly repeated) colors $c_0, c_1, \dots, c_{n-1}$ such that $0 \leq c_j \leq k-1$ ($0 \leq j \leq n-1$). We say that σ is *compatible with a k -colored graph G* if, for every $0 \leq i \leq k-1$, color i occurs $|V_i|$ times in σ . Let S be a k -colored set. Throughout the paper we always assume that the points of S have different x -coordinates (if not we can rotate the plane so to achieve this condition). Let $p_0, p_1, \dots, p_{n-1}$ be the points of S with $x(p_0) < x(p_1) < \dots < x(p_{n-1})$. The k -colored sequence $col(p_0), col(p_1), \dots, col(p_{n-1})$ is called the *k -colored sequence induced by S* , and is denoted as $seq(S)$.

A graph G is *Hamiltonian* if it has a simple cycle that contains all its vertices; such a cycle is called a *Hamiltonian cycle* of G . If G is a k -colored graph and $\sigma = c_0, \dots, c_{n-1}$ is a k -colored sequence compatible with G , a *k -colored Hamiltonian cycle of G consistent with σ* is a Hamiltonian cycle $v_0, v_1, \dots, v_{n-1}$ such that $col(v_i) = c_i$ ($0 \leq i \leq n-1$). If such a cycle exists, G is said to be *k -colored Hamiltonian consistent with σ* . Let G be a planar k -colored graph and let σ be a k -colored sequence compatible with G . It is always possible to augment G with dummy edges so that the resulting (not necessarily planar) graph has a k -colored Hamiltonian cycle $\mathcal{H}$ consistent with σ and including all dummy edges. Let Ψ

be a planar embedding of G and suppose that Γ is a drawing of $G \cup \mathcal{H}$ such that: (i) The drawing of G in Γ preserves Ψ ; (ii) no two dummy edges of $\mathcal{H}$ cross; (iii) each edge e of G crosses at most d times the dummy edges of $\mathcal{H}$. Each crossing between an edge e of G and a dummy edge of $\mathcal{H}$ is replaced in Γ with a dummy vertex, which we call a *division vertex* for e , and we say that $\mathcal{H}$ is an *augmenting k -colored Hamiltonian cycle of G consistent with σ with at most d division vertices per edge*. We also say that $\mathcal{H}$ (with at most d division vertices per edge) *is constructed on Ψ* .

Lemma 1. *Let G be a planar k -colored graph and let σ be a k -colored sequence compatible with G . If G admits a point-set embedding with at most $b \in \mathbb{N}^+$ bends per edge on every k -colored set S such that $\text{seq}(S) = \sigma$, then G admits an augmenting k -colored Hamiltonian cycle consistent with σ and with at most $b - 1$ division vertices per edge constructed on some planar embedding of G .*

Sketch of Proof: Let S be a set of points $p_0, p_1, \dots, p_{n-1}$ that lie on the x -axis such that $x(p_j) < x(p_{j+1})$ for $0 \leq j < n - 1$ and $\text{seq}(S) = \sigma$. By hypothesis G admits a point-set embedding on S with at most b bends per edge. Assume vertex v_i of G is placed at point p_i for all i . For all $0 \leq i < n - 1$ we add straight line edges (v_i, v_{i+1}) if they are not adjacent in G . If v_{n-1} and v_0 are not adjacent in G we connect them with a polyline as follows: from v_{n-1} we draw a horizontal line segment sufficiently far to the right, from v_0 we draw a horizontal line segment sufficiently far to the left, and we complete the polyline by connecting the two line segments to a point sufficiently high above the drawing of G . It can be shown that if an edge of G is intersected more than $b - 1$ times, then this edge is drawn with at least $b + 1$ bends. $\square$

Lemma 2. *Let G be a planar k -colored graph and let σ be a k -colored sequence compatible with G . If G has a planar embedding on which an augmenting k -colored Hamiltonian cycle consistent with σ and with at most $b \in \mathbb{N}^+$ division vertices per edge can be constructed, then G admits a point-set embedding with at most $2b + 1$ bends per edge on every k -colored set S such that $\text{seq}(S) = \sigma$.*

