A Procedure for Efficient Generation of $1/f^\beta$ Noise Sequences

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Abstract. This paper presents a simple, efficient and fast procedure for generation of $1/f^\beta$ noise sequences. The proposed procedure is based on the impulse invariance method applied to the impulse response of the ideal fractional order integrator whose order $\alpha = \beta/2$ is between 0 and 1. First, an optimal value for the initial value of the impulse response is obtained by minimizing a least squares error criterion and then any of the well-established signal modeling techniques can be employed for the parameterization of the discrete impulse response by pole-zero models. For a given model order, the approximation accuracy depends on the signal modeling technique used. An illustrative example is presented to demonstrate the effectiveness of the method.

Keywords: $1/f$ -noise, fractional order integration, impulse invariance method, pole-zero model, signal modeling.

1 Introduction

The $1/f^\beta$ noise belongs to a class of random processes whose Power Spectral Density (PSD) follows a power law with a real exponent β between 0 and 2 [1]. Such processes have been observed in various areas [2]. For the purpose of simulation studies, many algorithms have been developed to synthesize sample paths of random processes having power spectrum (at least over several decades of frequency) exhibiting $1/f$ behavior [3-6]. This paper presents a new procedure for generating $1/f^\beta$ noise sequences by filtering Gaussian white noise through a discrete-time fractional order integrator designed using impulse invariance method (excepting the first sample) and signal modeling techniques. From system theory viewpoint, fractional order integrals are computed as outputs of linear invariant systems described by the irrational transfer function $1/s^\alpha$. The case $0 < \alpha < 1$ is of special interest in the context of digital generation of $1/f$ -noise. Unfortunately, the application of fractional integral to the generation of $1/f^\beta$ noise sequences gives rise to two problems: the resulting sampled impulse response has an infinite number of

terms, and its first sample is infinite too. This paper suggests using minimization of a least squares error criterion to obtain the first sample and then applying any of the well-established signal modeling techniques such as Padé, Prony, Shanks methods [7] or Steiglitz-Mcbride technique [8] to find the parameters of a rational approximation model which can be easily and efficiently implemented as a recursive filter. In contrast to previous closely related methods, the proposed approach does not require the use of power series expansion [6, 9], continued fraction expansion [10] or truncation technique [11].

2 Background

Let $w(t)$ be a continuous-time signal. According to Riemann-Liouville definition, the α^{th} -order fractional integral of $w(t)$ is given by [12]

$$x(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} w(\tau) d\tau, \quad \alpha > 0. \quad (1)$$

The function $\Gamma(\cdot)$ is the gamma function. In terms of systems, the expression (1) represents a convolution integral which describes the input-output relationship between the input $w(t)$, the output $x(t)$ and the impulse response

$$h_{c\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} u(t). \quad (2)$$

The signal $u(t)$ is the unit-step function defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}. \quad (3)$$

The transfer function of the ideal fractional integrator, i.e., the Laplace transform of $h_{c\alpha}(t)$, is given by

$$H_{c\alpha}(s) = \frac{1}{s^\alpha}. \quad (4)$$

The frequency response can be obtained from (4) by the substitution $s = j2\pi f$ where f is the frequency in Hz.

3 Proposed Method

The analogue impulse response (2) is sampled with the sampling period T_s to yield the discrete impulse response [13]

$$h_\alpha(n) = T_s h_{c\alpha}(nT_s) = \frac{T_s^\alpha n^{\alpha-1}}{\Gamma(\alpha)}, \quad n = 0, 1, 2, \dots. \quad (5)$$

The discrete equivalent of (1) is then given by the discrete convolution

$$x(n) = \sum_{i=0}^{\infty} h_{\alpha}(i)w(n-i) . \quad (6)$$

This is expressed in the z-domain as

$$X(z) = H_{\alpha}(z)W(z) . \quad (7)$$

where $H_{\alpha}(z)$, the z-transform of $h_{\alpha}(n)$, is the transfer function of the discrete time fractional order integrator which performs the digital computation of (1). Applying the impulse invariance method gives rise to two problems here because the first sample of $h_{\alpha}(n)$ is infinite for $0 < \alpha < 1$, and the number of its samples is infinite too.

