Digital Flatness

Valentin E. Brimkov¹ and Reneta P. Barneva²

¹ Inst. of Math. and Comp. Science, Bulg. Acad. of Sci., Sofia 1113, Bulgaria

² SUNY Fredonia, Fredonia, NY 14063, USA

{brimkov,barneva}@cs.fredonia.edu

Abstract. In this paper we define and study the notion of *digital flatness*. We extend to dimension two various definitions and classical results about digital lines and rays. In particular, we resolve a conjecture of Maurice Nivat restricted to the case of digital planes, and define and characterize 2D Sturmian rays.

Keywords: Digital planarity, 2D Sturmian word, periodic array, digitization of planes, slope of digital planes

1 Introduction

Straight line/ray discretizations have been extensively studied over the last decades. Different aspects of “Digital straightness” have been found relevant to scientific disciplines as diverse as discrete geometry and topology, number theory, computer graphics, self-similarity studies in pattern recognition, periodicity studies in theory of words, as well as to some branches of physics and biology. Fundamental results characterizing digital lines and rays (in particular, their periodicity structure) have been obtained by Rosenfeld [16], Bruckstein [5], Brons [4], and others. Other deep theoretical results (e.g., related to properties of Sturmian words) have been obtained by Morse and Hedlund [13], Lunnnon and Pleasants [11], Coven and Hedlund [6]. For a nice survey on the subject the reader is referred to [15].

At the same time, very little is done towards extending the existing theory to higher dimensions. For instance, most of the above cited theoretical studies on discrete lines and rays do not have their plane counterpart, although the discrete plane is a very basic primitive, widely used in computer imagery. Moreover, “digital flatness” is relevant to all scientific disciplines listed above, as sometimes the possible applications are even more significant than in 1D. Thus developing a relevant theory for the case of digital planes is seen as an important task.

In the present work we propose a 2D extension of various concepts and results about digital rays. Some of these developments turn out to be quite perplexing due to certain intrinsic structural differences caused by the higher dimension.

The paper is organized as follows. In the next section, we recall some well-known basic knowledge from digital topology and combinatorics of 2D arrays, which we will need in order to describe our results. In Section 3, we propose a definition of a digital 2D ray and study its basic properties. In particular, we

present two basic theorems about periodicity of rational and irrational digital rays. In Section 4, we present other related results. In particular, we define and characterize 2D Sturmian rays and resolve a conjecture by Maurice Nivat for the case of digital 2D rays. Further tasks are commented in the final Section 6.

Since the proofs of the reported theorems considerably exceed the imposed space limit for the paper, most of them are omitted here and will be included in the full journal version. Detailed proofs of all theorems (from 1 through 8) together with a lot of illustrations, examples and additional references are available online in [2], as well as in [3].

2 Preliminaries

2.1 Basic Notions of Digital Topology and Digital Geometry

Discrete coordinate plane consists of unit squares (pixels), centered on the integer points of the two-dimensional Cartesian coordinate system in the plane. *Discrete coordinate space* consists of unit cubes (voxels), centered on the integer points of the three-dimensional Cartesian coordinate system in the space. The pixels'/voxels' coordinates are the coordinates of their centers. Sometimes they are called *discrete points*. The edges of a pixel/voxel are parallel to the coordinate axes. A set of discrete points is usually referred to as a *discrete object*.

We presume that any possible reader of this paper would be familiar with the basic notions of digital topology and geometry, such as pixel/voxel *adjacency* and *connectivity*, *separability* and *minimality* of a discrete object in another object, etc. For a detailed accounting of these and other basic concepts we refer to [10].

2.2 Periodicity, Repetitions and Tilings of Infinite Arrays

Consider the real plane R_+^2 , the integer lattice Z_+^2 , and the corresponding set of pixels with centers at the integer points and sides parallel to the coordinate axes. The pixels' sides form a *grid*. An *array* A on Z_+^2 over an alphabet Σ is a mapping from Z_+^2 to Σ , i.e.,

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ a_{2,0} & a_{2,1} & a_{2,2} & \dots \\ a_{1,0} & a_{1,1} & a_{1,2} & \dots \\ a_{0,0} & a_{0,1} & a_{0,2} & \dots \end{bmatrix}, \text{ where } a_{ij} \in \Sigma.$$

Array on Z^2 is defined analogously. A subset $s \subseteq Z_+^2$ is called a *shape*. Given an array A on Z_+^2 , by $A[s]$ we denote the restriction of A to s . $A[s]$ is *connected* if s is connected. We will call $A[s]$ *factor of A on shape s* . A rectangular factor of size $m \times n$ will be called *$m \times n$ -factor*. Below we follow in part [7], where periodicity in infinite 2D arrays is considered.

