Local Characterization of a Maximum Set of Digital (26, 6)-Surfaces^{*}

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Abstract. This paper provides a local characterization for a set of digital surfaces S^U defined in [6] by mean of continuous analogues. For this, we firstly identify the set of admissible plates for any surface $S \in S^U$ (i.e., the intersection $S \cap C$ of S with a unit cube C of $\mathbb{Z}^3$). Then, the characterization is given in terms of a graph representing the intersection of plates. In addition, we establish a further condition that detects the digital surfaces in S^U which are strongly separating objects.

The family S^U consists of all objects which are a digital surface in some homogeneous (26, 6)-connected digital space in the sense of [3]. Moreover, the subset of strongly separating surfaces of S^U contains the family of simplicity 26-surfaces and other surfaces in literature as well.

1 Introduction

One of the most interesting problems in Digital Topology is, probably, to obtain a general notion of digital surface, defined as a “thin” set of voxels (or, equivalently, points of $\mathbb{Z}^3$), that naturally extends to higher dimensions and such that these digital surfaces have properties similar to those held by topological surfaces. In this paper we restrict our interest to such a notion for the usual (26, 6)-adjacency in image processing. Morgenthaler and Rosenfeld gave in [10] the first definition of digital surface for this adjacency pair. Later on, Bertrand and Mourgouyres [4] introduced the family of *strong 26-surfaces* and showed that it strictly contains the set of Morgenthaler’s surfaces. More recently, Couprie and Bertrand [7] have proved that the strong 26-surfaces are still contained in a larger set of surfaces called *simplicity 26-surfaces*. Despite of these contributions the most general definition of digital surface for the (26, 6)-adjacency is not yet clear. For example, in [9] it is suggested that the (26, 6)-connected digital object

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shown Fig. 3(a) “should normally be interpreted as a simple surface”, however it is not a simplicity 26-surface. Thus, a new definition of digital surface is needed to include this kind of objects.

Within the framework for Digital Topology proposed in [3] we have recently found in [6] a homogeneous $(26, 6)$ -connected digital space E^U whose set of digital surfaces S^U is the largest in that class of digital spaces. In particular, the Jordan–Brouwer and Index Theorems, proved in [3] and [2] for digital manifolds of arbitrary dimensions, automatically hold for these surfaces. Moreover the set S^U strictly contains the family of simplicity 26-surfaces and the object in Fig. 3(a) as well.

In spite of these nice properties, the digital surfaces in S^U were defined in [6] by constructing a continuous analogue, and thus it might not be considered a completely digital notion. In this paper we give a purely digital characterization of the family S^U by the use of plates and graphs extending the Kong and Roscoe method in [8] (see Theorem 3).

The digital objects characterizing the surfaces in S^U are introduced in Section 2. In Section 3 we recall the basic elements of the framework in [3], and the definition of E^U , needed to obtain this characterization in Section 4.

Despite of all surfaces in S^U are *Jordan* objects, some of them might be considered as pathological examples since the deletion of one of their points might yield an object that still separates its complement into two 6-components. In Section 5, we state a characterization of the subset of non-pathological surfaces of S^U , which still strictly contains the family of simplicity 26-surfaces [6].

As a conclusion, the results in [6] and this work lead to a good and general enough approach to a notion of digital surface since:

1. it is suitable for image processing as it is given in terms of a graph;
2. these digital surfaces have properties similar to those of topological surfaces;
3. and, in a certain sense, they constitute a maximum family of digital surfaces that, as far as we know, contains any set of surfaces defined on $\mathbb{Z}^3$ using the graph-based approach in the literature.

2 A Set of Jordan Objects

In this section we introduce a family of objects in the discrete space $\mathbb{Z}^3$ that satisfies a Jordan property for the usual $(26, 6)$ -adjacency. These objects are made of small surface pieces, called *plates*, which are adequately glued to each other in the way defined by an *assembly graph*. In order to introduce these notions, we firstly recall some basic definitions of the graph–theoretical approach to Digital Topology.

Two voxels $\sigma = (\sigma_1, \sigma_2, \sigma_3), \tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{Z}^3$ are said to be 6-, 18- or 26-adjacent if $\max\{|\sigma_i - \tau_i|; 1 \leq i \leq 3\} \leq 1$ and they differ in, at most, one, two or three of their coordinates, respectively. Moreover, we say that two 18-adjacent voxels are *strictly 18-adjacent* if they are not 6-adjacent. A *unit cube* of $\mathbb{Z}^3$ is any subset C of eight mutually 26-adjacent voxels. Similarly, a *unit square* of $\mathbb{Z}^3$ is a subset of four mutually 18-adjacent voxels.

