A Class of Explicit Perfect Multi-sequences

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Abstract. In [7], perfect multi-sequences are introduced and a construction based on function fields over finite fields is given. In this paper, we explore the construction in [7] by considering rational function fields. Consequently a class of perfect multi-sequences are obtained.

1 Introduction

There are a lot of papers concerning linear complexity of one single sequence [2], [3], [4], [5], [8]. For multi-sequences, it seems that few results are known. One of these results is the Massey's algorithm for linear feedback shift register (LFSR) synthesis that was proven by Ding [1]. Recently perfect multi-sequences are introduced in [7]. This definition is a natural generalization of one single perfect sequence defined by Rueppel [4], [5].

We adopt notations from [7].

Definition 1. Let $\mathbf{F}_q$ be the finite field with q elements. The linear complexity of a sequence $\mathbf{c} = (c_1, c_2, \dots, c_n)$ of elements of $\mathbf{F}_q$ is defined to be the least k such that it is a k -th order LFSR sequence.

We now consider multi-sequences $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m$ of dimension m of length n . C denotes the multi-sequences $\{\mathbf{c}_i\}_{i=1}^m$ of length n .

Definition 2. The linear complexity of a multi-sequence set $C = \{\mathbf{c}_i = (c_{i1}, c_{i2}, \dots, c_{in})\}_{i=1}^m$ of length n is defined to be the smallest order k of LFSR $(f(T), k)$ generating all of sequences $\mathbf{c}_i$ for $i = 1, 2, \dots, m$.

The Massey's algorithm gives a way to find the shortest LFSR generating multi-sequences, and therefore the linear complexity of a multi-sequence set C is obtained.

Now we turn to sequences of infinite length.

Consider multi-sequences of dimension $m \geq 1$,

$$\begin{aligned}\mathbf{a}_1 &= (a_{11}, a_{12}, a_{13}, \dots) \in \mathbf{F}_q^\infty \\ \mathbf{a}_2 &= (a_{21}, a_{22}, a_{23}, \dots) \in \mathbf{F}_q^\infty \\ &\vdots \\ &\vdots \\ &\vdots \\ \mathbf{a}_m &= (a_{m1}, a_{m2}, a_{m3}, \dots) \in \mathbf{F}_q^\infty\end{aligned}$$

and let A be the multi-sequence set $\{\mathbf{a}_i\}_{i=1}^m$. We also denote by A_n the multi-sequences

$$\{(a_{i1}, a_{i2}, \dots, a_{in})\}_{i=1}^m$$

of length n and dimension m .

Definition 3. *The linear complexity profile of a multi-sequence set $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ is defined by the sequence of integral numbers*

$$\{\ell_n(A)\}_{n=1}^\infty,$$

where $\ell_n(A)$ denote the linear complexities of $A_n = \{(a_{i1}, a_{i2}, \dots, a_{in})\}_{i=1}^m$ of length n .

In [7], perfect multi-sequences are defined in the following way.

Definition 4. *The multi-sequence set $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ is called perfect if*

$$\ell_n(A) = \lceil \frac{mn}{m+1} \rceil$$

for all $n \geq 1$, where $\lceil v \rceil$ denotes the smallest integer bigger than or equal to the real number v .

Furthermore, it is proven in [7] that

Theorem 1. *The multi-sequence set $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ is perfect if and only if*

$$\ell_n(A) \geq \lceil \frac{mn}{m+1} \rceil$$

for all $n \geq 1$.

This paper is organized as follows. In Section 2, we recall the main result of [7] concerning constructions of perfect multi-sequences. In Section 3, a class of perfect multi-sequences are given based on the result of Section 2. Some explicit examples are illustrated in this section.

2 Construction of Perfect Multi-sequences

First we introduce some notations (see [6]).

For a divisor $D = \sum_P m_P P$, the *degree* of D is defined by $\deg(D) = \sum_P m_P \deg(P)$.