Definition 1. Let A be an array on Z_+^2 (or on Z^2). Let $S = A[s]$ be a factor of A on shape s (possibly, $s = Z_+^2$ and $S = A$). A vector v is a symmetry vector

for S if $A(i, j) = A(v + (i, j))$ for any point $(i, j) \in s$ such that $v + (i, j)$ is still in s . (If $s = \mathbb{Z}^2$, then clearly $v + (i, j) \in s$ for any point (i, j) .)

v is periodicity vector (or a period) for S if for any integer k , the vector kv is a symmetry vector for S .

Definition 2. An array A on $\mathbb{Z}_+^n$ is lattice periodic if there are two linearly independent vectors u and v such that $w = iu + jv$ is a period for A for any pair of integers i, j for which $w \in \mathbb{Z}_+^n$. A is line periodic if all periods of A are parallel vectors.

Note that the above definitions are similar but not equivalent to the well-known definitions from [1, 8].

Definition 3. Let A be lattice periodic array on $\mathbb{Z}_+^2$. The set of its symmetry vectors is a subset of (is extendable to) a sublattice Λ of $\mathbb{Z}^2$. Then any basis of Λ will be considered as a basis of A .

Definition 4. We say that an array A on $\mathbb{Z}_+^2$ is tiled by a tile W if it can be represented in the form $A = \begin{bmatrix} \dots & \dots & \dots \\ W & W & \dots \\ W & W & \dots \end{bmatrix}$, for certain rectangular block W .

We have the following proposition.

Proposition 1. Any lattice-periodic array on $\mathbb{Z}_+^2$ can be tiled by some tile W .

3 Digital Planes

3.1 Basic Definition

Consider the Euclidean plane

$$\mathcal{P}(\alpha_1, \alpha_2, \alpha_3, \beta) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta\}. \quad (1)$$

W.l.o.g., assume that $\mathcal{P}$ makes with the coordinate plane Ox_1x_2 an angle θ ,

$$0 \leq \theta \leq \arctan \sqrt{2}. \quad (2)$$

(See Figure 1.) Then the coefficient α_3 of x_3 in (1) will be nonzero. Dividing both sides of (1) by α_3 , we obtain the following equivalent formulation:

$$\mathcal{P}(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1 x_1 + a_2 x_2 + b\}, \quad (3)$$

where $a_1 = -\frac{\alpha_1}{\alpha_3}$, $a_2 = -\frac{\alpha_2}{\alpha_3}$, $b = \frac{\beta}{\alpha_3}$. We will consider digitizations of the plane $\mathcal{P}$ or its portions in the set of grid points $\mathbf{Z}^3 = \{(i, j, k) : i, j, k \in \mathbf{Z}\}$. In terms of representation (3), we will digitize the third coordinate x_3 over the integer grid points $\mathbf{Z}^2 = \{(i, j) : i, j \in \mathbf{Z}\}$ in the coordinate plane Ox_1x_2 .

Let us write (3) in a more general form:

$$\mathcal{P}^D(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1 x_1 + a_2 x_2 + b, (x_1, x_2) \in D \subseteq \mathbf{R}^2\}.$$

Fig. 1. A plane forming an angle $\arctan \sqrt{2}$ with the plane Ox_1x_2 .

We call $\mathcal{P}^D(a_1, a_2, b)$ a *restriction of $\mathcal{P}(a_1, a_2, b)$ to domain D* .

We have that $\mathcal{P}^D(a_1, a_2, b)$ is connected as long as D is connected. Also, $\mathcal{P}^D(a_1, a_2, b)$ is bounded (resp. unbounded) if and only if D is bounded/ unbounded. Note, however, that a 2D domain D admits many different shapes, whether D is bounded or not. (The possible unbounded shapes are even infinitely many.) As far as in our study periodicity properties of digitized planes are concerned, it is reasonable to restrict ourselves to a few cases. When D is a *finite domain*, we will assume that it is the rectangle $D = \{(x_1, x_2) : m_1 \leq x_1 \leq n_1, m_2 \leq x_2 \leq n_2, m_1, n_1, m_2, n_2 \in \mathbf{Z}\}$. Then the corresponding portion $\mathcal{P}^D(a_1, a_2, b)$ of $\mathcal{P}(a_1, a_2, b)$ will be a space rectangle. For an *infinite domain* D , one can consider the following three basic cases: (a) D is a quadrant; (b) D is a half-plane; (c) D is the whole plane. Note that the first case corresponds to a ray while the third one to a line in the plane. Therefore, if D is a quadrant, we will call $\mathcal{P}^D(a_1, a_2, b)$ a *2D ray*. The second case of a half-plane does not have a 1D counterpart. We will deal mostly with digitizations of 2D rays. The other cases can be handled in a similar (although not fully identical) way.

