On the Stability of Reconstructing Lattice Sets from X-Rays Along Two Directions

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Abstract. We consider the stability problem of reconstructing lattice sets from their noisy X-rays (i.e. line sums) taken along two directions. Stability is of major importance in discrete tomography because, in practice, these X-rays are affected by errors due to the nature of measurements. It has been shown that the reconstruction from noisy X-rays taken along more than two directions can lead to dramatically different reconstructions. In this paper we prove a stability result showing that the same instability result does not hold for the reconstruction from two directions. We also show that the derived stability result can be carried over by similar techniques to lattice sets with invariant points.

1 Introduction

The main problem in discrete tomography is to reconstruct lattice sets (or equivalently, binary pictures) from their X-rays, that is, from the number of its points lying along lines parallel to any set of prescribed directions. In practice, these X-rays are affected by noise. In [3] the authors investigate the stability of the problem of reconstruction from X-rays taken from $m \geq 3$ lattice directions. In this case they show that a small change in the data (of $2(m-1)$ measured in the ℓ_1 -norm) can lead to a dramatic change in the reconstruction. This instability persists even when the original is uniquely determined by its exact X-rays. This paper addresses the open question whether the reconstruction of lattice sets from X-rays along two directions is also an unstable task.

In this paper we analyze the case $m = 2$ with the same kind of data errors as in [3], i.e., where $2(m-1) = 2$. We show in Theorem 17 that lattice sets which are uniquely determined by their X-rays along two directions can only have stable reconstructions even if the X-rays are changed by 2 (in the ℓ_1 -norm). It still remains open what happens if there is a higher perturbation in the data. In any case, we show that a similar combinatorial reasoning leads to provable stability results (Theorem 25) when the requirement of uniqueness on the original set is weakened to the assumption that in every reconstruction there exists a suitable number of invariant points.

Other kinds of mathematically proved stability results can be found in [1], [2], [5] and [6]. Reports on the stable behavior of reconstruction algorithms which incorporate a-priori knowledge can be found in [11] and [15]. For details on discrete tomography we refer to [8]. More about $(0, 1)$ -matrices can be found in [4]. Invariant sets have been intensively studied by many authors (e.g. [7],[9],[13],[14]).

2 Notations and Statement of the Problem

The elements of $\mathcal{F}^2 = \{F \subset \mathbb{Z}^2 : F \text{ is finite}\}$ are called *lattice sets*, and the subspaces $\text{lin}\{v\} \subseteq \mathbb{R}^2$, $v \in \mathbb{Z}^2 \setminus \{0\}$ are called *lattice directions*. For a lattice direction S , we define $\mathcal{A}(S) = \{w + S : w \in \mathbb{Z}^2\}$. The (*discrete*) *X-ray* of $F \in \mathcal{F}^2$ parallel to S is the function $X_SF : \mathcal{A}(S) \rightarrow \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ defined by $X_SF(T) = |F \cap T|$, for each $T \in \mathcal{A}(S)$.

In this paper we always consider X-rays from two directions S_1, S_2 . By affine transformations of the grid we can assume in the following that $S_1 = \text{lin}\{(1, 0)\}$ and $S_2 = \text{lin}\{(0, 1)\}$, thus we speak of X-rays along horizontal and vertical lines, or even about row and column sums. It is clear that the X-rays can be assembled as a vector. Let two lattice sets F_1, F_2 be given. We will denote its vector containing the row sums by R_i and the vector containing the column sums by C_i , $i = 1, 2$, respectively. Notice that we assume that the i -th entries of R_1 and R_2 refer to the same (horizontal) line, which can be achieved by inserting zero-entries. We make the same assumption on the entries of C_1 and C_2 .

For $F_1 \in \mathcal{F}^2$ let $\mathcal{F}^2(F_1) := \{F_2 \in \mathcal{F}^2 : |F_1| = |F_2| \wedge \|R_1 - R_2\|_1 + \|C_1 - C_2\|_1 = 2\}$. So, we focus on the question: *Let $F_1 \in \mathcal{F}^2$, uniquely determined by R_1 and C_1 . What is the sharpest upper bound on $\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \Delta F_2|$?*

3 Preliminary Remarks

Since $F_1 \in \mathcal{F}^2$ is finite, it is contained in a rectangle whose rows and columns are non-empty. We refer to them also as the rows and the columns of F_1 . If p is a point of F_1 lying in the i th row and j th column, we write that $\text{row}(p) = i$ and $\text{col}(p) = j$. Sometimes we do not distinguish between the row itself and its index. We use the convention that the rows and columns are numbered starting from its left-upper corner (see Figure 1 (a)).

