11. Socio-Economic Modeling

Peter A. Markowich and Giuseppe Toscani[1]

A comparative empirical and statistical analysis of social and economic phenomena describing the collective behavior of human beings in different countries and markets leads to a strikingly large number of similarities. This motivates the basic idea that the collective behavior of a society composed of sufficiently many individuals (agents) can be modeled using the approach of statistical mechanics, which was originally developed for the description of physical systems consisting of many interacting particles. The details of the interactions between agents then characterize the emerging statistical phenomena.

In particular the evolution of wealth in a simple market economy has been studied extensively. A very interesting point of view in the representation of markets is the kinetic one, which leads to Boltzmann type equations for the evolution of the distribution of wealth [3–6, 12]. In these models, the market is represented by a gas of physical particles, where each particle is identified with an agent, and each trading event between two agents is considered to be a binary particle collision event, with collisional rules determined by the properties of the underlying market. The knowledge of the large-wealth behavior of the steady state density is of primary importance, since it characterizes the number of rich individuals in the society and can easily be used to determine *a posteriori* if the model fits known data of real economies.

More than a hundred years ago, the Italian economist Vilfredo Pareto [11] first quantified the large-wealth behavior of the income distribution in a society and concluded that it obeys a power-law. More precisely if $f = f(w)$ is the probability density function of agents with wealth w, and w is sufficiently large, then the fraction of individuals in the society with wealth larger than w is:

$$F(w) = \int_w^\infty f(w_*)\, dw_* \sim w^{-\mu} .$$

Pareto mistakenly believed the distribution function on the whole range of wealth (positive real axis) to be a power law with a universal exponent μ approximatively equal to 1.5.

Various statistical investigations with real data during the last ten years revealed that the tails of the income distributions indeed follow the above mentioned power law behavior. The numerical value of the so called Pareto index μ generally varies between 1 and 2.5 depending on the considered market (USA

[1] *http://www-dimat.unipv.it/toscani/*

~ 1.6, Japan ~ 1.8–2.2, [6]). It is also known from statistical studies that typically less than 20% of the population of any country own about 80% of the total wealth of that country. The top income group obeys the above Pareto law while the remaining low income population, in fact the majority (80% or more), follow a different distribution, which is typically Gibbs [6] or log-normal.

Kinetic models of the time evolution of wealth distributions can be described in terms of a Boltzmann-like equation which reads

$$\frac{\partial f}{\partial t} = Q(f, f)\,, \tag{11.1}$$

where $f = f(v, t)$ is the probability density of agents of wealth $v \in \mathbb{R}_+$ at time $t \geq 0$, and Q is a bilinear operator which describes the change of f due to binary trading events among agents. We shall refer to this equation in the sequel as kinetic Pareto–Boltzmann equation.

The involved binary tradings are described by the rules

$$v^* = p_1 v + q_1 w\,; \quad w^* = p_2 v + q_2 w\,, \tag{11.2}$$

where (v, w) denote the (positive) moneys of two arbitrary individuals before the trading and (v^*, w^*) the moneys after the trading. The transaction coefficients p_i, q_i, $i = 1, 2$ are either given constants or random variables, with the obvious constraint of non-negativity. Also, they have to be such that the transformation from the money states before trading and after trading is non-singular. Among all possible kinetic models of type (11.1), (11.2) the *conservative* models are characterized by the property

$$\langle p_1 + p_2 \rangle = 1\,, \quad \langle q_1 + q_2 \rangle = 1\,,$$

where $\langle \cdot \rangle$ denotes the probabilistic expectation. This guarantees conservation of the total expected wealth of the market (which is the first order moment of the distribution function, multiplied by the total number of individuals).

In weak form the *collision* operator $Q(f, f)$ is defined by

$$\int_{\mathbb{R}_+} Q(f,f)(v)\phi(v)\,dv$$
$$= \frac{1}{2}\left\langle \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \big(\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)\big) f(v) f(w) dv\,dw \right\rangle\,. \tag{11.3}$$

Here ϕ is a smooth test function with compact support in the non-negative reals.

Note that the *collision* operator is assumed to be of so-called Maxwellian type, i.e. the scattering kernel does not depend on the relative wealth of collisions and can therefore be accounted for in the computation of the statistical expectation by choosing the probability space appropriately.

In their pioneering paper A. Chakraborty and B.K. Chakrabarthi [3] started out by stating that the agents taking part in trading exchange their money according to the rule

$$v^* = v + \Delta(v, w) \; ; \quad w^* = w - \Delta(v, w) \; . \tag{11.4}$$

Here $\Delta(v, w)$ represents the amount of money to be exchanged, which has to be such that the agents always keep some money in their hands after trading. The ratio of saving to all of the money held is usually denoted by s and called the saving rate. Taking $0 < s < 1$ constant, the amount of money to be exchanged can be modeled as

$$\Delta(v, w) = (1 - s)\,[(\varepsilon - 1)v + \varepsilon w]\,, \tag{11.5}$$

where $0 \leq \varepsilon \leq 1$ is a random fraction. This model was further developed in B.K. Chakrabarthi's research group by assuming that agents feature a random saving rate [4]. Clearly, choosing a random value for s does not change the type of collision events.