Selecting the first sample of $h_{\alpha}(n)$

An attempt was made initially to select the initial value using the initial value theorem which states that

$$h_{\alpha}(0) = \lim_{z \rightarrow \infty} H_{\alpha}(z) . \quad (8)$$

Although the transfer function $H_{\alpha}(z)$ is unknown, it can approximately be determined by considering its relationship with (4) in the frequency range $0 \leq f \leq 1/2T_s$ [13]

$$H_{\alpha}(e^{j2\pi f T_s}) \approx \frac{1}{T_s} H_{c\alpha}(j2\pi f) . \quad (9)$$

Such approximation can be achieved by replacing s in (4) with an appropriate s-to-z transform such those of Euler, Tustin, Al-Alaoui's [14] or Tseng [15] given by $1/s \approx T_s/(1-z^{-1})$, $1/s \approx (T_s/2)(1+z^{-1})/(1-z^{-1})$, $1/s \approx (7T_s/8)(1+z^{-1}/7)/(1-z^{-1})$ and $1/s \approx (T_s/6)(7+16z^{-1}+z^{-2})/(3-2z^{-1}-z^{-2})$, respectively. From the above s-to-z transforms, the normalized initial value $h_{\alpha}(0)/T_s^{\alpha}$ has the following form

$$\frac{h_{\alpha}(0)}{T_s^{\alpha}} = C^{\alpha} . \quad (10)$$

while C is a positive real constant. The problem now is how to select the s-to-z transform which yields the best approximation of the desired magnitude response of (4). To solve this problem, we propose a least squares magnitude approximation. Note that according to (13) we are interested here in magnitude response only. The square error for magnitude approximation is defined as [16]

$$E(C) = \sum_{k=1}^N \left[\left| H(e^{j2\pi f_k T_s}) \right| - (2\pi f_k)^{-\alpha} \right]^2 . \quad (11)$$

where $f_k = k \frac{1}{2NT_s}$, $k = 1, 2, \dots, N$ are the frequency samples taken in the range

$0 < f \leq 1/2T_s$ and $H(e^{j2\pi f T_s})$ is the rational approximation defined in (12) whose coefficients a_i and b_i are estimated using signal modeling techniques. It is clear that the optimal value of C which minimizes the error criterion (11) does not necessarily derive from known s-to-z transforms; the latter served only to gain insight into the problem of selecting the initial value $h_{\alpha}(0)$. In figure 1 is plotted the square error as a

function of the normalized initial value C^α for $\alpha = 0.5$, $T_s = 0.001$, $L = 1000$ and $p = q = 5$ using MATLAB's m-file prony. The computation of the square error has been made by varying one of the parameters α , T_s or (p, q) while holding the others constant. In all the tests carried out, with $0.1 \leq C \leq 4$, the square error presents a unique minimum corresponding to a normalized optimal initial value located near 1. The value $C_0^\alpha = 1$ will then be taken as an initial guess for the optimization function.

Fig. 1. Square error $T_s = 0.001$, $L = 1000$ and $p = q = 5$. $\alpha = 0.4$

Approximating $H_\alpha(z)$

There are two approaches that can be used for the approximation of $H_\alpha(z)$: 1) direct truncation of (5) to N terms and, 2) signal modeling techniques using the first L samples of (5). The former suffers from the Gibbs phenomenon and requires lengthy FIR filter as the decay of $h_\alpha(n)$ is slow. The latter can lead to IIR filter approximation whose transfer functions possess the desired properties, namely, stability, minimum-phase, and zeros and poles distributed in alternating fashion on the real axis. Given the fractional order α , the sampling period T_s , the first L samples of (5) and the degrees p and q of the approximating rational transfer function (12)

$$H(z) = \frac{X(z)}{W(z)} = \sum_{i=0}^q b_i z^{-i} \left/ \left(1 + \sum_{i=1}^p a_i z^{-i} \right) \right. . \quad (12)$$

The unknown coefficients, a_i and b_i , are to be determined such that $H(z)$ best approximates $H_\alpha(z)$ in some sense. The generated sequences $x(n)$ will have PSDs proportional to the frequency response squared magnitude, that is

$$S_{xx}(f) = \sigma_w^2 \left| H_\alpha(e^{j2\pi f T_s}) \right|^2 . \quad (13)$$

where σ_w^2 is the variance of the Gaussian white noise $w(n)$. These PSDs approximate, over a finite range of frequencies, the perfect PSD of the $1/f^\beta$ noise.