Below we explain how one can digitize $\mathcal{P}^D(a_1, a_2, b)$ when D is a 2D ray, i.e., the *first quadrant* of the plane. Formally, we have

$$\mathcal{P}^D(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b, (x_1, x_2) \in D\},$$

where $D = \{(x_1, x_2) : 0 \leq x_1, x_2 \leq \infty\}$. We discretize $\mathcal{P}^D(a_1, a_2, b)$ in $\mathbf{Z}_+^3 = \{(i, j, k) : i, j, k \in \mathbf{Z}_+\}$, where $\mathbf{Z}_+$ is the set of nonnegative integers. We discretize the third coordinate x_3 over the nonnegative integer grid points $\mathbf{Z}_+^2 = \{(i, j) : i, j \in \mathbf{Z}_+\}$ in the first quadrant *QuadI*. Consider an array on $\mathbf{Z}_+^2$

$$\rho = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \rho_{2,0} & \rho_{2,1} & \rho_{2,2} & \dots \\ \rho_{1,0} & \rho_{1,1} & \rho_{1,2} & \dots \\ \rho_{0,0} & \rho_{0,1} & \rho_{0,2} & \dots \end{bmatrix},$$

whose elements are the intersection points of $\mathcal{P}^D(a_1, a_2, b)$ with the vertical *grid lines*. Let $(i, j, I_{i,j}) \in \mathbf{Z}^3$ be the grid point *nearest* to $\rho_{i,j}$. If there are two nearest points (i.e., in a vertical distance $\frac{1}{2}$ from $\rho_{i,j}$), we choose the upper one. Formally, we have that the *discretization* of $\mathcal{P}^D(a_1, a_2, b)$ over $\mathbf{Z}_+^3$ is $I_{a_1, a_2, b} = \{(i, j, I_{i,j}) :$

$i, j \geq 0, I_{i,j} = \lfloor a_1 x_1 + a_2 x_2 + b + \frac{1}{2} \rfloor$. It has a *slope vector* (a_1, a_2) and *intercept* b . The discretization of a 2D ray R will alternatively be denoted $\text{discr}(R)$.

The following theorem is an analog of a result about digital rays [16].

Theorem 1. *A discretization of a plane $\mathcal{P}$ is 2-minimal in $\mathbf{Z}_+^3$.*

Corollary 1. *A discretization of a 2D ray is 2-minimal in $\mathbf{Z}_+^3$.*

The plane $\mathcal{P}$ intersects the coordinate planes Ox_1x_3 and Ox_2x_3 in straight lines with equations $x_3 = a_1x_1 + b, x_2 = 0$ and $x_3 = a_2x_2 + b, x_1 = 0$, respectively. Considered in the plane Ox_1x_3 , the first line has slope a_1 , while the second has slope a_2 . The slope vector of the plane has the slopes of these two lines as coordinates.

Now we define a *digital 2D ray* $r_{a_1, a_2, b}$ with a *slope vector* (a_1, a_2) and *intercept* b , as follows:

$$r_{a_1, a_2, b} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ r_{a_1, a_2, b}(2, 0) & r_{a_1, a_2, b}(2, 1) & r_{a_1, a_2, b}(2, 2) & \dots \\ r_{a_1, a_2, b}(1, 0) & r_{a_1, a_2, b}(1, 1) & r_{a_1, a_2, b}(1, 2) & \dots \\ r_{a_1, a_2, b}(0, 0) & r_{a_1, a_2, b}(0, 1) & r_{a_1, a_2, b}(0, 2) & \dots \end{pmatrix},$$

where $r_{a_1, a_2, b}(i, j)$ are called *cell codes* and defined for $i, j \geq 0$, as follows:

Let us set $r_{a_1, a_2, b}(0, 0) = I_{0,0}$.