A somewhat different trading law was considered by S. Cordier, L. Pareschi and G. Toscani in [5]. Their trading model reads

$$v^* = sv + (1 - s)w + \eta v \; ; \quad w^* = (1 - s)v + sw + \tilde{\eta} w \; , \tag{11.6}$$

where $0 < s < \frac{1}{2}$. Here η and $\tilde{\eta}$ are independent equally distributed random variables with variance σ^2 and mean zero. Provided both η and $\tilde{\eta}$ take values in the interval $[-s, s]$, the trade (11.6) is such that the random coefficients p_i, q_i, $i = 1, 2$ are nonnegative. Note that this trade is conservative only in the mean, since $p_1 + p_2 = 1 + \eta \neq 1$, whereas $\langle p_1 + p_2 \rangle = 1$. The last terms in the trading laws describe the spontaneous growth or decrease of wealth due to random investments in the stock market and other macro-economic factors. This mechanism corresponds to the effects of an open market economy where typically the rich get richer and the poor get poorer.

Non-conservative models have been recently considered by F. Slanina [12], who introduced a model with increasing total wealth based on the collision coefficients:

$$p_1 = s \,, \quad q_1 = 1 - s + \varepsilon \; ; \quad p_2 = 1 - s + \varepsilon \,, \quad q_2 = s \; . \tag{11.7}$$

In (11.7) ε is a fixed positive constant, so that the total money put into the trade increases, since

$$v^* + w^* = (1 + \varepsilon)(v + w) \; .$$

This type of trade intends to introduce the feature of a strong economy, which is such that the total mean wealth is increasing in time. We remark that the

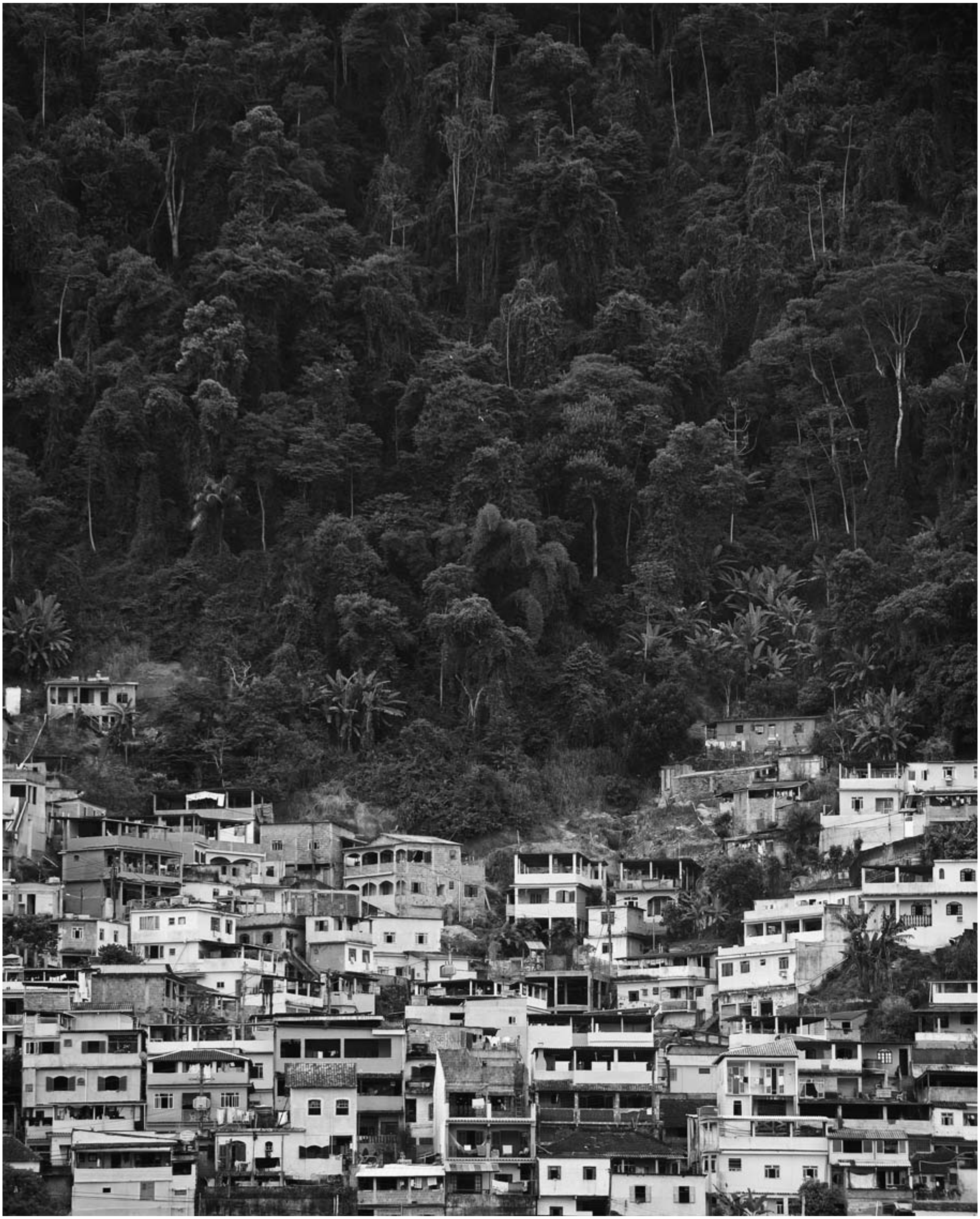

Fig. 11.1. Angra dos Reis, Brazil: on the $w = o(1)$-part of the Pareto distribution