We can change the coordinates of each point in the rectangle by first permuting the columns so that C_1 is a non increasing vector, and then permuting the rows so that R_1 is also a non increasing vector. This is a one-to-one function on the points of the rectangle yielding a triangular shape when the set is uniquely determined by its X-rays (it is a *maximal matrix* with non increasing row sums, see [12]). So, we shall assume that F_1 has a triangular shape, like in Figure 1 (a).

The following remarks concern the X-ray errors. Proposing an X-ray error of 2 means that the error occurs on exactly two lines of a single direction. This follows from the assumption $|F_1| = |F_2|$. Indeed the sum of the X-ray values according to one direction equals the cardinality of the set to be reconstructed,

Fig. 1. An illustration of sets F_1 (black points) and F_2 (white points). Points of $F_1 \cap F_2$ are colored half white and half black

and so if there is exactly one line where the error is -1 (which will in the following mean that there is one point more of $F_2 \setminus F_1$ than of $F_1 \setminus F_2$ on the line), then there exists exactly one line with a $+1$ -error (meaning that there is one point more of $F_1 \setminus F_2$ than of $F_2 \setminus F_1$ on the line). Furthermore, by possibly rotating F_1 and F_2 we may assume that the error occurs along horizontal lines (rows). If F_2 contains a single point on a horizontal line, then we can assume without loss of generality that it is row 1.

Remark 1. In summary, we assume in the rest of the paper that F_1 is a maximal matrix with sorted rows (or equivalently, row and column sum vectors are monotone and uniquely determine F_1), $|F_1| = |F_2|$, and the error occurs in exactly two rows (consequently in no columns).

Remark 2. Let p be any point of $F_2 \setminus F_1$ and let q be any point of $F_1 \setminus F_2$.

- (a) From the shape of F_1 we have:
  - if $col(p) = col(q)$, then $row(p) > row(q)$;
  - if $row(p) = row(q)$, then $col(p) > col(q)$.
- (b) Since no column error occurs: the point p exists if and only if the point q exists, with $col(p) = col(q)$. Similarly, in the rows in which no error occurs: p exists if and only if q exists, with $row(p) = row(q)$.

Lemma 3. *Let i be the row where the -1 error occurs, and let j be the row where the $+1$ error occurs. Then $i > j$.*

Proof. Suppose that $i < j$; then we show that there is an infinite sequence of points in $F_1 \triangle F_2$ starting with a point p of $F_2 \setminus F_1$ in the i th row. Since the -1 error occurs in the i th row, at least a point p of $F_2 \setminus F_1$ exists in this row. By Remark 2 (a) and (b), for any p , a point q of $F_1 \setminus F_2$ exists such that $col(p) = col(q)$ and $i = row(p) > row(q)$. Since there is no error in any row with index less than i , we conclude with Remark 2 (a) that there exists also a point

p' of $F_2 \setminus F_1$ in $\text{row}(q)$, then again a point q' of $F_1 \setminus F_2$, etc., all of which lie in a row with index less than i . This leads to an infinite sequence of points in $F_1 \triangle F_2$ which is not possible. $\square$

3.1 Staircases

Definition 4. Let $A, B \in \mathcal{F}^2$ with $A \cap B = \emptyset$. A staircase $T = (t_1, \dots, t_m)$ according to the columns is a sequence of an even number $m > 0$ of distinct points $t_{2i+1} \in A$ and $t_{2i+2} \in B$ for $0 \leq i \leq \frac{m}{2} - 1$ such that

- (i) $\text{col}(t_{2i+1}) = \text{col}(t_{2i+2})$ and $\text{row}(t_{2i+1}) > \text{row}(t_{2i+2})$ for $0 \leq i \leq \frac{m}{2} - 1$;
- (ii) $\text{row}(t_{2i}) = \text{row}(t_{2i+1})$ and $\text{col}(t_{2i}) < \text{col}(t_{2i+1})$ for $1 \leq i \leq \frac{m}{2} - 1$.