Defining the 0-th digitized row:

$$r_{a_1, a_2, b}(0, j+1) = I_{0, j+1} - I_{0, j} = \begin{cases} 0, & \text{if } I_{0, j+1} = I_{0, j} \\ 1, & \text{if } I_{0, j+1} = I_{0, j} + 1 \end{cases},$$

Defining the 0-th digitized column:

$$r_{a_1, a_2, b}(i+1, 0) = I_{i+1, 0} - I_{i, 0} = \begin{cases} 0, & \text{if } I_{i+1, 0} = I_{i, 0} \\ 1, & \text{if } I_{i+1, 0} = I_{i, 0} + 1 \end{cases},$$

Defining the i-th digitized row:

$$r_{a_1, a_2, b}(i, j+1) = I_{i, j+1} - I_{i, j} = \begin{cases} 0, & \text{if } I_{i, j+1} = I_{i, j} \\ 1, & \text{if } I_{i, j+1} = I_{i, j} + 1 \end{cases},$$

Alternatively, we can digitize the array columnwisely.

Defining the i-th digitized column:

$$r_{a_1, a_2, b}(i+1, j) = I_{i+1, j} - I_{i, j} = \begin{cases} 0, & \text{if } I_{i+1, j} = I_{i, j} \\ 1, & \text{if } I_{i+1, j} = I_{i, j} + 1 \end{cases}.$$

Note that the 0-th row and the 0-th column are the same both in the rowwise and the columnwise digitizations. Code 0 can be interpreted as a horizontal rowwise/columnwise grid increment, and 1 as a vertical rowwise/columnwise increment in the grid $\mathbf{N}^3$. Because of assumption (2), horizontal/vertical move

from one integer point to another in the domain D can increase the z -coordinate by at most 1. Once the 0-th row or column is generated, one can build the rest of the array either rowwisely or columnwisely. $r_{a_1, a_2, b}$ is called *digitization* of the 2D ray $\mathcal{P}^D$. The digitization of a 2D ray R will alternatively be denoted by $digit(R)$. We will also say that the digital 2D ray $r_{a_1, a_2, b}$ is *generated* by the 2D ray $x_3 = a_1x_1 + a_2x_2 + b$. If for two 2D rays $r_{a_1, a_2, b}$ and $r_{a_1, a_2, b'}$ the number $b - b'$ is integer, then clearly $r_{a_1, a_2, b} = r_{a_1, a_2, b'}$. Thus, without loss of generality we may assume that the intercepts are limited to $0 \leq b \leq 1$. For any b , the special digital rays $r_{0,0,b}$ and $r_{1,1,b}$ are composed entirely by 0's and 1's, respectively.

Let (a_1, a_2) be the slope-vector of a plane discretization $I_{a_1, a_2, b}$. $I_{a_1, a_2, b}$ (as well as the corresponding digital 2D ray $r_{a_1, a_2, b}$ and the Euclidean plane $x_3 = a_1x_1 + a_2x_2 + b$) is called *rational* if both a_1 and a_2 are rational numbers. Otherwise, it is called *irrational*.

The following theorem is an analog of a well-known result of Bruckstein about digital rays [5].

Theorem 2. *For irrational plane with a slope vector (a_1, a_2) , the plane discretization $I_{a_1, a_2, b}$ uniquely determines both (a_1, a_2) and b . For rational (a_1, a_2) , $I_{a_1, a_2, b}$ uniquely determines (a_1, a_2) , and b is determined up to an interval.*

3.2 Periodicity Properties of 2D Digital Rays

We consider separately the cases of rational and irrational 2D rays.

Rational Digital 2D Rays. Consider the rational 2D ray $R = \mathcal{P}^D(a_1, a_2, b)$, its discretization $discr(R) = I_{a_1, a_2, b}$, and the corresponding digital 2D ray $digit(R) = r_{a_1, a_2, b}$. The coefficients a_1, a_2, b are rational numbers. Without loss of generality we may assume that they are integer and that R contains integer points. These integer points belong to a 2-dimensional integer lattice $\Lambda \in \mathbf{Z}^3$ in the plane $P = \mathcal{P}(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b\}$. Consider a basis B for Λ , i.e., a linearly independent system of integer vectors $B = \{e^1, e^2\}$, such that $\{x : a_1x_1 + a_2x_2 + a_3x_3 = b, x \in \mathbf{Z}^n\} = \{e^0 + \lambda_1e^1 + \lambda_2e^2, \lambda_1, \lambda_2 \in \mathbf{Z}\}$ where e^0 is an arbitrary integer point in P . Note that Λ has different bases. For instance, in Figure 2a, any one of the pairs of vectors $B_1 = \{e^1, e^2\}$, $B_2 = \{-e^1, e^2\}$, $B_3 = \{e^1, -e^2\}$, and $B_4 = \{-e^1, -e^2\}$ constitutes a basis. Geometrically, for a given basis $\{e^1, e^2\}$, the whole plane P is partitioned into parallelograms spanned on the basis vectors. (See Figure 2a.) Any two parallelograms are equivalent up to translation. Every lattice point can be obtained from any other lattice point by consecutive passes along the vectors $e^1, e^2, -e^1$, or $-e^2$. The discretization $discr(P)$ and the digitization $digit(P)$ are periodic as well. $discr(P)$ has period vectors e^1 and e^2 , while $digit(P)$ has as period vectors the *projections* of e^1 and e^2 on the coordinate plane Ox_1x_2 .