Fig. 11.2. Salvador de Bahia, Brazil: on the $w = o(1)$-part of the Pareto distribution

same effect can also be obtained by simply allowing the random variables in the trading laws (11.6) to assume values on the whole real axis, and at the same time discarding those trades for which one of the post-trade wealths is non-positive.

A critical analysis of the discussed *collision=trading* rules reveals a deep analogy between the economic models described above and the granular material flows modeling framework of Chapter 3. They share the property that the steady (or, more generally, the self-similar asymptotic) states are different from the classical Maxwell distribution of the Boltzmann equation of gas dynamics presented in Chap. 1. Another analogy becomes evident when looking at the non-conservative properties of the *economic* and *granular* Boltzmann equations, resulting from inelastic binary collision models.

Conservative exchange dynamics between individuals redistribute the wealth among people. Without conservation, the best way to extract information on the large-time behavior of the solution relies on scaling the solution itself to keep the average wealth constant after scaling. Nevertheless, the explicit form of the limit distribution of the kinetic equation remains extremely difficult to recover, and often requires the use of suitable numerical methods.

A complementary method to extract information on the steady state distribution was linked in [5] to the possibility of obtaining particular asymptotics, which mimic the characteristics of the solution of the original problem for large times. The main result in this direction was to show that the kinetic model converges (under appropriate assumptions) in a suitable scaling limit to a par-

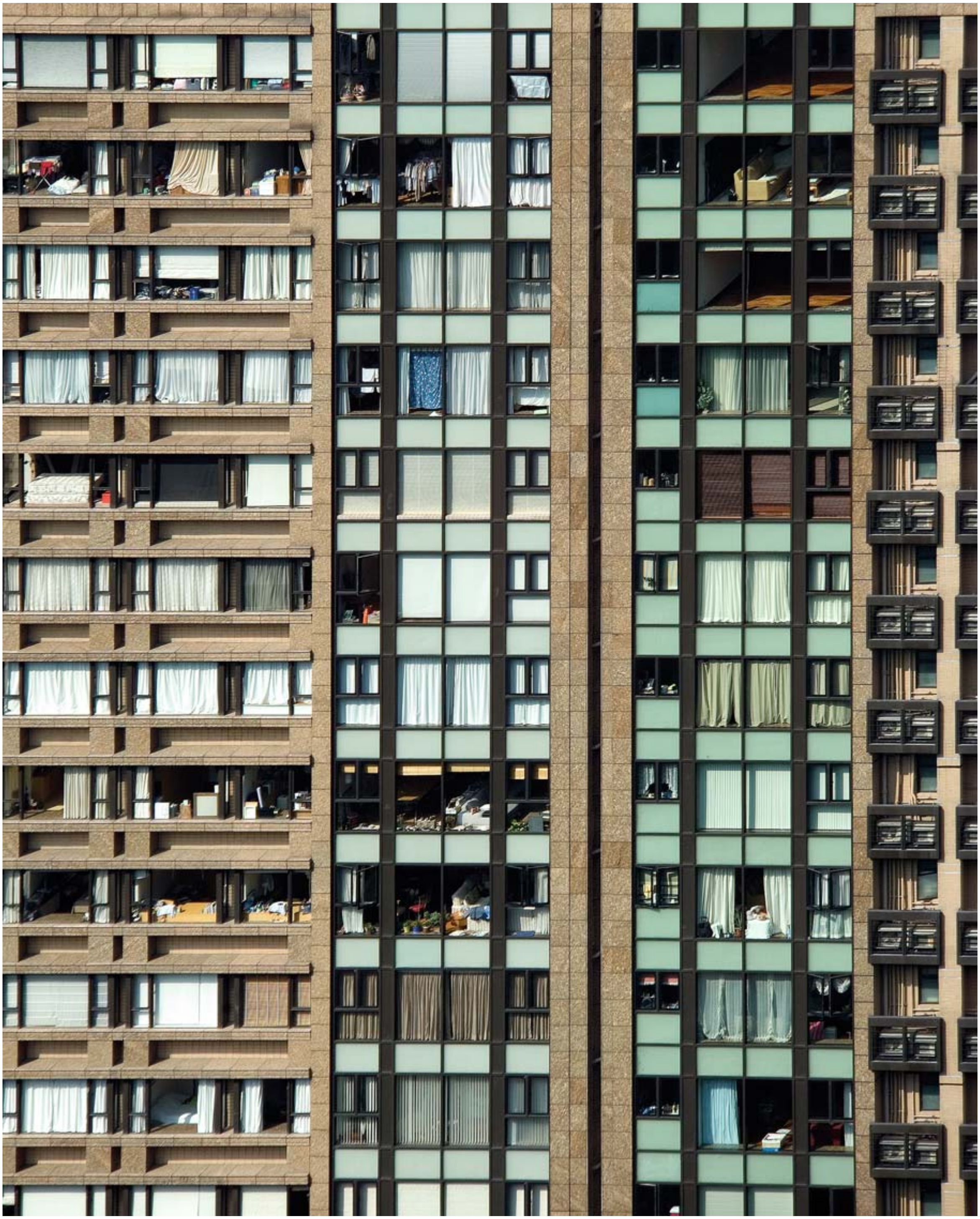

Fig. 11.3. Hongkong: in the thin $w = O(1)$-part of the Pareto distribution

tial differential equation of Fokker–Planck type for the distribution of money among individuals. This diffusion-convection equation reads:

$$\frac{\partial f}{\partial t} = \frac{\lambda}{2}\frac{\partial^2}{\partial v^2}\left(v^2 f\right) + \frac{\partial}{\partial v}\left((v-m)f\right) . \tag{11.8}$$

In (11.8) m is the mean wealth,

$$m = \int_{\mathbb{R}_+} v f(v,t)\, dv ,$$

which is time-conserved assuming that f has been scaled to be a probability density. The same Fokker–Planck equation was obtained in [2] as the mean field limit of a stochastic equation, as well as in [9, 14] in the context of generalized Lotka–Volterra dynamics.

The equilibrium state of the Fokker–Planck equation can be computed explicitly and is of Pareto type, namely it is characterized by a power-law tail for the richest individuals. By assuming for simplicity $m = 1$ we find:

$$f_\infty(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{\exp\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}} \tag{11.9}$$

where

$$\mu = 1 + \frac{2}{\lambda} > 1 .$$

We remark that the tails of the Pareto steady state of the Fokker–Planck equation are related to the coefficients s and σ^2 which appear in the collision rule (11.6), with $\sigma^2/s = \lambda$!

Another important field in which microscopic kinetic models describing the collective behavior and self-organization in a society [16] can be fruitfully employed is the modeling of opinion formation (cfr. [1, 13, 15] and the references therein).

In these studies, formation of opinion is described by mean field model equations. They are in general systems of ordinary differential equations or partial differential equations of diffusive type. In [1], attention is focused on two aspects of opinion formation, which in principle could be responsible for the formation of coherent structures. The first one is the remarkably simple compromise process, in which pairs of agents reach a fair compromise after exchanging opinions. The second one is a diffusion process, which allows individual agents to change their opinions in a random diffusive fashion. While the compromise process has its basis in the human tendency to settle conflicts, diffusion accounts for the possibility that people may change opinion through access to information. At present

