An Elementary Algorithm for Digital Line Recognition in the General Case

Lilian Buzer

A2SI Laboratory, ESIEE, 2 bd Blaise Pascal,
Cité Descartes, - BP 99, 93162 Noisy-Le-Grand Cedex, France
Buzerl@esiee.fr
Institut Gaspard Monge, Unité Mixte CNRS-ESIEE, UMR 8049

Abstract. This paper is concerned with the naive and, more generally, α -thick digital line recognition problem. Previous incremental algorithms deal with the 8-connected case [DR95] or with sophisticated machinery coming from Linear Programming [Buz03]. We present the first elementary method [Buz02] that works with any set of points (not necessarily 8-connected) and we propose a linear time algorithm under some restrictions (which were implicitly assumed in [DR95]). This paper deals with implementation details giving pseudo-code of our method. We insist on linking the recognition problem to the intrinsic properties of convex hulls.

Keywords: Digital line, incremental, recognition, convex hull, thickness, implementation.

1 Introduction

When one processes digital images, one also sometimes wants to know how to recognize basic geometric entities. In this way appears the recognition problem of digital lines of variable thickness (see Fig. 1). In this paper, we set up the definition of α -thick digital line which allows to represent a wide variety of digital line segments. We show that the knowledge of some basic information on the convex hull of a set of points S (thickness or critical supporting lines) is sufficient to determine if S is a subset of an α -thick digital line segment. Then, we adapt this remark in order to create an elementary algorithm for the recognition problem. We exhibit its optimal complexity relatively to many different configurations (static, incremental, dynamic). Thus, we propose a short implementation of the linear incremental version which works when points are inserted in a given direction. This new method and the previous one based on Megiddo algorithm [Buz03] are able to incrementally process α -thick digital line with non-connected pixels. In [Buz03] we only deal with naive digital line (where $\alpha = 1$), but this previous method is able to recognize α -thick digital line by replacing the thickness of 1 by α . Conversely, the new approach can operate naive digital line when we set α equal to 1. If the two algorithms recognize the same objects using the same thickness notion, the way they work is completely

different. The Megiddo approach does not explicitly compute the new thickness value when a point is inserted. We can only know if the current thickness is below a given threshold. With the new method, we are able to determine the current thickness value at any moment and to add other criteria that can be estimated from the thickness : density of pixels or curvature. This entails a higher complexity $O(n \log n)$ but in most cases we will work in configurations that allow to use a linear and elementary version of this new approach. Throughout this paper, we will denote by O_x and O_y the axes of the Cartesian coordinate system, by e_x and e_y the associated units vectors and by (x, y) the Cartesian components of points.

2 α -Thick Digital Line

We hereafter present α -thick digital line. Its seminal definition was given by Reveillès in [Rev91]. Any digital line D in $\mathbb{Z}^2$ is described by a set of parameters: the *normal vector* $N = (a, b)$ in $\mathbb{Z}^2 \setminus \{0\}$ with $\gcd(a, b)$ equal to 1, the *inferior bound* γ in $\mathbb{Z}$ and the *arithmetic thickness* w in $\mathbb{Z}$. A point (x, y) of a digital line with parameters $(a, b), \gamma, w$ verifies the following diophantine inequality:

$$\gamma \leq a.x + b.y < \gamma + w \tag{1}$$

If we choose the arithmetic thickness to be equal to the infinity norm of the normal vector: $\|N\|_\infty = \sup\{|a|, |b|\}$, we obtain an 8-connected object called a *naive digital line* (see Fig. 1). We extend this definition to the α -thick digital line where α corresponds to a thickness ratio whose reference ($\alpha = 1$) is the naive digital line. A point (x, y) belongs to such a line if it verifies:

$$\gamma \leq a.x + b.y < \gamma + \alpha. \sup\{|a|, |b|\} \tag{2}$$

We can consider three different subclasses of digital lines depending on the components of the normal vector $N = (a, b)$. When $|a| = |b| = 1$, the digital line has a slope of $\pm 45^\circ$. When $|a| < |b|$ (resp. $|b| < |a|$), the resulting slope relative to the O_x axis (resp. O_y axis) is always comprised between -45° and $+45^\circ$ (see Fig. 2).