For an integer point $e^0 \in \mathcal{P}^D(a_1, a_2, b)$, one can obtain identical periodicity picture of R , $discr(R)$, and $r_{a_1, a_2, b} = digit(R)$. (See Figure 2b.) In view of the above discussion, one can consider $digit(P)$ and $digit(R)$ as tiled by a tile with

Fig. 2. Illustration to the proof of Theorem 3. a) The 2D integer lattice Λ in the plane P and some of its bases. b) The points of the integer lattice Λ of the 2D ray R .

a shape of parallelogram formed by the vectors of a given basis. It follows from Proposition 1 that $r_{a_1, a_2, b}$ and $r_{a_1, a_2, b}^D$ are also tiled by a rectangular tile of suitable size.

We have seen that the lattice of the integer points of a plane or a 2D ray can be generated by different bases which feature different parallelogram partitions (Figure 2). Nevertheless, it is a well-known fact from lattice theory that the lattice cells have the same area for all possible bases. It equals the value $\max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$, where $\alpha_1, \alpha_2, \alpha_3$ are the coefficients in the plane representation (1) with $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$. The above discussion leads us to the following 2D version of a result of Brons about digital rays [4].

Theorem 3. *Rational digital 2D rays are lattice-periodic. For a given basis of the lattice, the corresponding lattice cells are parallelograms. For all possible bases, the lattice cells have the same area which equals the maximal by absolute value coefficient in the plane representation $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta$ with $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$.*

Irrational Digital 2D Rays. This case is more involved than the previous one. Note that, unlike with an irrational *line* digitization which is always aperiodic, a *plane* digitization may be line-periodic. Let P be an irrational plane. The possible periodicity structure of $\text{digit}(P)$ depends on the integer (or rational) points which P contains. Careful analysis of all the possibilities leads us to the following theorem.

Theorem 4. *Irrational digital 2D rays are either aperiodic or line-periodic.*

The proof relies particularly on the fact that a plane P with irrational coefficients can either contain no integer points, or exactly one integer point, or an infinite set of integer points belonging to a line which is on P or parallel to P .

4 Sturmian Planes and 2D Rays

Remember that an $m \times n$ factor of a 2D array A (finite or infinite) is any $m \times n$ subarray of A . We define *complexity function* $P_A(m, n)$ of A as the number of the different $m \times n$ factors of A . In particular, we have $P_A(0, 0) = 1$ (the empty word is the unique factor in this case), while $P_A(1, 1)$ is the size of the alphabet. Thus for a binary alphabet $\{0, 1\}$ we have $P_A(1, 1) = 2$. Further we will consider arrays on the alphabet $\{0, 1\}$.

Definition 5. We call a digital 2D ray r Sturmian if $P_r(m, n) = mn + 1$.

The following properties as a matter of fact are based on the Kronecker theorem.

Theorem 5. All digital planes/2D rays with an irrational slope vector (a, b) contain the same set of rectangular factors.

Theorem 6. Let r be an irrational digital 2D ray. Then every rectangular factor appearing in r , appears in it infinitely many times.

Theorem 7. Any rectangular factor of an irrational digital plane is a factor of a certain rational digital plane.

We now present a fundamental theorem.

Theorem 8. Let r be a digital 2D ray.

(a) If $P_r(m, n) \leq mn$ for some integers $m, n \geq 0$, then r has at least one periodicity vector.

(b) If r is rational, then it is lattice-periodic and has at least two linearly independent periodicity vectors. In this case, $P_r(m, n)$ is bounded for any $m, n \geq 0$. Also, $P_r(m, n) \leq mn - k$ always holds for some $m, n, k \geq 0$, $0 \leq k \leq mn$.