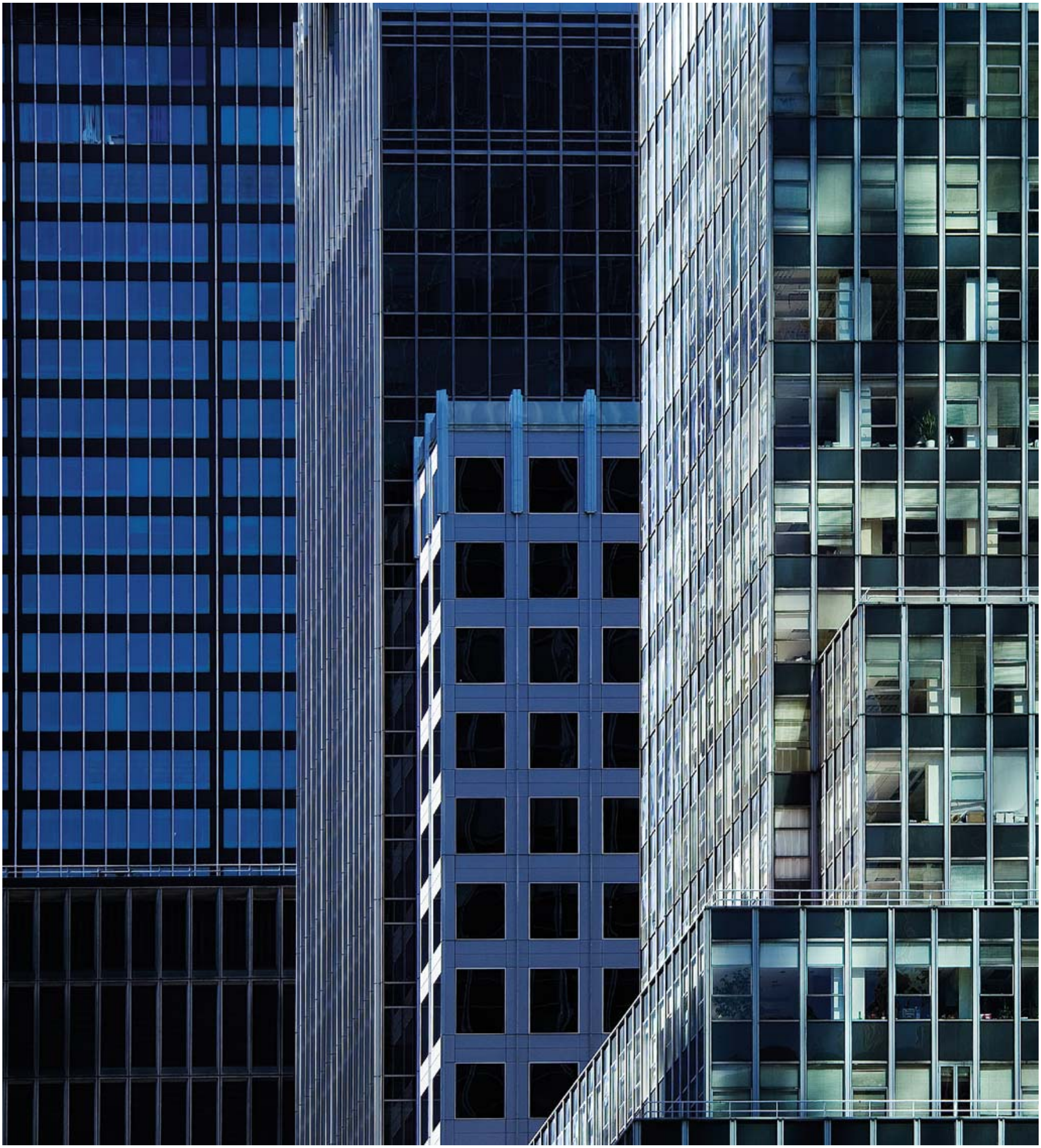

Fig. 11.4. Manhattan, New York: in the fat Pareto tail


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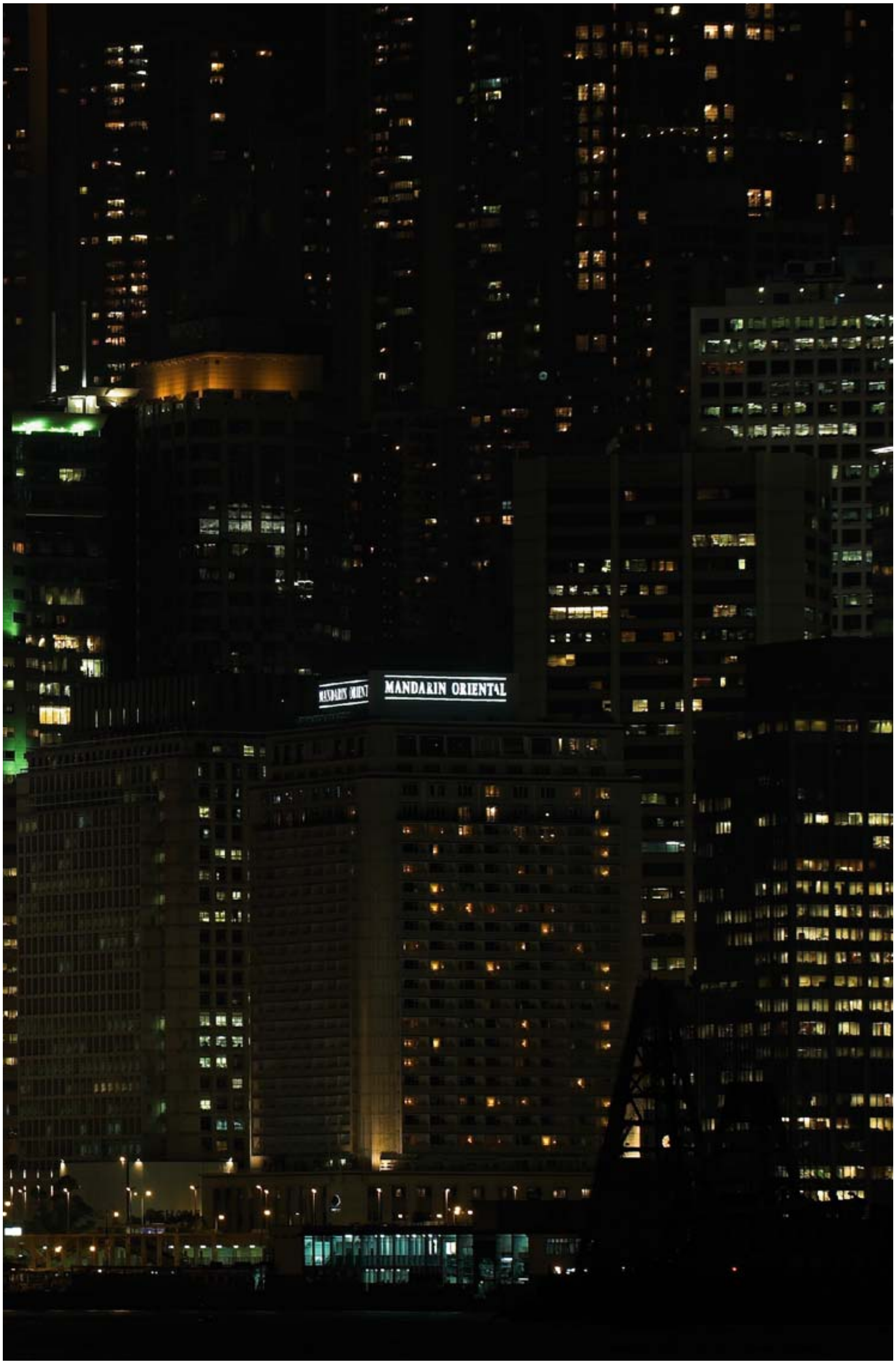

Fig. 11.5. Hongkong, China: where the fat Pareto tail is made

Fig. 11.6. Shanghai, China: some are left behind on the $w = o(1)$-part of the Pareto distribution (courtesy of Andrea Baczynski)

this aspect is gaining importance due to emerging new ways of global access to and exchange of information (among them electronic mail and web navigation).