Fig. 1. The recognition problem using different thickness for the digital lines

3 Thickness Criterion

3.1 Introducing the Notion of Thickness

The definition of a digital line is intrinsically algebraic. We use an equivalent characterization which is more linked to the field of Euclidian geometry:

Lemma 1. *A set of points is a piece of an α -thick digital line if and only if these points can be covered by a strip of rational slope and of horizontal or vertical thickness strictly inferior to α .*

Fig. 2. Different types of digital line orientations

Fig. 3. Height and vertical thickness of a convex hull

3.2 Thickness and Convex Hull

In this subsection, we show that the notion of thickness (more precisely the thickness of the convex hull of the input points) plays an important role in the recognition problem.

Definition 1. *The height at abscissa β of a convex set C is defined to be the length of the segment resulting from the intersection of C with the vertical line $x = \beta$. We call the vertical thickness of C the maximum reached by $\text{height}(x)$ (see Fig. 3). The width and the horizontal thickness are symmetrically defined.*

Lemma 2. *A convex polygon N has a vertical thickness strictly less than α iff there exists a strip of vertical thickness strictly less than α that covers N .*

► **Proof:** let x denote the abscissa which corresponds to the maximum height of N . The upper and lower border of N at abscissa x can be linked to either a vertex or an edge. We can consider three different configurations:

1. **edge-edge:** this case only appears if both edges are parallel (see Fig. 4.b). If this were not the case, it would exist a greater value for the vertical thickness of N (see Fig. 4.a). As N is convex, it is included in the strip defined by these two edges. So N can be covered by a strip of correct thickness.
2. **edge-vertex:** as the maximum is achieved at this abscissa, the line passing through this vertex and parallel to this edge is tangent to N (see Fig. 4.c). We use this line and the previous edge to build a valid strip.

Fig. 4. The different configurations of the vertical thickness location

3. vertex-vertex: this case is a bit more complicated. Let e_{ul}, e_{ur}, e_{ll} and e_{lr} denote the emerging edges from these two vertices. As the polygon is convex, we have $\text{slope}_{e_{ur}} \leq \text{slope}_{e_{ul}}$ and $\text{slope}_{e_{lr}} \geq \text{slope}_{e_{ll}}$. The maximum height reached at this abscissa implies $\text{slope}_{e_{ul}} \geq \text{slope}_{e_{ll}}$ and $\text{slope}_{e_{ur}} \leq \text{slope}_{e_{lr}}$. If we choose e_{ll} as a border of our strip, e_{ul} and e_{lr} lies inside. Assume that $\text{slope}_{e_{ur}} \leq \text{slope}_{e_{ll}}$. In this case (see Fig. 4.d) N is included in a valid strip. In the opposite case where $\text{slope}_{e_{ur}} > \text{slope}_{e_{ll}}$ (see Fig. 4.e), this edge can not be chosen as a border for our strip. Hopefully, e_{ur} is a correct choice. Indeed, we have $\text{slope}_{e_{ur}} \leq \text{slope}_{e_{ul}}$ and $\text{slope}_{e_{ur}} \leq \text{slope}_{e_{lr}}$ by assumptions; as $\text{slope}_{e_{ur}} > \text{slope}_{e_{ll}}$ this finally implies that N is included in a strip of vertical thickness strictly less than α . ◀

Using lemmas 1 and 2, we finally obtain the next property which links the thickness of a convex hull to the α -thick digital line recognition problem:

Property 1. A set of points is a piece of an α -thick digital line if and only if its convex hull has an horizontal or vertical thickness strictly inferior to α .