Sketch of Proof: The proof is similar to a technique used in [10]. Let S be a k -colored set such that $\text{seq}(S) = \sigma$. By hypothesis it is possible to find an augmenting k -colored Hamiltonian cycle $\mathcal{H}$ of G consistent with σ such that each edge has at most b division vertices. We add all division vertices to G and color these vertices with color k . Let G' denote the new graph, $\mathcal{H}'$ the new Hamiltonian cycle and n' the number of vertices of G' . Add $n' - n$ elements to S to create S' . We can do this in such a way that $\mathcal{H}'$ is consistent with S' . We first draw $\mathcal{H}'$ on S' from left to right, and the final edge from the last point of S' to the first point of S' is connected to a point drawn sufficiently high. Only the last edge of $\mathcal{H}'$ has a bend. The remaining edges of G' are either inside or outside $\mathcal{H}'$. The inside edges are drawn above S' , the others are drawn below S' . As is shown in [10], we can add these edges in a planar fashion by placing the bends sufficiently far up or down. This results in a drawing with at most one

Fig. 1. (a) A planar 2-colored graph G . An augmenting 2-colored Hamiltonian cycle $\mathcal{H}$ of G with at most one division vertex per edge is highlighted in bold. Dummy edges are dashed lines and dummy vertices are squares. The interior of $\mathcal{H}$ is shown as a shaded area. (b) A point-set embedding with at most 3 bends per edge. Edge $(3, 5)$ has 3 bends (one in correspondence of the dummy vertex).

bend per edge of G' , which implies that there is a drawing of G with at most $2b + 1$ bends per edge. For an illustration, see Figure 1. $\square$

A graph is *outerplanar* if it admits a planar embedding such that all vertices are on the external face. A graph is *2-outerplanar* if it admits a planar embedding such that removing all vertices on the external face, results in all the remaining vertices being on the external face. The following result is proved in [5].

Theorem 1. [5] *For every $n \geq 4$ there exists a 2-outerplanar 2-colored graph G with $2n$ vertices and a 2-colored set compatible with G such that the maximum number of bends per edge of every point-set embedding of G on S is $\Omega(n)$.*

3 Outerplanar 2-Colored Graphs

Motivated by Theorem 1, in this section we prove that every outerplanar 2-colored graph G admits a point-set embedding with at most 5 bends per edge on any 2-colored set S compatible with G .

Let Ψ be a planar embedding of G with all vertices on the external face. We prove that G admits an augmenting 2-colored Hamiltonian cycle constructed on Ψ , consistent with $\sigma = \text{seq}(S)$, and with at most 2 division vertices per edge. The result then follows from Lemma 2. Since every outerplanar graph can be made biconnected by adding edges while maintaining the outerplanarity, we can assume, without loss of generality, that G is biconnected. In this case the boundary of the external face of G is a simple cycle $\mathcal{C}$ containing all vertices of G . The edges of G that are not in $\mathcal{C}$ are called *chords*. We start by proving that every simple cycle $\mathcal{C}$ admits an augmenting 2-colored Hamiltonian cycle $\mathcal{H}$ consistent with σ and with at most 1 division vertex per edge. To this aim we describe an algorithm that computes $\mathcal{H}$. We then shall describe how it is possible to obtain from $\mathcal{H}$ an augmenting 2-colored Hamiltonian cycle of G consistent with σ and with at most two division vertices per edge. The idea is illustrated in Figure 2.

Let $\sigma = c_0, \dots, c_{n-1}$. We order the vertices of $\mathcal{C}$ counterclockwise starting from an arbitrary vertex whose color is c_0 . More formally, let v_0 be a vertex

Fig. 2. (a) An outerplanar 2-colored graph G . (b) An augmenting 2-colored Hamiltonian cycle of the external boundary of G , with at most 1 division vertex per edge. (c) An augmenting 2-colored Hamiltonian cycle of G with at most 2 division vertices per edge.

of $\mathcal{C}$ such that $col(v_0) = c_0$; we walk counterclockwise along $\mathcal{C}$ starting from v_0 and write $u < v$ if u is encountered before v . Also, given a subset of vertices $U \subseteq V(\mathcal{C})$ we write $first(U)$ to denote the first vertex of U according to $<$ and with $last(U)$ the last vertex of U according to $<$. Given a vertex $v \in U$, $next(v)$ denotes the vertex of U that is immediately after v according to $<$.