Let z be a non-zero element of F , define the *zero divisor* of z by

$$(z)_0 = \sum_{\nu_P(z) > 0} \nu_P(z) P$$

and the *pole divisor* of z by

$$(z)_\infty = - \sum_{\nu_P(z) < 0} \nu_P(z) P.$$

Then $(z)_0$ and $(z)_\infty$ are two effective divisors and $\deg(z)_0 = \deg(z)_\infty$. The *principal divisor* of z is defined by

$$\operatorname{div}(z) = (z)_0 - (z)_\infty.$$

For a divisor D , the linear space $L(D)$ is defined by

$$L(D) = \{f \in F \mid \operatorname{div}(f) + D \geq 0\}.$$

It is an $\mathbf{F}_q$ -linear space of finite dimension. For two divisors $D = \sum_P m_P P$ and $G = \sum_P n_P P$, we denote by $D \vee G$ the divisor $\sum_P \max\{m_P, n_P\} P$.

For a place Q of degree m of F , the integral ring of Q is

$$\mathcal{O}_Q = \{z \in F \mid \nu_Q(z) \geq 0\}.$$

This is a local ring with the maximal ideal

$$\wp_Q = \{z \in \mathcal{O}_Q \mid \nu_Q(z) > 0\}.$$

The *residue class field* $\mathcal{O}_Q/\wp_Q$ is an extension over $\mathbf{F}_q$ of degree $m = \deg(Q)$. For an element of $z \in \mathcal{O}_Q$, the residue class $\bar{z}$ of z in $\mathcal{O}_Q/\wp_Q$ is denoted by $z(Q)$.

From now on in this section, we have the following notations and assumptions.

$F/\mathbf{F}_q$ – a global function field with the full constant field $\mathbf{F}_q$;

Q – a place of degree m of F ;

$x_1, x_2, \dots, x_m - m$ elements of $\mathcal{O}_Q$ satisfying that $x_1(Q), x_2(Q), \dots, x_m(Q)$ form an $\mathbf{F}_q$ -basis of $\mathcal{O}_Q/\wp_Q$;

t – a local parameter of Q with $\deg(t)_\infty = m + 1$;

y – an element of $\mathcal{O}_Q$ satisfying $y \notin \bigoplus_{i=1}^m \mathbf{F}_q(t)x_i$.

Then y has a local expansion at Q with the following form,

$$y = \sum_{j=0}^{\infty} \left(\sum_{i=1}^m a_{ij} x_i \right) t^j,$$

where $a_{i,j} \in \mathbf{F}_q$.

Denote

$$\mathbf{a}_i(y) = (a_{i1}, a_{i2}, a_{i3}, \dots) \in \mathbf{F}_q^\infty$$

for any $1 \leq i \leq m$. We form a multi-sequence set

$$A(y) = \{\mathbf{a}_i(y)\}_{i=1}^m. \quad (1)$$

It is proved in [7] that $A(y)$ is a perfect multi-sequence set if $m = \deg((y)_\infty \vee (x_1)_\infty \vee (x_2)_\infty \vee \dots \vee (x_m)_\infty)$.

We organize the above result as follows.

Theorem 2. *Let G be a positive divisor of degree m with $Q \notin \text{Supp}(G)$. Suppose that $x_1, x_2, \dots, x_m, y$ are elements of $L(G)$ such that $x_1(Q), x_2(Q), \dots, x_k(Q)$ form an $\mathbf{F}_q$ -basis of $\mathcal{O}_Q/\wp_Q$ and $y \notin \bigoplus_{i=1}^m \mathbf{F}_q(t)x_i$. Then $A(y) = \{\mathbf{a}_i(y)\}_{i=1}^m$ constructed as (1) is perfect.*

3 Explicit Perfect Multi-sequences

In this section, we present a class of perfect multi-sequences based on the results of Section 2. The function fields used are the rational function fields.

Throughout this section, F denotes the rational function field $\mathbf{F}_q(x)$. We also fix a place Q of degree m and a rational place P of F with $Q \neq P$. Let $t \in F$ be a function with $(t)_0 = Q + P$.

