Stopping Criterion for Adaptive Algorithm

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Abstract. Adaptive algorithm consists of many different parameters. For example, adaptive index, adaptive criterion and stopping criterion. The adaptivity index drives the adaptive algorithm by selecting some elements for further refinement. Apart from the driving force, another important aspect of an algorithm is its stopping criteria. We present a new stopping criterion for adaptive algorithm.

1 Introduction

Convergence rate of finite volume on uniform meshes depends on the regularity or singularity of the solution. We develop finite volume on adaptive meshes. And, we present pointwise or infinity convergence of the developed method. It is shown that the convergence of the presented adaptive method is independent of the regularity or singularity of the underlying problem. An adaptive techniques depend on several factors such as error indicator and adaptive algorithm. We present a simple adaptive criterion and adaptive algorithm.

Now let us consider the steady state pressure equation of a single phase flowing in a porous medium Ω

$$-\operatorname{div}(\mathbf{K} \operatorname{grad} p) = f \quad \text{in } \Omega, \quad (1)$$

$$p(x, y) = p^D \quad \text{on } \partial\Omega_D. \quad (2)$$

Here, Ω is a polyhedral domain in $\mathbb{R}^2$, the source function f is assumed to be in $L^2(\Omega)$, and the diagonal tensor coefficient $\mathbf{K}(x, y)$ is positive definite and piecewise constant. The coefficient $\mathbf{K}$ is allowed to be discontinuous in space. In porous media flow [7,4,1], the unknown function $p = p(x, y)$ represents the pressure of a single phase, $\mathbf{K}$ is the permeability or hydraulic conductivity of the porous medium, and the velocity $\mathbf{u}$ of the phase is given by the Darcy law as: $\mathbf{u} = -\mathbf{K} \operatorname{grad} p$. The next section presents finite volume method and adaptive algorithm.

2 Finite Volume Discretization and Adaptive Algorithm

For solving partial differential equations (PDEs) in a domain, by numerical methods such as finite volume, the domain is divided into smaller elements called

Fig. 1. Computation of flux across an edge

finite volumes or cells. Finite volume discretization of the Equation (1) for a finite volume is given as [7]

$$\sum_{i=1}^4 \mathcal{F}_i = \int_V f \, d\tau . \quad (3)$$

Here, $\mathcal{F}_i$ is the flux through the interface i . Now let us compute the flux for the interface MN shared by the cells 1 and 2 (see Fig. 1(a)). The flux [12,13,14,7] through the edge MN is given as

$$F_{MN} = \Phi_{MN} (p_2 - p_1) , \quad (4)$$

where the scalar Φ_{MN} is referred to as the transmissibility of the interface MN and is given as

$$\Phi_{MN} = K_1 K_2 \left(\frac{l}{h_1 h_2} \right) \frac{1}{(K_1/h_1 + K_2/h_2)} . \quad (5)$$

Here, K_1 and K_2 refers to the permeability of the cells 1 and 2 in Fig. 1(a). The perpendicular distance of the interface MN from the center of cell 1 is h_1 . Similarly, h_2 is the perpendicular distance of the interface MN from the center of cell 2. The length of interface MN is l . Adaptive discretization can result in a non-matching grid as shown in Fig. 1(b). We are using the same flux approximation for computing flux on a non matching grid.

We are using the following expression for computing the error from the cell i in a mesh [7],

$$\epsilon_i \stackrel{\text{def}}{=} \left[\|f\|_{L^2(\Omega_i)} |\Omega_i|^{\frac{1}{2}} + \|(K \nabla p_h) \cdot \hat{\mathbf{n}}\|_{L^2(\partial\Omega_i)} |\partial\Omega_i|^{1/2} \right] . \quad (6)$$

Here, $|\Omega_i|$ is the area of the finite volume, $|\partial\Omega_i|$ is the circumference of the finite volume, and $\hat{\mathbf{n}}$ is the unit outward normal. The quantity

$$\|(K \nabla p_h) \hat{\mathbf{n}}\|_{L^2(\partial\Omega_i)} |\partial\Omega_i|^{1/2}$$

is the total flux associated with cell i . Let us further define a quantity named adaptivity index for cell i in a mesh,

$$\eta_i \stackrel{\text{def}}{=} \left[\frac{\epsilon_i}{\max_{j \in \text{cells}} \epsilon_j} \right]. \quad (7)$$