The construction of $\mathcal{H}$ begins with v_0 and adds one vertex of $\mathcal{C}$ per step (plus possibly one division vertex). The vertex of $\mathcal{C}$ added at Step i will be denoted as v_i . Also, we denote as H_i the set of vertices added to $\mathcal{H}$ up to Step i and as G_i the augmented graph constructed up to Step i , i.e. the graph consisting of $\mathcal{C}$ plus all dummy edges and division vertices possibly added during Steps $0, 1, \dots, i$. Also we define the following sets:

- $NB_i = \{v \mid v \in V(\mathcal{C}), v \notin H_i, first(H_i) < v < last(H_i)\}$
- $F_i^c = \{v \mid v \in V(\mathcal{C}), last(H_i) < v, col(v) = c\}$, where $c = 0, 1$ (notice that $F_i^c \cap H_i = \emptyset$)

Sets NB_i and F_i^c partition the set of vertices of $\mathcal{C}$ that must be still added to $\mathcal{H}$ at the end of Step i . Intuitively, the vertices in NB_i have been already encountered moving counterclockwise on $\mathcal{C}$, while the vertices in F_i^c have yet to be encountered. At the end of Step i , the following invariants are maintained:

Invariant 1. All vertices of NB_i have the same color.

Invariant 2. All vertices of NB_i are on the external face of G_i .

Invariant 3. Vertex v_i is on the external face of G_i .

Invariant 4. If $v_i \neq last(H_i)$, then for each vertex u such that $v_i < u < last(H_i)$, we have $u \in H_i$.

At Step $i + 1$ the algorithm chooses the vertex v_{i+1} of $\mathcal{C}$ to be added to $\mathcal{H}$; the addition of v_{i+1} may require us to add to $\mathcal{H}$ some dummy edges and one division vertex. We say that a dummy edge is added *inside* $\mathcal{C}$ if it is inserted on the left-hand side when walking counterclockwise around $\mathcal{C}$. In order to choose and add v_{i+1} , the algorithm distinguishes between the following cases:

Case 1: $NB_i \neq \emptyset$ and $col(NB_i) = c_{i+1}$. The algorithm chooses $v_{i+1} = \text{last}(NB_i)$.

If v_i and v_{i+1} are not adjacent in $\mathcal{C}$, a dummy edge (v_i, v_{i+1}) is added on the external face of G_i (see, e.g., the addition of vertices v_4 and v_5 in Figure 2(b)).

Case 2: $NB_i = \emptyset$ or $col(NB_i) \neq c_{i+1}$. The algorithm chooses $v_{i+1} = \text{first}(F_i^{c_{i+1}})$.

Vertices v_i and v_{i+1} are connected in $\mathcal{H}$ according to the following sub-cases:

Case 2.a: $v_i = \text{last}(H_i)$. If $v_{i+1} = \text{next}(\text{last}(H_i))$, then v_i and v_{i+1} are adjacent in $\mathcal{C}$ and therefore no dummy edge needs to be added (see, e.g., the addition of vertex v_7 in Figure 2(b)). If $v_{i+1} \neq \text{next}(\text{last}(H_i))$, a dummy edge (v_i, v_{i+1}) is added inside $\mathcal{C}$ (see, e.g., the addition of vertex v_8 in Figure 2(b)).

Case 2.b: $v_i \neq \text{last}(H_i)$ and $v_{i+1} = \text{next}(\text{last}(H_i))$. A dummy edge (v_i, v_{i+1}) is added on the external face of G_i (see, e.g., the addition of vertex v_6 in Figure 2(b)).