The definition of staircase according to the rows is obtained by exchanging the words “row” by “column” and “A” by “B”.

A staircase can be interpreted geometrically. It is a rook path of alternating points of A and B with end points t_1 and t_m . We refer to (t_{2i}, t_{2i+1}) as a *horizontal step* and to (t_{2i+1}, t_{2i+2}) as a *vertical step* of T . Since A and B are finite, every staircase has a finite number of points. A staircase which is not a proper subset of another staircase is called a *maximal staircase*. Below we will show that there is only one single maximal staircase if $A = F_2 \setminus F_1$ and $B = F_1 \setminus F_2$. The points of A will be called *white points*, and the points of B will be called *black points*.

4 Technical Lemmas

In the following we will speak about staircases in $F_1 \triangle F_2$, which implicitly means that $A = F_2 \setminus F_1$ and $B = F_1 \setminus F_2$. Now, we are going to show that the symmetric difference of F_1 and F_2 is a maximal staircase in $F_1 \triangle F_2$.

Remark 5. Notice that for every $p \in F_1 \triangle F_2$ there exists a staircase $T = (t_1, \dots, t_m)$ (possibly constituted by two points) such that p is an element of T . This can be easily deduced from Remark 2.

Lemma 6. Any two maximal staircases in $F_1 \triangle F_2$ have the same starting point and the same end point.

Proof. Let $T_1 = (t_1, \dots, t_m)$ and $T_2 = (s_1, \dots, s_n)$ be any two maximal staircases in $F_1 \triangle F_2$. We are going to show that $t_1 = s_1$ and $t_m = s_n$. Since the -1 error occurs in exactly one row (say i), it follows that $\text{row}(t_1) = \text{row}(s_1) = i$. If $\text{col}(t_1) \neq \text{col}(s_1)$, a black point, say q , exists in the i th row. By Remark 2 a white point p exists such that $\text{col}(p) = \text{col}(q)$ and $\text{row}(p) > i$. Since there is no error in any row $k > i$ this leads to an infinite sequence of points in $F_1 \triangle F_2$.

Analogously, since the $+1$ error occurs in exactly one row (say j), we have $\text{row}(t_m) = \text{row}(s_m) = j$, and $\text{col}(t_m) = \text{col}(s_n)$ because otherwise (with Remark 2) there is an infinite sequence of black and white points in $F_1 \triangle F_2$. $\square$

Remark 7. By Lemma 6 every maximal staircase starts in a white point t_1 and ends in a black point t_m . So, there is exactly one white point in $\text{row}(t_1)$ and $\text{col}(t_1)$ and one black point in $\text{row}(t_m)$ and $\text{col}(t_m)$. Moreover there is no black or white point outside the rectangle which is made up of the rows between $\text{row}(t_1)$ and $\text{row}(t_m)$, and the columns between $\text{col}(t_1)$ and $\text{col}(t_m)$.

Lemma 8. *Any two maximal staircases in $F_1 \triangle F_2$ coincide.*

Proof. Let $T_1 = (t_1, \dots, t_m)$ and $T_2 = (s_1, \dots, s_n)$ be any two maximal staircases according to the columns in $F_1 \triangle F_2$. We are going to show that $m = n$ and $t_i = s_i$ for $i = 1, \dots, m$. Since $t_1 = s_1$ there is no white point other than t_1 in $\text{col}(t_1)$ by Remark 7. It follows that exactly one black point lies on this column, that is, $t_2 = s_2$. Analogously, we can conclude that $t_{m-1} = s_{n-1}$. Consider now $T_1 \setminus \{t_1, t_m\}$ and $T_2 \setminus \{s_1, s_n\}$. They are two staircases according to the rows. Proceeding as before, we conclude that $t_3 = s_3$ and $t_{m-2} = s_{n-2}$. We repeat the procedure alternatively on a staircase according to the rows and one according to the columns until an empty set is achieved. So, $t_i = s_i$ for $i = 1, \dots, m$. $\square$

The previous lemmas prove the following proposition.