Proposition 1. *Let G be a positive divisor of degree m such that $\text{Supp}(G) \cap \{Q, P\} = \emptyset$. Let $\{x_1 = 1, x_2, \dots, x_m, y\}$ be a basis of $L(G)$. Then $\{x_1 = 1, x_2, \dots, x_m, y\}$ is also a basis of F over $\mathbf{F}_q(t)$,*

Proof. Since the degree $[F : \mathbf{F}_q(t)]$ of the extension $F/\mathbf{F}_q(t)$ is equal to the degree of the zero divisor of t , we have $[F : \mathbf{F}_q(t)] = \deg(t)_0 = \deg(Q + P) = m + 1$. This means that F is a vector space of dimension $m + 1$ over $\mathbf{F}_q(t)$. Hence it is sufficient to prove that $x_1 = 1, x_2, \dots, x_m, y$ are linearly independent over $\mathbf{F}_q(t)$. Suppose that there exist $m + 1$ functions $u_1(t), \dots, u_m(t), u(t) \in \mathbf{F}_q(t)$ such that not all of them equal to zero and

$$u(t)y + \sum_{i=1}^m u_i(t)x_i = 0. \quad (2)$$

By multiplying a polynomial in $\mathbf{F}_q[t]$, we may assume that all $u_i(t)$ and $u(t)$ are polynomials in $\mathbf{F}_q[t]$. Furthermore, dividing $u(t)$ and all $u_i(t)$ by the greatest common divisor $(u(t), u_1(t), \dots, u_m(t))$ of $u(t), u_1(t), \dots, u_m(t)$, we can assume that $(u(t), u_1(t), \dots, u_m(t)) = 1$. Thus not all of $u(0), u_1(0), \dots, u_m(0)$ are equal to zero. As $x_1 = 1, x_2, \dots, x_m, y$ are a basis of $L(G)$, we have $u(0)y + \sum_{i=1}^m u_i(0)x_i \neq 0$.

Rewrite (2) into the form

$$(u(t) - u(0))y + \sum_{i=1}^m (u_i(t) - u_i(0))x_i = - \left(u(0)y + \sum_{i=1}^m u_i(0)x_i \right). \quad (3)$$

Look at the left side of (3), we find that $Q + P = (t)_0$ is less than or equal to the zero divisor of the function $u(0)y + \sum_{i=1}^m u_i(0)x_i$. Hence $u(0)y + \sum_{i=1}^m u_i(0)x_i$ is a nonzero element of $L(G - (Q + P))$. However $L(G - (Q + P)) = \{0\}$ since $\deg(G - (Q + P)) = -1$. This contradiction shows that $x_1 = 1, x_2, \dots, x_m, y$ are linearly independent over $\mathbf{F}_q(t)$.

Theorem 3. *Let the divisor G and functions $x_1, x_2, \dots, x_m, y$ be as in Proposition 1. If there is a positive divisor G_1 with $\deg(G_1) = m - 1$ such that $G_1 \leq G$ and $x_1, x_2, \dots, x_m$ form a basis of $L(G_1)$, then the multi-sequence set $A(y)$ constructed as (1) is perfect.*

Proof. Since $\{x_1 = 1, x_2, \dots, x_m, y\}$ is a basis of F over $\mathbf{F}_q(t)$ by Proposition 1, we have $y \notin \bigoplus_{i=1}^m \mathbf{F}_q(t)x_i$.

Suppose that $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbf{F}_q$ satisfy

$$\sum_{i=1}^m \alpha_i x_i(Q) = 0.$$

Then $\sum_{i=1}^m \alpha_i x_i$ is an element of $L(G_1 - Q)$. Thus $\sum_{i=1}^m \alpha_i x_i$ is equal to 0 since $\deg(G_1 - Q) = -1$. This implies that $x_1(Q), x_2(Q), \dots, x_m(Q)$ form an $\mathbf{F}_q$ -basis of $\mathcal{O}_Q/\wp_Q$.

Applying Theorem 2, we have that $A(y)$ is perfect as $\deg(G) = m$.

Example 1. Let $Q(x)$ be an irreducible polynomial of degree m in $\mathbf{F}_q[x]$ and α an element of $\mathbf{F}_q$ satisfying $Q(\alpha) \neq 0$. Put

- (a) $t = Q(x)(x - \alpha)$;
- (b) $x_i = x^{i-1}$ for all $1 \leq i \leq m$;
- (c) $y = x^m$.

Then it follows from Theorem 3 that $A(y)$ is a perfect multi-sequence set of dimension m since $\{x_1, x_2, \dots, x_m\}$ is a basis of $L((m-1)\infty)$ and $\{x_1, x_2, \dots, x_m, y\}$ is a basis of $L(m\infty)$, where ∞ is the pole of x .

(a) binary perfect multi-sequences of dimension $m = 3$.