This line of thought is at the basis of kinetic models of opinion formation [15], based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals.

The goal of kinetic models of opinion formation is to describe the evolution of the distribution of opinions in a society by means of *microscopic* interactions between agents which exchange information. To fix ideas, we associate opinion with a variable which varies continuously from -1 to 1, where -1 and 1 denote the two (extreme) opposite opinions. We assume binary interactions which are such that the bounds of the admissible opinion-interval are maintained. This crucial rule emphasizes the difference between *social* interactions, where not all interaction outcomes are permitted, and collisions of molecules in the kinetic theory of rarefied gases.

Let $\mathbb{I} = [-1, +1]$ denote the interval of admissible opinions. From a microscopic view point, a binary interaction is described by

$$\begin{aligned} v^* &= v - sP(|v|)(v - w) + \eta D(|v|) ; \\ w^* &= w - sP(|w|)(w - v) + \tilde{\eta} D(|w|) , \end{aligned} \tag{11.10}$$

where the pair (v, w), denotes the opinions of two arbitrary individuals before the interaction and (v^*, w^*) their opinions after exchanging information between them and with the exterior world. Opinions are not allowed to cross boundaries, and thus the interaction takes place only if both $v^*, w^* \in \mathbb{I}$. In (11.10) the coefficient $s \in (0, 1/2)$ is a given constant (the analogue of the saving rate in (11.5), while η and $\tilde{\eta}$ are equally distributed random variables with variance σ^2 and zero mean. The constant s and the variance σ^2 measure the compromise propensity and, respectively, the modification of opinion due to diffusion. Finally, the functions $P(\cdot)$ and $D(\cdot)$ describe the local relevance of the compromise and diffusion for a given opinion.

In analogy to kinetic modeling of market economies, the binary interactions (11.10) are used to construct a Boltzmann-like equation similar to (11.1), where now

$$\int_{\mathbb{I}} Q(f, f)(v)\phi(v)\, dv =$$

$$\frac{1}{2} \left\langle \int_{\mathbb{I}} \int_{\mathbb{I}} (\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)) f(v) f(w) dv\, dw \right\rangle . \tag{11.11}$$

A suitable asymptotic analysis allows to obtain a Fokker–Planck equation with variable coefficients from this Boltzmann equation [15]:

$$\frac{\partial f}{\partial t} = \frac{\lambda}{2} \frac{\partial^2}{\partial v^2} \left(D(|v|)^2 f \right) + \frac{\partial}{\partial v} \left(P(|v|)(v - m(t)) f \right) . \tag{11.12}$$

In (11.12) $m(t)$ is the mean opinion at time t,

$$m(t) = \int_{-1}^{+1} v f(v,t)\, dv \,.$$

The long-time behavior of the Fokker–Planck equation is very rich, and depends on the interaction dynamics of the Boltzmann equation. As for economic interactions, the constant λ in the Fokker–Planck equation (11.12) is related to the coefficients s and σ^2 which appear in the collision rule (11.10), with $\sigma^2/s = \lambda$. The structure of the steady state represents the formation of opinion contingent to the choice of the interaction dynamics. To show results in some simple case, we fix $P(|v|) = 1$, which implies conservation of the average opinion, again assuming that f has been scaled to be a probability density and that f and D vanish at the extreme opinions $v = +1, -1$. If in addition

$$D(|v|) = 1 - v^2 \,,$$

then the steady state distribution of opinion solves the equation

$$\frac{\lambda}{2} \frac{\partial}{\partial v} \left((1 - v^2)^2 f \right) + (v - m) f = 0 \qquad (11.13)$$

where m is a given constant (the average initial opinion). The solution of (11.13) is easily found:

$$f_\infty(w) = c_{m,\lambda} (1 + v)^{-2+m/(2\lambda)} (1 - v)^{-2-m/(2\lambda)} \exp\left\{ -\frac{1 - mv}{\lambda(1 - v^2)} \right\} \,. \qquad (11.14)$$

Here the constant $c_{m,\lambda}$ has to be fixed such that the mass of f_∞ is equal to the mass of the initial state, which is 1 by assumption, implying $-1 < m < +1$. Note that the presence of the exponential assures that $f_\infty(\pm 1) = 0$. The solution is regular, but not symmetric unless $m = 0$. Hence, the initial opinion distribution impacts on the steady state through its mean (opinion) value. In any case, the stationary distribution has two peaks (on the right and on the left of zero) with intensities depending on λ.