It can be seen from the above definition of adaptivity index. For a cell with zero error ($\epsilon = 0$), the adaptivity index η is zero, and for a cell with maximum error η is 1. Thus for any cell, the adaptivity index η_i will be in the range $[0, 1]$. It can be seen in the Algorithm 1 that the driving force for the Algorithm is the adaptivity index η .

The adaptivity index (7) drives the Algorithm 1 by selecting some finite volumes for further refinement. Apart from the driving force, another important aspect of an algorithm is its stopping criteria. The two obvious stopping criteria of an adaptive algorithm are the maximum allowable degrees of freedom ($\text{DOF}_{\max}$) or the maximum allowed mesh refinement, and the maximum allowed adaptive iteration steps “ $\text{Iter} \leq \text{Iter}_{\max}$ ”.

For defining third criterion, let us compute maximum error associated with a finite volume or cell on a mesh formed after k iterative steps of the algorithm. Let this error be ξ_k . Thus,

$$\xi_k = \max_{i \in \text{cells}} \epsilon_i ,$$

Thus, ξ_0 is the maximum error of a cell on the initial mesh. Our third stopping criterion is defined as

$$\frac{\xi_k}{\xi_0} \geq \text{tol} .$$

The third criterion “ $\xi_k/\xi_0 \geq \text{tol}$ ” is the error reduction after k iteration steps of the adaptive algorithm. Here, ξ_k denotes the maximum error (maximum value of ϵ_i on a mesh) on an adaptively refined mesh after k iteration steps of the adaptive Algorithm 1. The quantity ξ_k/ξ_0 , which measures the reduction of the posteriori error estimate ϵ_i , provides information of the relative error reduction. Thus, ξ_k/ξ_0 can be used as a stopping criterion apart from the maximum number of degrees of freedom. The degrees of freedom and maximum iteration of the adaptive algorithm do not provide information about the error reduction.

Algorithm 1 is used for adaptive refinement. When a finite volume is selected for further refinement based on the value of the adaptivity index (7), this finite volume is divided into four equal finite volumes. During the adaptive refinement process all finite volumes Ω_i in a mesh, for which the adaptivity index η_i is greater than a given tolerance δ , are refined. The tolerance δ lies between the values 0 and 1. Tolerance δ equal to 0 means uniform refinement (refine all finite volumes), and tolerance δ equal to 1 means that the adaptive algorithm will refine a single finite volume per iteration step which can be costly. Both of these values can be computationally expensive and may not be optimal. A small δ will

Algorithm 1. Adaptive Algorithm with a new stopping criterion $[\xi_{Iter}/\xi_0] \geq tol$.

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1 Mesh the domain;
2 Compute  $\xi_0$ ;
3 Set Iteration Counter  $Iter = 0$ ;
4 while  $DOF \leq DOF_{\max}$  or  $Iter \leq Iter_{\max}$  or  $[\xi_{Iter}/\xi_0] \geq tol$  do
5     Discretize the PDE on the mesh;
6     Solve the discrete system to a given tolerance;
7     forall Finite Volumes  $j$  in the Mesh do
8         if  $\eta_j \geq \delta$  then
9             Divide the Finite Volume  $j$  into Four Elements;
10        end
11    end
12    Form a new mesh;
13     $Iter^{++}$ ;
14    Compute  $\xi_{Iter}$ ;
15 end
    
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refine many finite volumes and thus introduce many new cells per iteration step of the adaptive algorithm. On the other hand, a large value of δ will refine fewer cells and thus introduce fewer new finite volumes per iteration step. It should be kept in mind that during each iteration step of the adaptive algorithm a discrete system needs to be solved. Typically a value of $\delta = 0.5$ is used [15].