Taking $Q(x) = x^3 + x + 1 \in \mathbf{F}_2[x]$, $t = xQ(x)$, and $x_1 = 1, x_2 = x, x_3 = x^2, y = x^3$, we obtain the local expansion of y at Q :

$$y = (x_1 + x_2) + (x_1 + x_3)t + (x_1 + x_2 + x_3)t^2 + x_1 t^3 + (x_1 + x_2)t^4 + x_1 t^5 + x_1 t^6 + (x_1 + x_3)t^8 + \dots$$

Put

$$\mathbf{a}_1(y) = (1, 1, 1, 1, 1, 1, 0, 1, \dots),$$

$$\mathbf{a}_2(y) = (0, 1, 0, 1, 0, 0, 0, 0, \dots),$$

$$\mathbf{a}_3(y) = (1, 1, 0, 0, 0, 0, 0, 1, \dots).$$

By Theorem 3, $\mathbf{a}_1(y), \mathbf{a}_2(y), \mathbf{a}_3(y)$ are perfect multi-sequences of dimension 3.

(b) ternary perfect multi-sequences of dimension $m = 2$.

Taking $Q(x) = x^2 + 1$, $t = x(x^2 + 1)$, and $x_1 = 1, x_2 = 2, y = x^2$, we get the local expansion of y at Q :

$$y = 2x_1 + 2x_2t + 2x_1t^2 + x_2t^3 + 2x_1t^4 + 2x_1t^6 + 2x_2t^9 + \cdots.$$

Put

$$\begin{aligned}\mathbf{a}_1(y) &= (0, 2, 0, 2, 0, 2, 0, 0, \cdots), \\ \mathbf{a}_2(y) &= (2, 0, 1, 0, 0, 0, 0, 0, \cdots).\end{aligned}$$

By Theorem 3, $\mathbf{a}_1(y), \mathbf{a}_2(y)$ are perfect multi-sequences of dimension 2.

Example 2. Let $Q(x)$ be an irreducible polynomial of degree m in $\mathbf{F}_q[x]$ and α an element of $\mathbf{F}_q$ satisfying $Q(\alpha) \neq 0$. Put

- (a) $t = Q(x)/x^{m+1}$;
- (b) $x_i = (x - \alpha)^{1-i}$ for all $1 \leq i \leq m$;
- (c) $y = (x - \alpha)^{-m}$.

Then it follows from Theorem 3 that $A(y)$ is a perfect multi-sequence set of dimension m since $\{x_1, x_2, \dots, x_m\}$ is a basis of $L((m-1)P)$ and $\{x_1, x_2, \dots, x_m, y\}$ is a basis of $L(mP)$, where P is the zero of $x - \alpha$.

(a) binary perfect multi-sequences of dimension $m = 2$.

Taking $Q(x) = x^2 + x + 1$, $t = (x^2 + x + 1)/x^3$, and $x_1 = 1, x_2 = 1/x, y = 1/x^2$, we obtain the local expansion of y at Q :

$$y = (x_1 + x_2) + (x_1 + x_2)t + x_2t^2 + (x_1 + x_2)t^3 + t^4 + t^6 + \cdots.$$

Put

$$\begin{aligned}\mathbf{a}_1(y) &= (1, 0, 1, 1, 0, 1, \cdots), \\ \mathbf{a}_2(y) &= (1, 1, 1, 0, 0, 0, \cdots).\end{aligned}$$

By Theorem 3, $\mathbf{a}_1(y), \mathbf{a}_2(y)$ are perfect multi-sequences of dimension 2.

(b) ternary perfect multi-sequences of dimension $m = 2$.

Taking $Q(x) = x^2 + 2x + 2$, $t = (x^2 + 2x + 2)/x^3$, and $x_1 = 1, x_2 = 1/x, y = 1/x^2$, we obtain the local expansion of y at Q :

$$y = (x_1 + 2x_2) + (2x_1 + 2x_2)t + (x_1 + 2x_2)t^2 + x_1t^3 + (x_1 + 2x_2)t^4 + \cdots.$$

Put

$$\begin{aligned}\mathbf{a}_1(y) &= (2, 1, 1, 1, \cdots), \\ \mathbf{a}_2(y) &= (2, 2, 0, 2, \cdots).\end{aligned}$$

By Theorem 3, $\mathbf{a}_1(y), \mathbf{a}_2(y)$ are perfect multi-sequences of dimension 2.

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