Case 2.c: $v_i \neq \text{last}(H_i)$ and $v_{i+1} \neq \text{next}(\text{last}(H_i))$. The algorithm splits edge $(\text{last}(H_i), \text{next}(\text{last}(H_i)))$ by means of a division vertex d , which is added to $\mathcal{H}$ between v_i and v_{i+1} . A dummy edge (v_i, d) is added on the external face of G_i and a dummy edge (d, v_{i+1}) is added inside $\mathcal{C}$. For example, in Figure 2(b), the addition of vertex v_{10} is done by inserting a division vertex d that splits (v_8, v_{11}) , and the two dummy edges (v_9, d) , (d, v_{10}) .

Lemma 3. *Let $\mathcal{C}$ be a 2-colored simple cycle and let σ be a 2-colored sequence compatible with $\mathcal{C}$. Then $\mathcal{C}$ admits an augmenting 2-colored Hamiltonian cycle consistent with σ and with at most 1 division vertex per edge.*

Lemma 4. *Let G be an outerplanar 2-colored graph and let Ψ be a planar embedding of G having all vertices on the external face. Let σ be a 2-colored sequence compatible with G . Then G admits an augmenting 2-colored Hamiltonian cycle constructed on Ψ , consistent with σ , and with at most 2 division vertices per edge.*

Proof. Since every outerplanar graph can be made biconnected by adding edges while maintaining outerplanarity, we assume that G is biconnected. Let $\mathcal{C}$ be the boundary of the external face of G in Ψ . We remove all the chords from G and compute an augmenting 2-colored Hamiltonian cycle of $\mathcal{C}$ consistent with σ and with at most one division vertex per edge, by using the algorithm described above. If we add back the chords of G to the graph G_{n-1} , i.e. the augmented graph constructed at the end of the algorithm, these edges will cross the edges of $\mathcal{H}$ that are inside $\mathcal{C}$. For each crossing between an edge $e_{\mathcal{H}}$ of $\mathcal{H}$ and a chord e_{ch} of G we add a division vertex d that splits both $e_{\mathcal{H}}$ and e_{ch} . Thus we obtain an augmenting 2-colored Hamiltonian cycle of G . To complete the proof we must show that every chord e_{ch} is split by at most two division vertices. To this aim observe that an edge $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ must cross e_{ch} only if an endvertex v of e_{ch} is such that $u < v < w$. We show that for each vertex v there can be at most one edge $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ such that $u < v < w$. Since edge $e_{\mathcal{H}}$ is inside the cycle $\mathcal{C}$, it is a dummy edge added at some Step i ($0 \leq i \leq n-1$) when

Cases 2.a or 2.c apply. After the addition of this edge, vertex v and all the other vertices between v and w become vertices of NB_i . According to the algorithm the vertices of NB_i are added to $\mathcal{H}$ by means of edges that are either edges of $\mathcal{C}$ or dummy edges on the external face of G_i (Case 1). This implies that any other dummy edge incident on a vertex between v and w is not inside $\mathcal{C}$ and therefore edge $e_{\mathcal{H}} = (u, w)$ is the only edge of $\mathcal{H}$ such that $u < v < w$. Since each chord e_{ch} has two endvertices, there is at most one division vertex on e_{ch} for each of them and therefore at most two division vertices for each chord. $\square$

Theorem 2. *Every outerplanar 2-colored graph G admits a point-set embedding with at most 5 bends per edge on any 2-colored set compatible with G . Such a point-set embedding can be computed in $O(n \log n)$ time, where n is the number of vertices of G .*

4 Outerplanar 3-Colored Graphs

Since, by Theorem 2, outerplanar 2-colored graphs are 2-colored point-set embeddable with $O(1)$ bends per edge, one may wonder whether the number of bends per edge remains constant for values of k larger than 2. Unfortunately, this may not be the case even for 3 colors.