Comments on the Images 11.1 to 11.6 The Italian political economist Vilfredo Pareto (1848–1923)[2] is the originator of the so called empirical Pareto law[3] which in a simplified form states that – in any given country – less than 20% of the population own 80% of the total wealth. Although this was not considered a moral issue by Pareto himself, it is very hard not to think of morality when traveling through third world countries and being in direct contact with the huge number

[2] *http://cepa.newschool.edu/het/profiles/pareto.htm*

[3] *http://www.it-cortex.com/Pareto_law.htm*

Fig. 11.7. Opinion forming in Shanghai, China (courtesy of Andrea Baczynski)

of people, who are not part of the large-income Pareto tail. Mathematically speaking, a more general form of the Pareto law is represented by the fact that the large w(ealth)-tails, which correspond to the density of the rich individuals, of the large-time asymptotic states of the kinetic Pareto–Boltzmann equation (at least after an appropriate scaling limit) decay only algebraically as the wealth variable w tends to infinity, leading to so-called *heavy* or *fat* tails. The precise decay rate depends on properties of the market under consideration. Statistical data confirm the 80–20 wealth distribution rule as a surprisingly universal outcome, consistent with the algebraic decay law.

Also we remark that Pareto's work on efficiency and optimality of economic systems[4] has deep implications on mathematical game theory[5], which was turned into a precise mathematical theory in the first half of the 20th century, mainly by John von Neumann[6] and John Nash[7]. We refer to the book [17] for an excellent introduction to mathematical game theory, mainly in the context of biological systems.

Comments on the Images 11.7–11.10 Mathematical opinion formation models are based on quantifying the outcome of social interactions in the society under

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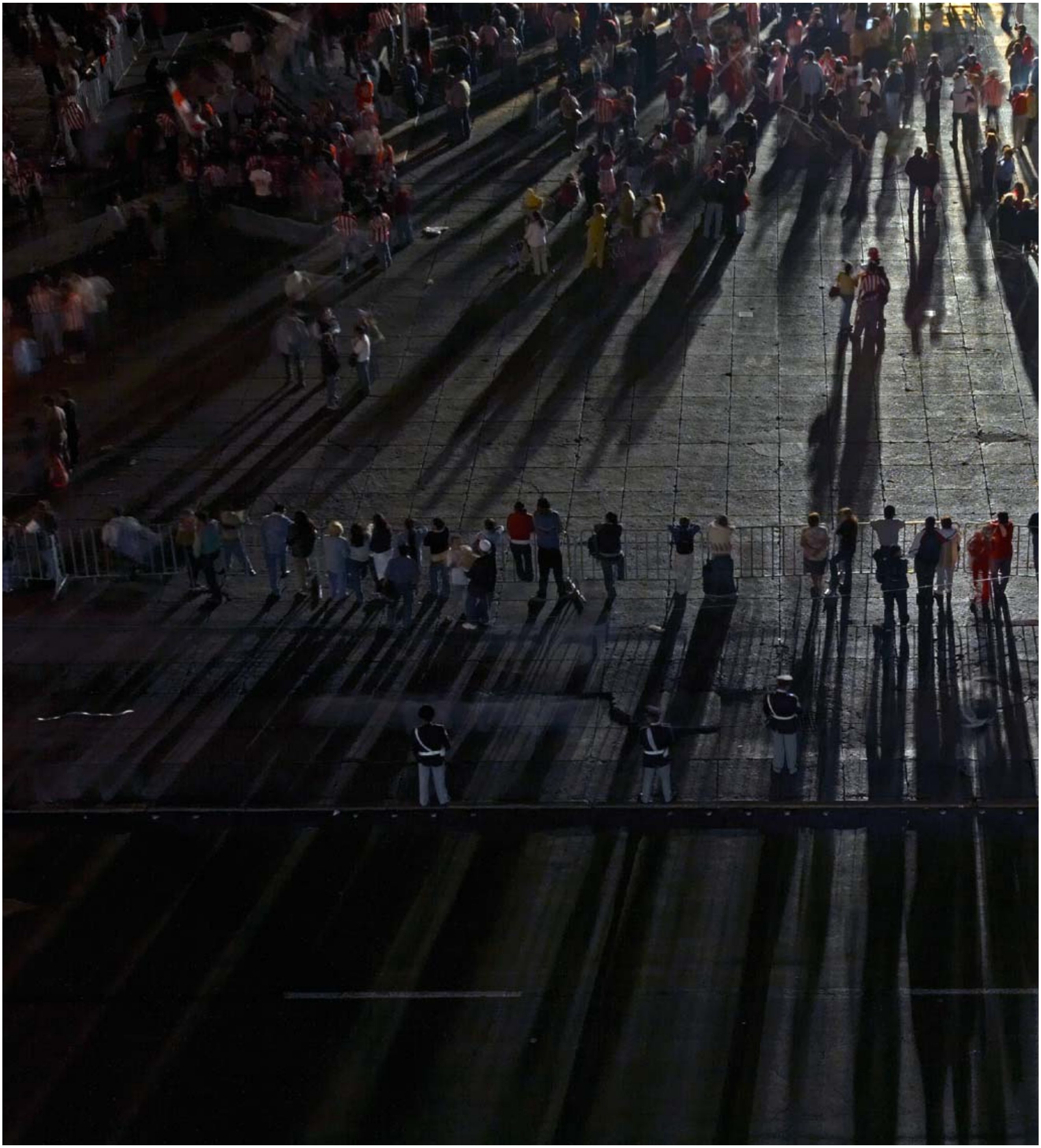
4 *http://en.wikipedia.org/wiki/Pareto_efficiency*
5 *http://en.wikipedia.org/wiki/Game_theory*
6 *http://en.wikipedia.org/wiki/John_von_Neumann*
7 *http://nobelprize.org/economics/laureates/1994/nash-autobio.html*