Proposition 9. *The points of $F_1 \triangle F_2$ constitute a maximal staircase.*

5 Bounds

In this section we give an upper bound for the number of points on any maximal staircase, when we fix F_1 but may vary F_2 . This gives a sharp bound on $|F_1 \triangle F_2|$, since the maximal staircase contains exactly the points of $F_1 \triangle F_2$.

Let $T = (t_1, \dots, t_m)$ denote this staircase, and let $\mathcal{R} = \{1, \dots, a\} \times \{1, \dots, b\}$ be the rectangle containing F_1 having non-empty rows and columns. Clearly, there is at most one point t_1 of T outside of $\mathcal{R}$, and for this point we have $1 \leq \text{col}(t_1) \leq b$, while all the other points of T are inside of $\mathcal{R}$. Without loss of generality we can assume that all points of $F_2 \subseteq \{0, \dots, a\} \times \{1, \dots, b\}$. So, we have $R_1 \in \mathbb{N}_0^a$ and $C_1 \in \mathbb{N}_0^b$.

Proposition 10. *Let $R_1 \in \mathbb{N}_0^a$ and $C_1 \in \mathbb{N}_0^b$ uniquely determine $F_1 \in \mathcal{F}^2$. Then,*

$$\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| \leq 2 \min(a+1, b).$$

Proof. Since $F_1 \triangle F_2$ forms a staircase T , it is immediately clear that an upper bound for the symmetric difference can be obtained by counting two times the number of vertical steps in T that, in turn, is less than b . But another bound is given by adding 2 (for including t_1 and t_m) to two times the number of horizontal steps. Since t_1 is the only point that can lie outside $\mathcal{R}$, we have $\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| \leq \min(2a+2, 2b) = 2 \min(a+1, b)$. $\square$

Proposition 11. *Under the same hypothesis of Proposition 10, let l be the number of pairwise different row sums of F_1 . Then,*

$$\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| \leq 2l.$$

Proof. Clearly, $|T|$ equals two times the number of vertical steps in T . Since by definition of staircase, $\text{row}(t_{2i}) = \text{row}(t_{2i+1})$ and $\text{col}(t_{2i+2}) = \text{col}(t_{2i+1})$ with $t_{2i}, t_{2i+2} \in F_1 \setminus F_2$ and $t_{2i+1} \in F_2 \setminus F_1$, we know that the number of points of F_1 in $\text{row}(t_{2i})$ is less than the number of points of F_1 in $\text{row}(t_{2i+2})$. So the maximal number of vertical steps in any staircase T for F_1 equals l . $\square$

The next two lemmas provide lower bounds to the symmetric difference of F_1 and F_2 . These are used later on to show that the derived bounds are sharp.

Lemma 12. *For every $n \in \mathbb{N}$ there exist $F_1, F_2 \in \mathcal{F}^2$ with $|F_1| = |F_2| = \frac{1}{2}n(n+1)$ such that $F_1 \triangle F_2$ is a staircase with $2n$ points.*

Proof. Taking the sets $F_1 = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq n+1-i\}$ and $F_2 = \{(i, j) : 1 \leq i \leq n-1, 1 \leq j \leq n-i\} \cup \{(i, n+2-i) : 2 \leq i \leq n+1\}$ one can easily verify that the desired properties are fulfilled. $\square$

Lemma 13. *For every $k, n \in \mathbb{N}$ with $0 < k < n$, there exist $F_1, F_2 \in \mathcal{F}^2$ with $|F_1| = |F_2| = \frac{1}{2}n(n-1) + k$ such that $F_1 \triangle F_2$ is a staircase with $2(n-1)$ points.*

Proof. We define (see Figure 1 (b))

$$\begin{aligned} S_1 &= \{(i, j) : 1 \leq i \leq n-2, 1 \leq j \leq n-i-1\}, S_2 = \{(i, n-i) : 1 \leq i \leq k\}, \\ B_1 &= \{(i, n-i+1) : 1 \leq i \leq k\}, B_2 = \{(i, n-i) : k+1 \leq i \leq n-1\}, \\ W_1 &= \{(i+1, n-i+1) : 1 \leq i \leq k\}, W_2 = \{(i+1, n-i) : k+1 \leq i \leq n-1\}. \end{aligned}$$