In this section we describe an infinite family of outerplanar 3-colored graphs such that for any integer $h \geq 0$ there exists a graph in the family and a set of points S such that any point-set embedding of the graph on S contains an edge having more than h bends. Our family of outerplanar 3-colored graphs is parametric with n ; every member of this family, denoted as G_n , has $3n$ vertices and is defined as follows. G_n consists of a simple cycle formed by n vertices of color 0, followed (in the counterclockwise order) by n vertices of color 1, followed by n vertices of color 2 (notice that G_n actually has $3n$ vertices). The vertex of color 1 adjacent in the cycle to a vertex of color 0 is denoted as v_1 ; the vertex of color 2 adjacent in the cycle to a vertex of color 1 is denoted as v_2 ; the vertex of color 0 adjacent in the cycle to a vertex of color 2 is denoted as v_0 . Also, in G_n every vertex colored i is adjacent to v_i ($i = 0, 1, 2$) and vertices v_0, v_1, v_2 form a 3-cycle. Figure 3(a) is an example of G_n for $n = 12$.

To prove our lower bound, we consider all possible planar embeddings of G_n ; for each planar embedding Ψ of G_n we will prove that every augmenting k -colored Hamiltonian cycle constructed on Ψ and consistent with $seq(S)$ has more than h division vertices for some edge of G_n .

For a planar embedding Ψ of G_n and a cycle $C \in G_n$ we say that C *separates* a subset V' from a subset V'' of the vertices of G_n if all vertices of V' lie in the interior of the region bounded by C and all vertices of V'' are in the exterior of this region.

Lemma 5. *Let $h > 0$ be a positive integer and let G_n be such that $n = (h + 1)(25h^2 + 2) + 25h^2$. Let Ψ be a planar embedding of G_n and let C be the cycle v_0, v_1, v_2 . If C does not separate any two vertices, then at least one of the following conditions holds for Ψ :*

Fig. 3. (a) Graph G_{12} . (b) An illustration for the proof of Lemma 5. Cycle C' is highlighted in bold.

C.1 *There exists a face f in Ψ having at least $25h^2$ vertices of each color and containing vertices v_0 , v_1 , and v_2 .*

C.2 *There exists a cycle C' containing vertices v_0 , v_1 , and v_2 and such that: (i) C' has at most $25h^2 + 2$ edges, (ii) at least $(h + 1)(25h^2 + 2)$ vertices of a color i ($i = 0, 1, 2$) are inside C' , (iii) every vertex of $V - \{v_0, v_1, v_2\}$ with color different from i is outside C' .*

Sketch of Proof: We first observe that in every planar embedding of G_n there are exactly two faces whose boundary contains vertices of the three colors. If C does not separate any two vertices, one of these two faces has C as its boundary. Let f be the face of Ψ containing vertices of the three colors whose boundary is not C . For each color i in G_n , exactly two vertices of color i are adjacent to vertices of different color and each vertex of color i except v_i has degree three and is adjacent to two vertices of the same color; this implies that all vertices of f having color i are consecutive along the boundary of f ($i = 0, 1, 2$).

If Condition **C.1** does not hold, then the number of vertices of at least one color, say 0, along the boundary of f is $k < 25h^2$. Let v'_0 be the vertex of f having color 0 and adjacent to v_0 ; if v_0 is the only vertex of f having color 0, let $v'_0 = v_1$. Refer to Figure 3(b). Consider the cycle $C' = v_0, v'_0, \pi(v'_0, v_1), v_1, v_2$, where $\pi(v'_0, v_1)$ is the path along the boundary of f that goes from v'_0 to v_1 . Cycle C' has at most $25h^2 + 2$ vertices (and hence edges). By using the argument that C does not separate any two vertices, it follows that C' separates at least $n - 25h^2 = (h + 1)(25h^2 + 2)$ vertices of color 0 from all vertices of color 1 and 2 distinct from v_1 and v_2 . $\square$

Theorem 3. *For every $h > 0$, there exists an outerplanar 3-colored graph G with $3 \cdot ((h + 1)(25h^2 + 2) + 25h^2)$ vertices and a set of points S compatible with G , such that every point-set embedding of G on S has an edge with more than h bends.*

Sketch of Proof: Let n be $((h + 1)(25h^2 + 2) + 25h^2)$ and let S be any set of points compatible with G_n such that $\text{seq}(S)$ is an alternating sequence of the 3 colors (i.e., $\sigma = 0, 1, 2, 0, 1, 2, \dots, 0, 1, 2$). We denote $\text{seq}(S)$ as σ .