3.3 The Importance of Convex Hull

A convex set C has a vertical thickness strictly less than α if and only if C and its translation by $\alpha \cdot e_y$ have an empty intersection. Consider the case where a point P is inserted on the right of C (see Fig. 5.a). Let u (resp. l) denote the point lying on the upper (resp. lower) border through which a *supporting line* (a line passing through P and tangent to C) passes. If the triangle ulP intersects $C + \alpha \cdot e_y$, then the vertical thickness of the new convex set is greater or equal to α . Conversely, if the intersection is empty, the vertical thickness of ulP is strictly less than α , so the intersection between ulP and $ulP + \alpha \cdot e_y$ is empty. It follows:

- (a) ulP does not intersect $CH + e_y$ (b) area delimited by the four critical supporting lines

- (c) When a point P lies in A , ulP can not intersect $CH + e_y$

- (d) Representation of $\mathbb{D}$

- (e) Example 1 - a piece of a naive digital line

- (f) Example 2 - a piece of an α -thick digital line with $\alpha = 2.5$

Fig. 5. Domain of the points that maintain an α -thick digital line

Property 2. We can insert a point to a convex set C while maintaining its vertical thickness strictly below α if and only if the two supporting lines do not intersect $C + \alpha.e_y$ and $C - \alpha.e_y$.

For the convex hull CH of a given set of points, we want to determine the domain $\mathbb{D}$ of the points that can be inserted while preserving the vertical thickness of CH strictly below α . We call *critical supporting lines (CSL)* between two convex objects C_1 and C_2 the two lines tangent to both convex objects such that C_1 and C_2 lie on opposite sides of each line. We show that the area $\mathbb{A}$ (see 5.b) delimited by the four CSL between $CH + \alpha.e_y$ and CH and between CH and $CH - \alpha.e_y$ plays an important role. Indeed, according to the previous property, when a point lying in $\mathbb{A}$ is inserted the new vertical thickness is always valid. We now examine the area $\mathbb{B}$ above the two CSL t_1 and t_2 and under $CH + e_y$ (see Fig. 5.c). Let csl_1 and csl_2 denote the vertices of $CH + e_y$ that support t_1 and t_2 . We show that when a point P lying in $\mathbb{B}$ is inserted, the associated supporting segments p_1 and p_2 between P and CH do not intersect $CH + e_y$. By assumptions, P is above t_1 and t_2 . As p_1 (resp. p_2) is tangent to CH , it must cross t_1 (resp. t_2). So $slope_{p_1} \leq slope_{t_1}$ and $slope_{p_2} \geq slope_{t_2}$. By convexity properties, the slopes of all the edges of CH between csl_1 and csl_2 are ranging from the slope of t_1 to the slope of t_2 . Thus p_1 and p_2 can not cross $CH + e_y$. Finally, the upper border of $\mathbb{D}$ is delimited by the CSL lines emerging from csl_1 and csl_2 and the edges of $CH + e_y$ located between these two vertices (see Fig. 5.d). We symmetrically define the lower border of $\mathbb{D}$. We present this result for a naive digital line (see Fig. 5.e) and an α -thick digital line with $\alpha = 2.5$ (see Fig. 5.f). The CSL can be computed in optimal linear time with the rotating calipers approach [Tou83].

4 Algorithm Design

4.1 Computing the Vertical Thickness of a Convex Hull

We recall that our definition of thickness is different from the one usually used in Computational Geometry. In this field, the thickness is linked to the strip of minimal width (computed with the L_2 -norm) that encloses all the points (see Fig. 6). The rotating caliper algorithm [HoT88] can be used to compute this value in optimal linear time in the number of vertices. In the same way, we can determine the vertical thickness in linear time by using a double-traversal of the upper and lower borders of the convex hull. We set up a new approach based on convexity properties. Let $f(x)$ and $g(x)$ denote the functions defined by the upper and lower borders of the convex hull (see Fig. 7). The vertical thickness is always linked to a hull vertex (the edge-edge case is always associated to a vertex-edge or vertex-vertex case). So we only have to check the value of $height(x)$ at the abscissas of the hull vertices. Notice that f is concave and g is convex, so the difference $height(x) = f(x) - g(x)$ is a concave function and we can apply a binary search to find its optimum. For example, suppose we have k vertices on U the upper border of the hull. We compute the value $height(U_{k/2})$ and