Then, $F_1 = S_1 \cup S_2 \cup B_1 \cup B_2$ and $F_2 = S_1 \cup S_2 \cup W_1 \cup W_2$ are sets with $|F_1| = |F_2| = \frac{1}{2}n(n-1) + k$. It is easy to see that F_1 has a triangular shape. The points $F_1 \triangle F_2 = B_1 \cup B_2 \cup W_1 \cup W_2$ form a staircase with $2(n-1)$ points, namely $T = (p_1, q_2, p_2, q_2, \dots, p_{n-2}, q_{n-2})$ with

$$p_i = \begin{cases} (n+1-i, i) \in W_2 & : 1 \leq i \leq n-k-1 \\ (n-i+1, i+1) \in W_1 & : n-k \leq i \leq n-1 \end{cases}$$

and

$$q_i = \begin{cases} (n-i, i) \in B_2 & : 1 \leq i \leq n-k-1 \\ (n-i, i+1) \in B_1 & : n-k \leq i \leq n-1. \end{cases} \quad \square$$

The next lemma is used in the following for bounding the number of different consecutive row sums for a given set F_1 .

Lemma 14. *For any $n+j$ integers $r_1 \geq \dots \geq r_{n+j} \geq 1$ with $n \in \mathbb{N}$, $j \in \mathbb{N}_0$ and*

- (i) $j \geq 1$ and $\sum_{i=1}^{n+j} r_i = \frac{1}{2}n(n+1)$, or
- (ii) $j = 0$ and $\sum_{i=1}^{n+j} r_i < \frac{1}{2}n(n+1)$

there are at most $n-1$ pairwise different r_i 's.

Proof. Suppose there are more than $n - 1$ pairwise different r_i 's, which means that in $r_1 \geq \dots \geq r_{n+j}$ we have at least $n - 1$ times a strict inequality. This implies $r_i \geq n - i + r_{n+j}$ for $1 \leq i \leq n - 1$, and $r_i \geq r_{n+j}$ for $n \leq i \leq n + j$. Summation yields the contradiction

$$\begin{aligned} \sum_{i=1}^{n+j} r_i &\geq (n+j)r_{n+j} + n(n-1) - \sum_{i=1}^{n-1} i \geq n+j + n(n-1) - \frac{1}{2}n(n-1) \\ &= \frac{1}{2}n(n+1) + j. \end{aligned} \quad \square$$

Lemma 15. *Let $F_1 \in \mathcal{F}^2$ with $|F_1| = \frac{1}{2}n(n+1)$ for an $n \in \mathbb{N}$. Then, we have $\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| = 2n$.*

Proof. By Lemma 12 we have $\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| \geq 2n$. If F_1 has $n+j$ non-empty rows, where $j \geq 1$, then we have by Lemma 14 (i) at most $n-1$ different consecutive row sums. This leads only, by Proposition 11, to a staircase with at most $2(n-1)$ points. If F_1 has less than $n+1$ non-empty rows, then we again conclude (by Proposition 10) that any staircase contains at most $2n$ points. $\square$

Lemma 16. *Let $F_1 \in \mathcal{F}^2$ with $\frac{1}{2}n(n-1) < |F_1| < \frac{1}{2}n(n+1)$ for an $n \in \mathbb{N}$. Then, $\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| = 2(n-1)$.*

Proof. Because of Lemma 13 we have $\max_{F_2 \in \mathcal{F}^2(F_1)} |F_1 \triangle F_2| \geq 2(n-1)$ for any F_1 with $\frac{1}{2}n(n-1) < |F_1| < \frac{1}{2}n(n+1)$. If F_1 has $n+j$ non-empty rows ($j \geq 0$), then we have, by Lemma 14 (ii), at most $n-1$ different row sums. Consequently, by Proposition 11, this leads to a staircases with at most $2(n-1)$ points. If F_1 has less than n non-empty rows, then we again conclude (by Proposition 10) that any staircase contains at most $2(n-1)$ points. $\square$

Now, we can summarize the results in the following theorem.