To measure the effectiveness of the adaptivity index (7) in selecting the cells with maximum error, we use the relation

$$\Gamma \stackrel{\text{def}}{=} \frac{\text{Cell number with } \eta = 1.0}{\text{Cell number with maximum point-wise error } |p - p_h|} . \quad (8)$$

Here, Γ is the robustness of the indicator η . If Γ is close to 1, the cells with the maximum point-wise error and the cells with the maximum error given by the error indicator (6) are the same. We compute the robustness quantity Γ of the adaptive index during each iteration step of the adaptive Algorithm 1.

3 Numerical Examples

Let $\mathbf{p}$ be the exact solution vector, and $\mathbf{p}_h$ be the finite volume solution vector on a mesh. Let us further assume that p^k be the exact pressure at the center of the cell k and p_h^k be the discrete pressure by the finite volume approximation for the same location. Error in the L_∞ norm is defined as

$$\|\mathbf{p} - \mathbf{p}_h\|_{L_\infty} \stackrel{\text{def}}{=} \max_{k \in \text{cells}} [|p^k(x) - p_h^k(x)|] , \quad (9)$$

The finite volume solution is enforced inside the domain by the Dirichlet boundary condition and the source term. For solving the discrete systems of

equations formed on the sequence of adaptive and uniform meshes, we are using the ILU preconditioned Conjugate Gradient (CG) iterative solver unless mentioned otherwise. Let the domain be $\Omega = [-1, 1] \times [-1, 1]$, and it is divided into four sub-domains according to the permeability K (see Fig. 2). Let the permeability in the sub-domain Ω_i be K_i . It is assumed that the permeability in the sub-domain Ω_1 is equal to the permeability in the sub-domain Ω_3 , and the permeability in the sub-domain Ω_2 is equal to the permeability in the sub-domain Ω_4 . That is $K_1 = K_3$ and $K_2 = K_4$. Let us further assume that $K_1 = K_3 = R$ and $K_2 = K_4 = 1$. The parameter R is defined below. Let the exact solution in polar form be

$$p(r, \theta) = r^\gamma \eta(\theta) , \quad (10)$$

[8,9]. The parameter γ denotes the singularity in the solution [9], and it depends on the permeability distribution in the domain. For the singularity $\gamma = 0.1$, the Fig. 3 presents permeability distribution. $\eta(\theta)$ is given as

$$\eta(\theta) = \begin{cases} \cos[(\pi/2 - \sigma)\gamma] \cos[(\theta - \pi/2 + \rho)\gamma] , & \theta \in [0, \frac{\pi}{2}] , \\ \cos(\rho\gamma) \cos[(\theta - \pi + \sigma)\gamma] , & \theta \in [\frac{\pi}{2}, \pi] , \\ \cos(\sigma\gamma) \cos[(\theta - \pi - \rho)\gamma] , & \theta \in [\pi, \frac{3\pi}{2}] , \\ \cos[(\pi/2 - \rho)\gamma] \cos[(\theta - 3\pi/2 - \sigma)\gamma] , & \theta \in [\frac{3\pi}{2}, 2\pi] , \end{cases} \quad (11)$$

and the parameters R , γ , ρ and σ satisfy the following nonlinear equations

$$\begin{cases} R &= -\tan[(\pi - \sigma)\gamma] \cot(\rho\gamma) , \\ 1/R &= -\tan(\rho\gamma) \cot(\sigma\gamma) , \\ R &= -\tan(\sigma\gamma) \cot[(\pi/2 - \rho)\gamma] , \end{cases} \quad (12)$$

under the following nonlinear constraints

$$\begin{aligned} 0 &< \gamma < 2 , \\ \max\{0, \pi\gamma - \pi\} &< 2\gamma\rho < \min\{\pi\gamma, \pi\} , \\ \max\{0, \pi - \pi\gamma\} &< -2\gamma\rho < \min\{\pi, 2\pi - \pi\gamma\} . \end{aligned} \quad (13)$$

The constrained nonlinear Equations (12) can be solved for the parameters R , σ , and ρ by Newton's iteration algorithm for different degrees of singularity γ . The analytical solution $p(r, \theta)$ satisfies the usual interface conditions: p and $\mathbf{K} \frac{\partial p}{\partial \mathbf{n}}$ are continuous across the interfaces. It can be shown that the solution p belongs in the fractional Sobolev space $\mathbf{H}^{1+\kappa}(\Omega)$ ($\kappa < \gamma$) [10].