Fig. 6. The common thickness definition

Fig. 7. The vertical thickness

$height(U_{k/2+1})$. By another binary search on the lower border, we determine which vertices are faced to $U_{k/2}$ and $U_{k/2+1}$ in $O(\log n)$ time. Suppose that $height(U_{k/2}) < height(U_{k/2+1})$. Then by convexity property all the vertices of indices less than $k/2$ can not define the optimum. So we are able to reject one half of the upper vertices. We perform in the same way for the vertices of the lower hull. So we have removed at least one half of all the vertices in $O(\log n)$ time. By continuing the binary search on $height(x)$, we finally obtain the vertical thickness in $O(\log^2 n)$ time.

4.2 The Incremental Approach

This is an on-line version where no information about insertions are known in advance. At each iteration, thickness and covering strip are computed. When a new point is inserted, we compute the resulting convex hull using classical incremental convex hull algorithm [PS85,BKS95] in $O(\log n)$ time. When the point lies inside the current strip, the thickness is unchanged. When the point lies outside, we have to compute the new thickness. Using the previous method, we would obtain an $O(n \log^2 n)$ algorithm, but we improve the global complexity by simplifying the binary search. We only consider the case where the point is inserted on the right of the hull (see Fig. 8.a), others cases are similar (see Fig. 8.b). The light grey area corresponds to a triangle; its thickness can be computed in constant time. The other area is delimited by a piece of the upper border and a tangent. Thus, all the vertices of this area are linked to the same segment and we can perform a simple binary search to find in $O(\log n)$ time the maximum height of this area. Finally, we can update the vertical thickness in $O(\log n)$ time per insertion.

Fig. 8. Computing the new thickness in the incremental method

4.3 A Special Case

We often know in real applications that points are inserted in the same direction. Previous algorithms [DR95] take advantage of this in order to obtain a linear time complexity. We present an elementary implementation designed for this special case. First, we recall that the incremental convex hull under this assumption can be processed in $O(n)$ time using classical methods [Mlk87]. We choose to insert points by increasing abscissa (see Fig. 9). Notice that insertions on the right of the convex hull shifts the vertical thickness to the right. Let β denote the abscissa where the previous vertical thickness was located. The dimmed area is delimited by the line $x = \beta$, a border of the previous strip and a tangent. It covers the left part of the new convex hull. In this triangle, all values of $height(x)$ are less or equal to $height(\beta)$. So the vertical thickness must be located on the right of this area. We do not use a binary search anymore. We just traverse each vertex on the right up to the maximal height. Each trial has a $O(1)$ cost, and the total number of trials is bounded by the total amount of inserted points. So the incremental recognition of a digital line of thickness d can be computed in $O(n)$ time if we insert points in a given order.

Fig. 9. Insertion on the right shifts vertical thickness to the right

Fig. 10. Notations of the algorithm

4.4 Pseudo-Code

We present (see Fig. 13) the main function used to compute the vertical thickness of a convex hull: *UPDATE_VERTICAL_THICKNESS*. The hull will consist of two lists U and L corresponding to its upper and lower borders. Let n_U and n_L denote the size of U and L respectively. We call i_U and i_L the indices of the couple of vertices lying on U and L respectively that define the vertical thickness. When the case edge-vertex appears, we select the left vertex of the edge. The other vertex is necessarily the one of U_{i_U+1} and L_{i_L+1} with minimum abscissa (function *HEIGHT*). We call $new\ i_U$ and $new\ i_L$ (see Fig. 10) the indices of vertices that are associated with the next discontinuity of $height(x)$. The implementation presented here processes points entered from left to right (the insertion order in the vertical direction is unimportant). The same function can be used in the three others directions by symmetrizing the (x, y) coordinates.