(c) If r is irrational, then the inequality $P_r(m, n) \leq mn - k$ may hold for some $m, n, k \geq 0$, $0 \leq k \leq mn - \max(m, n)$. This may happen only if r is line-periodic. If r is aperiodic, then $P_r(m, n) = mn + 1$, i.e., r is Sturmian.

(d) If r is irrational, then $P_r(m, n)$ is always unbounded, even if $P_r(m, n) \leq mn - k$ for some $m, n, k \geq 0$.

For example, the rational digital 2D ray $\text{digit}_{\text{col}}(R_{\frac{1}{2}, \frac{1}{3}})$ is lattice-periodic, as illustrated below. (Bullet signs indicate lattice elements.)

	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
7	0	0	0	0	0	0	0
6	•	0	1	•	0	1	•
5	1	0	1	0	1	0	1
4	0	0	0	0	0	0	0
3	•	0	1	•	0	1	•
2	1	0	1	0	1	0	1
1	0	0	0	0	0	0	0
0	•	0	1	•	0	1	•
	0	1	2	3	4	5	6
	7	8					

Acknowledgments

We would like to thank Azriel Rosenfeld who encouraged this research. We are grateful to Reinhard Klette who read an extended preliminary version of the paper, suggested certain improvements, and proposed to post it as a technical report on the CITR web site. Thanks go also to Eric Andres, Alberto Apostolico, and Stefan Dantchev for some helpful discussions. We thank the referees for some useful remarks and suggestions.

The reported research was done in part while the first author was visiting the Laboratory on Signal, Image and Communication, CNRS, University of Poitiers, France, and supported by a grant from the University of Poitiers.

References

1. Amir, A., G. Benson, Two-dimensional periodicity and its applications, Proc. 3rd ACM-SIAM Symp. on Discrete Algorithms (1992) 440-452
2. Brimkov, V.E., Digital flatness and related combinatorial problems, CITR-TR-120, University of Auckland, New Zealand (2002) 44 pages, <http://www.citr.auckland.ac.nz/techreports/?year=2002>
3. Brimkov, V.E., Notes on digital flatness, TR 2002-01, Laboratory on Signal, Image and Communication, CNRS, University of Poitiers, France, July 2002, 52 pages
4. Brons, R., Linguistic methods for description of a straight line on a grid, *Computer Graphics Image Processing* **2** (1974) 48-62
5. Bruckstein, A.M., Self-similarity properties of digitized straight lines, *Contemp. Math.* **119** (1991) 1-20
6. Coven, E., G. Hedlund, Sequences with minimal block growth, *Math. Systems Theory* **7** (1973) 138-153
7. Epifanio, Ch., M. Koskas, F. Mignosi, On a conjecture on bidimensional words, <http://dipinfo.math.unipa.it/mignosi/periodicity.html>
8. Galil, Z., K. Park, Truly alphabet-independent two-dimensional pattern matching, Proc. 33rd IEEE Symp. Found. Computer Science (1992) 247-256
9. Grünbaum, B., G.C. Shephard, *Tilings and patterns*, Freeman & Co, New York, 1987
10. Kong, T.Y., A. Rosenfeld, Digital topology: introduction and survey, *Comput. Vision Graphics Image Processing* **48** (1989) 357-393
11. Lunnon, W.F., P.A.B. Pleasants, Characterization of two-distance sequences, *J. Austral. Math. Soc. (Ser. A)* **53** (1992) 198-218
12. Mignosi, F., G. Perillo, Repetitions in the Fibonacci infinite words, *RAIRO Theor. Inform. Appl.* **26** (1992) 199-204
13. Morse, M., G.A. Hedlund, Symbolic dynamics II: Sturmian sequences, *Amer. J. Math.* **61** (1940) 1-42
14. Nivat, M., Invited talk at ICALP'97
15. Rosenfeld, A., R. Klette, Digital straightness, *Electronic Notes in Theoretical Computer Science* **46** (2001) URL: <http://www.elsevier.nl/locate/entcs/volume46.html>
16. Rosenfeld, A., Digital straight line segments, *IEEE Trans. Computers* **23** (1974) 1264-1269
17. Smeulders, A. W. M., L. Dorst, Decomposition of discrete curves into piecewise segments in linear time, *Contemporary Mathematics* **119** (1991) 169-195
18. Wu, A. Y., On the chain code of a line, *IEEE Trans. Pattern Analysis Machine Intelligence* **4** (1982) 347-353