Theorem 17. *For any two lattice directions and any two sets $F_1, F_2 \in \mathcal{F}^2$ with the three properties*

- (i) F_1 is uniquely determined by its X-rays along the two prescribed directions;
- (ii) $|F_1| = |F_2|$;
- (iii) $\|R_1 - R_2\|_1 + \|C_1 - C_2\|_1 = 2$,

we have:

$$|F_1 \triangle F_2| \leq \begin{cases} 2n & : \text{ if } |F_1| = \frac{1}{2}n(n+1) \text{ with } n \in \mathbb{N} \\ 2(n-1) & : \text{ if } \frac{1}{2}n(n-1) < |F_1| < \frac{1}{2}n(n+1) \text{ with } n \in \mathbb{N}. \end{cases}$$

These bounds are sharp and imply

$$|F_1 \triangle F_2| \leq 2\sqrt{2|F_1| + \frac{1}{4}} - 1.$$

Proof. The first bound results from Lemma 15 and Lemma 16. The sharpness of the bounds follows from the constructions given in Lemma 12 and Lemma 13. The second bound follows when n is expressed in terms of $|F_1|$. $\square$

6 A Generalization to Lattice Sets with Invariant Points

In this section we study the stability problem under the weaker condition that the X-rays do not uniquely determine the set F_1 , but we have some "invariant" points.

Let $\mathcal{U}(R_1, C_1)$ denote the class containing lattice sets having row and column sum vectors R_1 and C_1 . The class is *normalized* if R_1 and C_1 are monotone. If R_1 and C_1 do not determine F_1 , then more than one set belongs to $\mathcal{U}(R_1, C_1)$. In this context it is meaningful to study the case where $\mathcal{U}(R_1, C_1)$ has some *invariant points* (these are points belonging to every set in $\mathcal{U}(R_1, C_1)$, or to none of these sets).

It is well-known ([4]) that the normalized class $\mathcal{U}(R_1, C_1)$ has invariant points if and only if the lattice sets in $\mathcal{U}(R_1, C_1)$ are of the form illustrated in Figure 2 (a). To be more precise, let $\mathcal{R} = \{1, \dots, a\} \times \{1, \dots, b\}$ be the rectangle containing F_1 ; there exist pairwise disjoint subsets $K_1, \dots, K_h \subseteq \{1, \dots, a\}$ and pairwise disjoint subsets $L_1, \dots, L_h \subseteq \{1, \dots, b\}$ such that $\mathcal{R} \setminus \bigcup_{u=1}^h K_u \times L_u$ contains only invariant points. E.g., the black points in the Figure 2 (a) are invariant points belonging to every set in $\mathcal{U}(R_1, C_1)$ (also called 1-invariant points), while the smaller dots indicate invariant points belonging to no set of $\mathcal{U}(R_1, C_1)$ (also called 0-invariant points).

Fig. 2. The illustrations used in Section 6

Remark 18. We assume that $\mathcal{U}(R_1, C_1)$ is normalized, $|F_1| = |F_2|$, and the X-ray error is 2, so again we assume that the error occurs in two rows.

Let p be any point of $F_2 \setminus F_1$ and let q be any point of $F_1 \setminus F_2$. Clearly, statement (a) and (b) of Remark 2 hold when p or q is not in $\bigcup_{u=1}^h K_u \times L_u$.

Remark 19. Consider a so called *non-trivial component* $K_u \times L_u$, $u \in \{1, \dots, h\}$, and suppose that no error occurs on the lines intersecting this component.

- Let $q \in F_1 \setminus F_2$ with $\text{row}(q) \in K_u$ and $\text{col}(q) \notin L_u$ (implying that q is an 1-invariant point). Then, there exists either one point $p' \in F_2 \setminus F_1$ such that $\text{row}(p') \in K_u$ and $\text{col}(p') \notin L_u$, or one point $q' \in F_1 \setminus F_2$ such that $\text{col}(q') \in L_u$ and $\text{row}(q') \notin K_u$. Otherwise we would obtain a contradiction to the assumption about the shape of $\mathcal{U}(R_1, C_1)$, and the assumption that there is no error in the rows and columns intersecting $K_u \times L_u$.
- Similarly, let $p \in F_2 \setminus F_1$ with $\text{row}(p) \in K_u$ and $\text{col}(p) \notin L_u$ (meaning that p is a 0-invariant point). Then, there exists either one point $q' \in F_1 \setminus F_2$ such that $\text{row}(q') \in K_u$ and $\text{col}(q') \notin L_u$, or one point $p' \in F_2 \setminus F_1$ such that $\text{col}(p') \in L_u$ and $\text{row}(p') \notin K_u$.