4.5 Complexity of the Algorithms

We may use *dynamic* convex hull (incremental and decremental) that induces an $O(n \log^2 n)$ overhead [PS85]. We recall the complexities of our methods:

case	convex hull	incremental	in one direction	dynamic
thickness computation	$O(\log^2 n)$	$O(n \log n)$	$O(n)$	$O(n \log^2 n)$

4.6 Applications of α -Thick Lines

We are able to convert any set of points into a polygonal chain of digital line segments, even if no assumptions on connectedness can be done and no information about thickness is known a priori. We present in Fig. 11 a first test on a low quality digitization. We may adjust the thickness in order to obtain a compromise between the number of segments and the conversion quality.

Pixel Traversal Order. First, we have to choose the pixel order that will be used in our recognition process. Pixels are stored in a data structure L that allows efficient range searching. Suppose we have a function *STARTING_POINT* that arbitrary selects a starting pixel. This point represents the first level and it can be removed from L . To fill the next level we only have to select pixels that are in the neighborhood of the pixels in the current level (using a particular distance function). When no pixels are found, we widen the current neighborhood until we find one or more points. When a maximal thickness δ is allowed for the recognition, it is judicious to limit the maximal neighborhood to a width δ . Thus, the holes of at most δ pixels can be filled. The pixels of the same level are arbitrary ordered. We recall the entire process in the following algorithm and present an example in figure 12.

The Overall Method. We process pixels one after the other relative to the predefined order. When the thickness of the current digital segment exceeds the given threshold δ , we backtrack to the previous pixel and select the corresponding digital segment and the associated thickness (that may be between 0 and δ). When we want to obtain a polygonal chain, we may insert the last pixel of the previous digital segment in the next one. The method presented here is quite elementary and is based on a greedy approach. Thus the result is completely dependent of the chosen pixel traversal order and no uniqueness can be obtained.

Open Problems. When we have worked with digital segments, no considerations have been done about the quality of the junction between two consecutive digital segments. In fact, pieces of them can cover themselves at the junction. Optimizations relatively to the number of digital segments and to the visual quality remain to solve. Thus many questions are left untreated and their management would require an improved method that goes beyond the scope of this paper.

Fig. 11. A low quality digitization of the Italian coast converted using α -thick digital segments ($\alpha = 2.5, 3$ and 4.5). We successively obtain 37, 30 and 20 digital segments

Fig. 12. Computing the pixel order traversal

Function ORDER_PIXELS ($L, Level, \delta$)

```

 $i_{CL} = 1$     - current level index
while (  $L.non\_empty$  )
     $Level[i_{CL}] = STARTING\_POINT(L)$ 
    while (  $Level[i_{CL}].non\_empty$  )
         $d = 1$ 
        repeat  $Level[i_{CL} + 1] = Neighborhood(L, Level[i_{CL}], d)$ 
             $d = d + 1$ 
        until (  $Level[i_{CL} + 1].non\_empty$  OR  $d > \delta$  )
         $L.remove(Level[i_{CL}])$ 
         $i_{CL} = i_{CL} + 1$ 

```

Function UPDATE_VERTICAL_THICKNESS ($U, n_U, L, n_L, i_U, i_L, h, P$)

```

if ( $n_U = 0$ ) {  $n_L = n_U = 2$ ;  $U_1 = U_2 = L_1 = L_2 = P$ ;  $h = 0$ ; return; }

    - backtracking to update the convex hull -
while ( $n_U > 1$ ) and ( $(U_{n_U-1} - U_{n_U}) \wedge (P - U_{n_U}) \leq 0$ ) {  $n_U = n_U - 1$  }
while ( $n_L > 1$ ) and ( $(L_{n_L-1} - L_{n_L}) \wedge (P - L_{n_L}) \geq 0$ ) {  $n_L = n_L - 1$  }

    - when a vertex that defines the previous thickness is removed -
if ( $i_U > n_U$ ) {  $i_U = n_U$  }
if ( $i_L > n_L$ ) {  $i_L = n_L$  }

    - insertion of the new point P -
 $n_L = n_L + 1$ ;  $L_{n_L} = P$ ;
 $n_U = n_U + 1$ ;  $U_{n_U} = P$ ;

    - update Height(x) -
 $h = HEIGHT(U, L, i_U, i_L)$ ;

    - traverse the hull while the height increases -
while ( $i_L \neq n_L - 1$ ) or ( $i_U \neq n_U - 1$ )
    if ( $(U_{i_U+1}.x < L_{i_L+1}.x)$ ) {  $new\ i_U = i_U + 1$ ;  $new\ i_L = i_L$ ; }
    else {  $new\ i_U = i_U$ ;  $new\ i_L = i_L + 1$ ; }
     $h_{next} = HEIGHT(U, L, new\ i_U, new\ i_L)$ ;
    if ( $h_{next} > h$ ) {  $h = h_{next}$ ;  $i_U = new\ i_U$ ;  $i_L = new\ i_L$ ; }
    else return;