consideration. Clearly, they have to take into account the various factors making up the social tissue of the society, which stem from the historical, religious, socio-economic, political etc. background. A lot of research in this direction has been carried out in the last years, and the interested reader can find information on the subject in the webpage of the Condensed Matter ArXiv[8].

8 *http://xxx.lanl.gov/find/cond-mat*



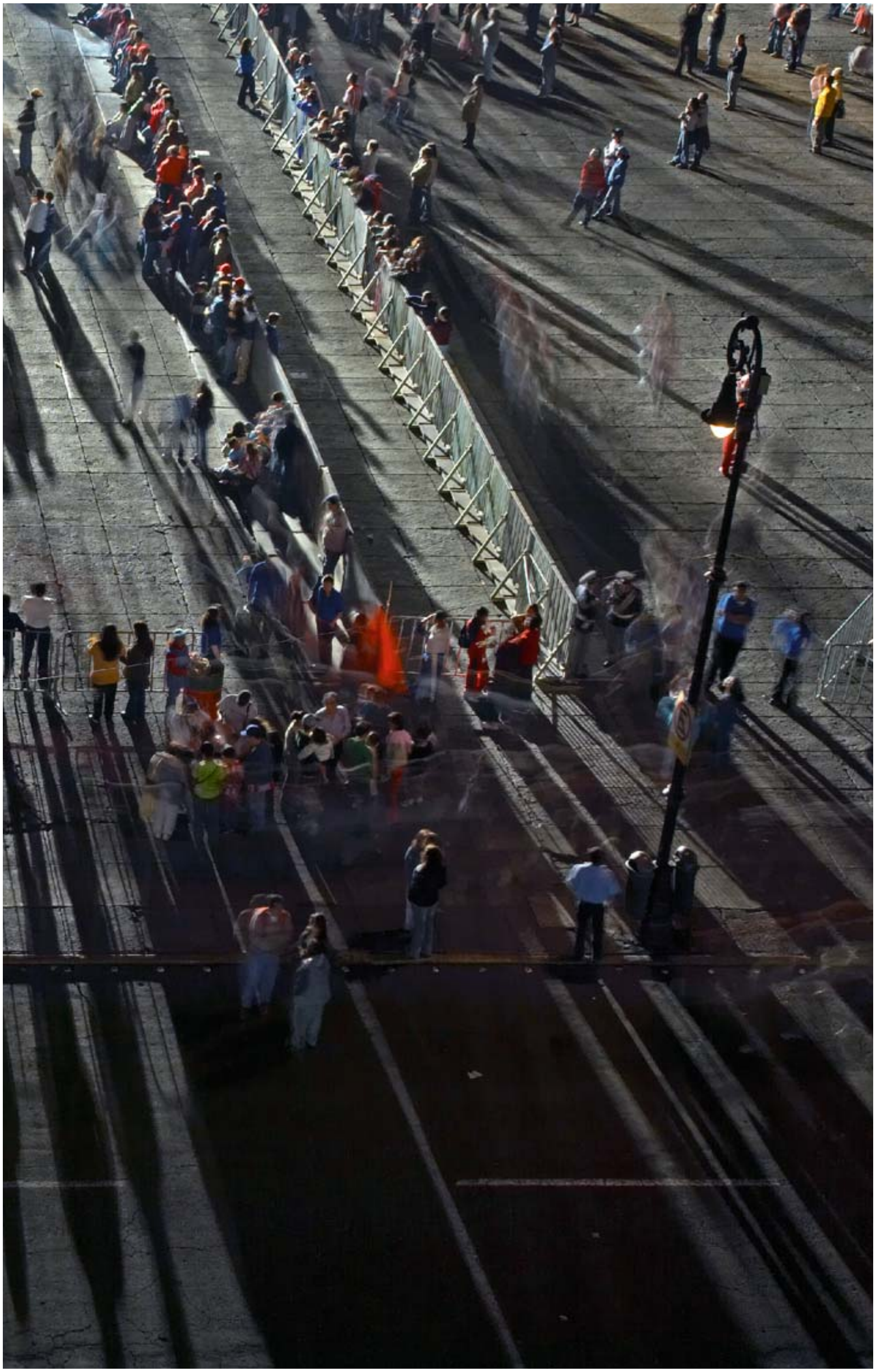


Fig. 11.8. Opinion forming on the Zocalo, Mexico City

Fig. 11.9. Opinion forming in Isfahan, Iran

Fig. 11.10. Beach in Salvador de Bahia, Brazil: what is the mean free path?

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[9] can be downloaded from *http://www-dimat.unipv.it/toscani/*