Determining the optimal initial value

The optimization of the error criterion (11) is a one dimensional unconstrained optimization problem. A suitable search method for this type of problems is the simplex method of Nelder and Mead which requires function evaluations only. This method is implemented in MATLAB by the function *fminsearch*. The MATLAB function to be provided to *fminsearch* is the following

```
function E = optiminvalue(C,Ts,L,p,q,r,f);
%function to be minimized to find the
%optimal value of the first sample
% r is the fractional order
n = 1:L;
h = [(C*T_s)^r, (T_s^r)*(n.^(r-1))/gamma(r)];
[b,a]=prony(h,q,p); % or stmcb
H = freqz(b,a,f,1/T_s);
Hid = (2*pi*f).^(-r); %ideal frequency %response
E = sum((Hid-abs(H)).^2);
```

Table 1. Approximate transfer functions to $1/s^\alpha$

$\alpha = 0.4$, prony	$\frac{0.05936 - 0.1588z^{-1} + 0.1483z^{-2} - 0.0545z^{-3} + 0.005409z^{-4} + 0.0003039z^{-5}}{1 - 3.155z^{-1} + 3.694z^{-2} - 1.939z^{-3} + 0.4258z^{-4} - 0.02601z^{-5}}$
$\alpha = 0.8$, prony	$\frac{0.003897 - 0.009623z^{-1} + 0.007922z^{-2} - 0.002234z^{-3} + 4.021e-006z^{-4} + 3.955e-005z^{-5}}{1 - 3.347z^{-1} + 4.205z^{-2} - 2.411z^{-3} + 0.5966z^{-4} - 0.04417z^{-5}}$
$\alpha = 0.4$, stmcb	$\frac{0.05936 - 0.2145z^{-1} + 0.2909z^{-2} - 0.1774z^{-3} + 0.04327z^{-4} - 0.001651z^{-5}}{1 - 4.092z^{-1} + 6.542z^{-2} - 5.067z^{-3} + 1.876z^{-4} - 0.2589z^{-5}}$
$\alpha = 0.8$, stmcb	$\frac{0.003683 - 0.01264z^{-1} + 0.01578z^{-2} - 0.008149z^{-3} + 0.001118z^{-4} + 0.0002107z^{-5}}{1 - 4.36z^{-1} + 7.514z^{-2} - 6.38z^{-3} + 2.659z^{-4} - 0.4324z^{-5}}$

The parameters α and T_s are application-dependent while L , p , q and the signal modeling techniques are to be chosen by the designer. Taking an initial value $C_0 = 1$, the transfer functions reported in Table I were obtained with $\alpha = 0.4$ and $\alpha = 0.8$, $T_s = 0.001$, $L = 1000$, $N = 2^{16}$ and $p = q = 5$ using Prony and Steiglitz-McBride methods. The latter is implemented in the MATLAB's m-file *stmcb*. The checking of the pole-zero maps show that the poles and zeros of the transfer functions in Table I are inside the unit circle and are distributed in alternating fashion on the real axis and, hence, they correspond to stable minimum phase systems. The magnitude responses are plotted in figure 2. This figure shows that the magnitude response is better approximated at low frequency using Steiglitz-McBride's method than Prony's method. Experiments have shown that increasing the model order results in slight improvement in the quality of the magnitude approximation. For example, the square error (11) was found to be equal respectively to 53.9906, 42.0917 and 32.5782 for $p = q = 4, 5$ and 6 with $\alpha = 0.8$ and the same values for the others parameters using Steiglitz-McBride's method.

Fig. 2. Magnitude frequency response for $T_s = 0.001$, $L = 1000$ and $p = q = 5$. Top: $\alpha = 0.4$, bottom: $\alpha = 0.8$.

4 Illustrative Example

The proposed procedure was used to generate 100 $1/f^\beta$ noise sequences of 2^{10} samples by filtering zero-mean, unit variance Gaussian white noise through discrete time fractional order integrator whose transfer functions are given in Table I. The periodogram of each sequence was computed by the use of FFT technique. Estimates of the PSDs were obtained as averages over the 100 periodograms and plotted in figure 3 along with the frequency response squared magnitude of the filter used for

Fig. 3. Average power spectral densities for $T_s = 0.001$, $L = 1000$ and $p = q = 5$. Top: $\alpha = 0.4$, bottom: $\alpha = 0.8$.

generation. As it can be seen from this figure, the PSD estimates of the generated noise sequences best fit the exact PSD over a frequency band of about four decade using Steiglitz-McBride's technique than Prony's method.

5 Conclusion

This paper describes a simple, efficient and fast method for the generation of $1/f^\beta$ noise sequences via impulse invariance method and signal modeling techniques. The first sample of the impulse response is optimally selected and then the resulting sampled one is modeled as the impulse response of a linear invariant system. The accuracy of the approximation is influenced by signal modeling technique used. The

proposed procedure can be easily implemented to design recursive filters for efficient generation of $1/f^\beta$ -noise sequences from white noise sequences. In contrast to other methods, the proposed approach does not require the use of power series expansion, continued fraction expansion or truncation technique.

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