```

Function HEIGHT (U, L, i_U, i_L)

```

if ( $(U_{i_u}.x < L_{i_l}.x)$ ) return  $VERT\_DIST(U_{i_u}, U_{i_u+1}, L_{i_l})$ ;
else return  $VERT\_DIST(L_{i_l}, L_{i_l+1}, U_{i_u})$ ;

```

Fig. 13. The core function of the recognition method based on thickness computation

5 Conclusion and Future Works

We introduce a new digital line recognition method based on the vertical and horizontal thickness. Our method is well adapted to recognize any set of points

where no assumptions on connectedness can be done. The internal machinery uses classical convex hull algorithms.

The knowledge of the convex hull of the current set of points allows to increase the thickness computation efficiency, to make easier any implementation in the two-dimensional case, to determine the points that can be inserted while preserving a valid digital line, to obtain an incremental or dynamic method and to extend our approach to points with real coordinates. We do not deal with the strategy used to order points during the recognition. At each step, the thickness is computed and we may suppose that the associated normal vector is close to the local curvature. Thus, we may take advantage of this information in order to enhance the overall conversion quality. Other criteria may be inserted like density. Indeed, we may consider the ratio between the number of points inside the hull and the hull surface. This would be an automatic criterion that could replace the usual predefined thickness threshold. Such strategies combined with the dynamic approach could lead to a new generation of adaptive algorithms...

The source code in C++ of our method as well as a basic geometry kernel, some examples and some tests can be downloaded from the web page of the author: www.esiee.fr/~buzerl/DG/.

References

- [BKS95] M. de Berg, M. van Kreveld, and S. Schirra. *A new approach to subdivision simplification*. In Proc. of Auto-Carto 12, pages 79-88, 1995.
- [Buz02] L. Buzer. *Reconnaissance des plans discrets - Simplification polygonale*. Thèse, Université d'Auvergne, Clermont-Ferrand, 2002.
- [Buz03] L. Buzer. *A linear incremental algorithm for naive and standard digital lines and planes recognition*, Graphical Models, 65(1-3), pp. 61-76, 2003.
- [DR95] I. Debled-Rennesson, J.-P. Reveillès. *A linear algorithm for segmentation of digital curves*. International Journal of Pattern Recognition and Artificial Intelligence, Volume **9**, N. 6, dec. 1995.
- [Hot88] M. E. Houle and G. T. Toussaint. *Computing the width of a set*. IEEE Trans. Pattern Anal. Mach. Intell., PAMI-10, 761-765, 1988.
- [Mlk87] A.A. Melkman. *On-line construction of the convex hull of a simple polyline*. Information Processing Letters, **25**:11-12, 1987.
- [PS85] F.P. Preparata, M.I. Shamos. *Computational Geometry*. Springer-Verlag, New-York, 1985.
- [Rev91] J.P. Reveillès. *Géométrie discrète, calculs en nombre entiers et algorithmique*. Thèse d'état, Université Louis Pasteur, Strasbourg, 1991.
- [Tou83] G.T. Toussaint. *Solving geometric problems with the rotating calipers*. Proceedings of IEEE MELECON'83, Athens, Greece, May 1983.