Suppose now that exactly one error occurs on a line intersecting $K_u \times L_u$. From the assumptions about the shape of $\mathcal{U}(R_1, C_1)$ there follows:

- If the -1 error occurs, then there exists $q' \in F_1 \setminus F_2$ such that $\text{col}(q') \in L_u$ and $\text{row}(q') \notin K_u$.
- If the $+1$ error occurs, then there exists $p' \in F_2 \setminus F_1$ such that $\text{row}(p') \in K_u$ and $\text{col}(p') \notin L_u$.

From the previous remark there easily follows that, if not both rows i (the row with -1 error) and j (the row with $+1$ error) intersect the same non-trivial component, then we have $i > j$ like in Lemma 3.

Definition 20. Let $A, B \in \mathcal{F}^2$ with $A \cap B = \emptyset$. An (A, B) -switching component (or a switching component for short) is a sequence of an even number $m > 0$ of distinct points $t_1, \dots, t_m$ such that $t_{2i+1} \in A$, $t_{2i+2} \in B$, $\text{col}(t_{2i+1}) = \text{col}(t_{2i+2})$ for $0 \leq i \leq \frac{m}{2}-1$, $\text{row}(t_{2i+2}) = \text{row}(t_{2i+3})$ for $0 \leq i \leq \frac{m}{2}-2$, and $\text{row}(t_1) = \text{row}(t_m)$.

Definition 21. Let $\mathcal{R}$ be a rectangle of size $a \times b$ containing the disjoint sets $A, B \in \mathcal{F}^2$, and let $K_1, \dots, K_h$ and $L_1, \dots, L_h$ be pairwise disjoint subsets of $\{1, \dots, a\}$ and $\{1, \dots, b\}$, respectively. An almost-staircase $T = (t_1, \dots, t_m)$ according to the columns in $\mathcal{R}$ is a sequence of an even number of $m > 0$ distinct points $t_{2i+1} \in A$, $t_{2i+2} \in B$, $0 \leq i \leq \frac{m}{2} - 1$ such that:

- (i) For every $i \in \mathbb{N}_0$ with $0 \leq i \leq \frac{m}{2} - 1$ it holds that $\text{col}(t_{2i+1}) = \text{col}(t_{2i+2})$ and, if $t_{2i+1} \notin \bigcup_{u=1}^h K_u \times L_u$ or $t_{2i+2} \notin \bigcup_{u=1}^h K_u \times L_u$, then $\text{row}(t_{2i+1}) > \text{row}(t_{2i+2})$;
- (ii) For every $i \in \mathbb{N}_0$ with $0 \leq i \leq \frac{m}{2} - 2$ it holds that $\text{row}(t_{2i+2}) = \text{row}(t_{2i+3})$ and, if $t_{2i+2} \notin \bigcup_{u=1}^h K_u \times L_u$ or $t_{2i+3} \notin \bigcup_{u=1}^h K_u \times L_u$, then $\text{col}(t_{2i+2}) < \text{col}(t_{2i+3})$;
- (iii) For every $i \in \mathbb{N}$ with $1 \leq i \leq m$ we have that t_i is no member of an (A, B) -switching component.

An almost-staircase is a staircase “almost everywhere” except that the properties in (i) and (ii) of Definition 4 are relaxed for points in components $K_u \times L_u$.