We prove that the statement is true for G_n and S . To this aim, based on Lemma 1, it suffices to show that, for each planar embedding Ψ of G_n , every augmenting 3-colored Hamiltonian cycle constructed on Ψ and consistent with σ has more than h division vertices for some edge of G_n . Let $\mathcal{H}$ be one such augmenting 3-colored Hamiltonian cycle and let C be the cycle v_0, v_1, v_2 . Note that in any planar embedding of G_n all vertices having a same color are either inside or outside the region bounded by C ; hence if C separates any two vertices u, v , it separates all vertices with the same color as u from all vertices with the same color as v (except v_0, v_1 , and v_2). We consider two cases.

Case 1. Ψ is such that C separates all vertices of a color i , except v_i , from all vertices of a different color j , except v_j .

Case 2. Ψ is such that C does not separate any two vertices of G_n . Without loss of generality, assume that all vertices of G_n are on or outside C .

In Case 1, since $\mathcal{H}$ is consistent with σ , each vertex of color i is adjacent in $\mathcal{H}$ to a vertex of color j . Therefore there are $n - 1$ edges of $\mathcal{H}$ that cross C in Ψ ; it follows that one edge of C has at least $\frac{n-1}{3} > h$ division vertices.

In Case 2, either Condition **C.1** or Condition **C.2** of Lemma 5 holds for Ψ . We sketch here the proof when Condition **C.2** holds. The proof for the remaining case is omitted for reasons of space. If Condition **C.2** holds, denote by U_i the set of vertices of color i inside the cycle C' in the statement of Lemma 5. Since $\mathcal{H}$ is consistent with σ then each vertex of U_i is adjacent to vertices with color different from i . Thus at least $(h + 1)(25h^2 + 2)$ edges cross cycle C' . Since C' has at most $25h^2 + 2$ edges, then at least one edge of C' has more than h division vertices. $\square$

5 Outerplanar k -Colored Graphs

In contrast with the result of Theorem 3, we prove that given any outerplanar k -colored graph G (for any constant $k > 2$), there exist infinite k -colored sets compatible with G for which a point-set embedding of G with a constant number of bends per edge is possible.

Let S be a k -colored set such that no two points have the same x -coordinate; assume that the points are ordered by increasing x -values. S is an *ordered k -colored set* if, for every color i ($0 \leq i \leq k - 1$), all points of color i appear consecutively in the ordering (that is, the colors never alternate in the ordering). The sequence of colors induced by an ordered k -colored set is called an *ordered k -colored sequence*.

We first describe an algorithm that, given a k -colored simple cycle $\mathcal{C}$ and a k -colored sequence σ compatible with $\mathcal{C}$, computes an augmenting k -colored Hamiltonian cycle $\mathcal{H}$ of $\mathcal{C}$ consistent with σ and with at most k division vertices. We then use this algorithm to obtain an augmenting k -colored Hamiltonian cycle of an outerplanar k -colored graph G consistent with σ and with at most $2k$ division vertices. Also in this case, the augmenting k -colored Hamiltonian cycle of G is constructed on a planar embedding of G having all vertices on the external face.

Let $\mathcal{C}$ be a k -colored simple cycle and let $\sigma = c_0, \dots, c_{n-1}$ be an ordered k -colored sequence consistent with $\mathcal{C}$. The algorithm computes $\mathcal{H}$ by performing at most k rounds. At each round, it walks along the cycle $\mathcal{C}$ either in the counterclockwise direction or in the clockwise direction. At each round we refer to the direction followed in that round as the *walking direction* of the round. We order the vertices of $\mathcal{C}$ from a starting vertex and according to the walking direction. For each walking direction, we use the relation $<$ and the notation $\text{first}(U)$, $\text{last}(U)$, and $\text{next}(v)$ defined in Section 3.