Let the singularity be $\gamma = 0.1$. Various parameters that satisfy the relations (12) under the constraints (13) are

$$R \approx 161.4476 , \quad \rho \approx 0.7854 \quad \text{and} \quad \sigma \approx -14.9225 .$$

The permeability distribution is shown in Figure 3. The exact solution belongs to the fractional Sobolev space $\mathbf{H}^{1+k}$ ($k < 0.1$). We have solved this problem on

Fig. 2. Domain Ω is divided into four sub-domains Ω_i , $i = 1 \dots 4$ according to the permeability

Fig. 3. Permeability distribution for the singularities $\gamma = 0.1$ and $\gamma = 0.1269$. The solution is singular at $O = (0, 0)$.

(a) Exact solution given by the equation (10) for $\gamma = 0.1$.

(b) Surface plot of the error $(u - u_h) / \|u\|_{L_\infty}$ for $\gamma = 0.1$.

Fig. 4. Surface plots of the exact solution and error for the singularity parameter $\gamma = 0.1$

adaptive and uniform meshes. The outcome of our numerical work is reported in Figs. 4 and 5. Figure 4(a) is a surface plot of the exact solution. The solution is singular at the origin. Figure 4(b) presents a surface plot of the error. It can be seen that the error is maximum at the singularity. Figure 5 compares the convergence behaviour on adaptive and uniform meshes in the L_∞ norm. We did not notice any convergence in the L_∞ norm on uniform meshes till one million degrees of freedom. A similar behaviour was also observed in [11] on uniform meshes for singular problems. It was suggested in [11] that adaptive meshes may be ideal for such solutions. On adaptive meshes, we are getting $\|\mathbf{p} - \mathbf{p}_h\|_{L_\infty} \approx \text{DOF}^{-P/2}$ with the convergence $P \approx 1$ (see Figure 5). Because of the regularity of the solution, this convergence is quasi optimal [9,8].

Fig. 5. On adaptive meshes $\|\mathbf{p} - \mathbf{p}_h\|_{L_\infty} \approx \text{DOF}^{-1/2}$

Fig. 6. On adaptive meshes $\|\mathbf{p} - \mathbf{p}_h\|_{L_\infty} \approx \text{DOF}^{-1/2}$

Let the singularity be $\gamma = 0.1269$. Various parameters that satisfy the relations 12 under the constraints 13 are

$$R \approx 99.999999 \text{ , } \rho \approx 0.7853982 \text{ and } \sigma \approx -11.59263 \text{ .}$$

The exact solution belongs in the fractional Sobolev space $\mathbf{H}^{1.126}$. Figure 6 is comparing the convergence behaviour of the finite volume on adaptive and uniform meshes. Again, we did not observe any convergence till one million degrees of freedom on uniform meshes (there is some convergence during the

Fig. 7. Decrease in the stopping criteria ξ_k/ξ_0 in the Algorithm 1 with adaptive refinement

Fig. 8. Robustness (defined by the Equation (8)) of the adaptivity index for finding cells with most error. Solutions are in the spaces $\mathbf{H}^{1.126902}$ and $\mathbf{H}^{1.1}$.

last refinement, see the Figure 6). While on the adaptive meshes, we are still getting $\|\mathbf{p} - \mathbf{p}_h\|_{L_\infty} \approx \text{DOF}^{-P/2}$ with $P \approx 1$. Figure 8 is a plot of the robustness against the iterations of the adaptive algorithm. It can be seen in the Figure 8, the robustness is almost always equal to 1.0 for all of the adaptive iterations. It means that the cells with the maximum point-wise error and cells with the maximum value of the error indicator given by the equation 6 are the same.

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