Remark 22. If for a staircase $T = (t_1, \dots, t_m)$ and a component $K_u \times L_u$ we have $\{t_1, \dots, t_m\} \cap (K_u \times L_u) = \{t_i, \dots, t_j\}$ with $i > 1$ and $j < m$ then, by definition of almost-staircases, the following cases can arise:

- If $t_i \in B$, then $t_{i-1} \in A$, and (a) $col(t_j) = col(t_{j+1})$ implies $t_{j+1} \in B$, whereas (b) $row(t_j) = row(t_{j+1})$ implies $t_{j+1} \in A$;
- If $t_i \in A$, then $t_{i-1} \in B$, and (c) $row(t_j) = row(t_{j+1})$ implies $t_{j+1} \in A$, whereas (d) $col(t_j) = col(t_{j+1})$ implies $t_{j+1} \in B$.

If $i = 1$, then either case (a) or (b) occurs, and if $j = m$, then either case (a) or (d) occurs. Figure 2 (b) illustrates the four configurations: the three kinds of rectangles represent $K_u \times L_u$.

The rectangle of configuration (b) in Figure 2 (b) is colored black because the number of black points inside is greater than the number of white points; rectangle of configuration (d) is colored white because the number of black points inside is smaller than the number of white points, and finally rectangles of configurations (a) and (c) have dotted edges because the number of white and black points inside is the same. Again, a *maximal almost-staircase* is an almost staircase which is no proper subset of another almost-staircase.

Lemma 23. *Any two maximal almost-staircases in $F_1 \triangle F_2$ have the same starting point and the same end point.*

Proof. We just stress the differences in the proof of Lemma 6 by using the same notations. The case that remains to be considered is the following: $row(t_1) = row(s_1) = i \in K_u \times L_u$, $col(t_1) > col(s_1)$, and a black point q on row i is in $K_u \times L_u$, and this point is not in a switching component (otherwise the error would be too large). Since there is no error on the columns a white point exists such that $col(p) = col(q)$ but we cannot claim that $row(p) > row(q)$. Anyway, points alternate each other such that this sequence visits a black point to the left of the rectangle or a white point to the bottom of the rectangle (leading to an infinite sequence), or it infinitely alternates within $K_u \times L_u$, or forms a switching component with q . In all cases, this is a contradiction to the assumptions. $\square$

Lemma 24. *Any two maximal almost-staircases in $F_1 \triangle F_2$ coincide.*

Proof. The proof follows like in Lemma 8. Indeed the case to analyze is that of $t_1 = s_1 \in K_u \times L_u$. One can easily show that if there is another white point in $col(t_1)$, then it belongs to the same $K_u \times L_u$, so proving that $t_2 = s_2$. Similarly, one deduces that $t_{m-1} = s_{n-1}$. This allows to apply the procedure used in the proof of Lemma 8. $\square$

Again, every $p \in F_1 \triangle F_2$ (outside of a switching component) is contained in an almost-staircase which is possibly constituted by two points (see Remark 19 and Remark 22). Because of the shape of F_1 and F_2 we also have that points in a switching component can only lie in a single component $K_u \times L_u$.

Let $\mathcal{P} := \{p \in F_1 \cup F_2 : p \text{ lies in an } (F_1, F_2) - \text{switching component}\}$ and $T_{\max}$ denote the maximal almost-staircase. Then, we can summarize the results as follows.

Theorem 25. *Let $F_i \in \mathcal{F}^2$ be given, with row sum vector R_i and column sum vector C_i , where $i = 1, 2$. Suppose the following properties are fulfilled*

- (i) $\mathcal{U}(R_1, C_1)$ has invariant points;
- (ii) $|F_1| = |F_2|$ and
- (iii) $\|R_1 - R_2\|_1 + \|C_1 - C_2\|_1 = 2$.

Then, $F_1 \triangle F_2 = T_{\max} \cup \mathcal{P}$.

This theorem can be used to obtain a bound (similar to Section 5) which depends only on $\sum_{u=1}^h |K_u \times L_u|$, R_1 , and on C_1 .

7 Conclusion

In this paper we have answered a question which was left open in [3]. Under the assumption of a small X-ray error and that the original set is uniquely determined by its X-rays, we proved that the reconstruction of lattice sets from two directions is a stable task. This is in contrast to its unstable counterparts, i.e., to the case of reconstruction from more than two directions. Additionally, we have shown that similar arguments can be carried over to the more general case where the original set has invariant points.

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