At Step 0, the algorithm adds to $\mathcal{H}$ any vertex v_0 of color c_0 . At each Step i ($i = 1, \dots, n-1$), a new vertex v_i of color c_i is added to $\mathcal{H}$; adding v_i may imply adding some division vertices. Denote by H_i the set of vertices added to $\mathcal{H}$ up to Step i and as G_i the augmented graph constructed up to Step i , i.e. the graph consisting of $\mathcal{C}$ plus all edges and division vertices added during Steps $0, 1, \dots, i$. For each Step i we define the following two sets:

- $F_i^c = \{v \mid v \in V(\mathcal{C}), v \notin H_i, \text{last}(H_i) < v, \text{col}(v) = c\}$, where $c = 0, \dots, k-1$
- $N_i(u, w) = \{v \mid v \in V(\mathcal{C}), v \notin H_i, u < v < w\}$, where $u < w$.

At the end of Step i , the following invariants are maintained:

Invariant 1. Any vertex $v \notin H_i$ is on the external face of G_i .

Invariant 2. Vertex v_i is on the external face of G_i .

Fig. 4. An illustration of different cases of the algorithm to compute $\mathcal{H}$ for a k -colored cycle. (a) Case 1 with $v_i = v_4$ and $v_{i+1} = v_5$. Edge (v_4, v_5) is added on the external face because $N_4(v_4, v_5) = \emptyset$. (b) Case 2 with $v_i = v_5$ and $v_{i+1} = v_6$. Sub-sequences σ_0 , σ_1 , and σ_2 are highlighted; dummy vertices are drawn as small squares.

At Step $i+1$ the algorithm chooses a vertex v_{i+1} of $\mathcal{C}$ to be added to $\mathcal{H}$. The addition of v_{i+1} to $\mathcal{H}$ may require the addition to $\mathcal{H}$ of some dummy edges and some division vertices. We say that a dummy edge is added *inside* $\mathcal{C}$ if it is added on the left-hand side when walking counterclockwise around $\mathcal{C}$. If $F_i^{c_{i+1}} \neq \emptyset$ we choose $v_{i+1} = \text{first}(F_i^{c_{i+1}})$, else invert the walking direction (which starts a new round) and repeat the test. Vertices v_i and v_{i+1} can be connected in $\mathcal{H}$ according to different cases:

Case 1: $N_i(v_i, v_{i+1}) = \emptyset$. Refer to Figure 4(a). If v_i and v_{i+1} are not adjacent in $\mathcal{C}$, a dummy edge (v_i, v_{i+1}) is added on the external face of G_i .

Case 2: $N_i(v_i, v_{i+1}) \neq \emptyset$. Refer to Figure 4(b). The vertices of $N_i(v_i, v_{i+1})$ may not form a consecutive sequence along $\mathcal{C}$, because there can be vertices of H_i intermixed with them. Let $\sigma_0, \sigma_1, \dots, \sigma_{l-1}$ be the maximal sub-sequences of consecutive vertices of $N_i(v_i, v_{i+1})$. For each sub-sequence σ_j , let $s_j = \text{first}(\sigma_j)$ and $t_j = \text{last}(\sigma_j)$. Also, let s'_j be the vertex (either a “real” vertex of $\mathcal{C}$ or a division vertex) that is immediately before s_j and let t'_j be the vertex (either a “real” vertex of $\mathcal{C}$ or a division vertex) that is immediately after t_j . Edges (s'_j, s_j) and (t_j, t'_j) are split by means of the division vertices a_j and b_j , respectively. The algorithm connects v_i to v_{i+1} in $\mathcal{H}$ by means of the path $v_i, a_0, b_0, a_1, b_1, \dots, a_{l-1}, b_{l-1}, v_{i+1}$. Edges (v_i, a_0) , (b_j, a_{j+1}) ($0 \leq j \leq l-1$) and (b_{l-1}, v_{i+1}) are added on the external face of G_i , while edges (a_j, b_j) ($0 \leq j \leq l-1$) are added inside $\mathcal{C}$.

Theorem 4. *Let G be an outerplanar k -colored graph and let S be an ordered k -colored set compatible with G . Then G admits a point-set embedding with at most $4k + 1$ bends per edge on S . Such a point-set embedding can be computed in $O(n \log n + kn)$ time, where n is the number of vertices of G .*

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