Fig. 1. Some patterns that may appear in a digital object $O \subseteq \mathbb{Z}^3$. Each picture represents a unit cube C of $\mathbb{Z}^3$, and the black dots are the set of voxels in $O \cap C$. The row above contains the six non-square plates that may appear in a digital object. The patterns in the row below cannot appear in a 26-presurface

For each subset $A \subseteq \mathbb{Z}^3$, the transitive closure of the n -adjacency, $n \in \{6, 18, 26\}$, defines an equivalence relation whose classes are called the n -components of A . Moreover, A is said to be n -connected if it has only one n -component.

Definition 1. Let $O \subseteq \mathbb{Z}^3$ be a digital object. A subset $P \subseteq O$ is said to be a plate in O if either P is a unit square of $\mathbb{Z}^3$ or $P = O \cap C$, where C is a unit cube of $\mathbb{Z}^3$, and P correspond (up to rotations and symmetries) to one of the patterns in the set $\mathfrak{C} = \{C_3^c, C_4^b, C_4^e, C_4^f, C_5^b, C_6^c\}$; see Fig. 1 and Remark 3 below.

For any voxel $\sigma \in O$ we denote by $\mathbf{P}(O, \sigma)$ the set of all plates in O containing σ , and $\mathbf{P}(O)$ the set of all plates in O .

From the set $\mathbf{P}(O)$ of plates in a given object O we construct a bipartite graph $G(O)$, termed the *assembly graph* of O , whose nodes are the elements of the set $\mathbf{P}(O) \cup \mathbf{E}(O)$, where each $q \in \mathbf{E}(O) = \cup_{\sigma \in O} \mathbf{E}(O, \sigma)$ is associated to a pair of plates p_1, p_2 in O such that $p_1 \cap p_2$ contains at least two voxels (notice that $p_1 \cap p_2$ consists of three voxels at most.) More precisely, the sets $\mathbf{E}(O, \sigma)$ and the edges of $G(O)$ are chosen according to the following criteria. Let $p_1, p_2 \in \mathbf{P}(O)$ be two plates, then:

1. If $p_1 \cap p_2 = \{\tau_1, \tau_2\}$ we consider a new node $q \in \mathbf{E}(O, \tau_1) \cap \mathbf{E}(O, \tau_2)$ and two edges $\langle p_i, q \rangle$, $i = 1, 2$, in $G(O)$.
2. If $p_1 \cap p_2 = \{\tau_0, \tau_1, \tau_2\}$ then we consider two new nodes q_1 and q_2 and four edges $\langle p_i, q_j \rangle$, $i, j \in \{1, 2\}$, in $G(O)$. Moreover, assume τ_0 is the only voxel in $p_1 \cap p_2$ which is 6-adjacent to both τ_1 and τ_2 , then we set $q_j \in \mathbf{E}(O, \tau_0) \cap \mathbf{E}(O, \tau_j)$, $j \in \{1, 2\}$.
3. For each $\sigma \in O$, the set $\mathbf{E}(O, \sigma)$ consists exclusively of the elements q and q_j introduced above for all intersections $p_1 \cap p_2$ containing σ .

For each voxel $\sigma \in O$ we define the *assembly graph of O around σ* , $G(O, \sigma)$, as the subgraph of $G(O)$ induced by the nodes in $\mathbf{P}(O, \sigma) \cup \mathbf{E}(O, \sigma)$.

Fig. 2. A 26-presurface S and its assembly graph $G(S)$. Each dot in (b) represents one of the eight unit cubes shown in (a), which are actually plates of S . Squares in (b) are the points that, together with plates, define the assembly graph of S ; each one represents the intersection between two plates

Definition 2. A digital object $S \subseteq \mathbb{Z}^3$ is said to be a 26-presurface if the following conditions hold for each voxel $\sigma \in S$.

1. For each unit cube C of $\mathbb{Z}^3$ the intersection $S \cap C$ does not corresponds (up to rotations and symmetries) to any pattern in $\mathfrak{B} = \{B_2^c, B_5^c, B_6^b, B_8\}$; see Fig. 1.
2. If $\tau \in S$ is strictly 18-adjacent to σ and no other voxel in S is 6-adjacent to both σ and τ then $\mathbf{P}(S, \sigma) \cap \mathbf{P}(S, \tau) \neq \emptyset$.
3. If $\tau \in S$ is 6-adjacent to σ then $\mathbf{P}(S, \sigma) \cap \mathbf{P}(S, \tau)$ consists exactly of two plates.
4. $\mathbf{P}(S, \sigma) \neq \emptyset$ and $G(S, \sigma)$ is a cycle.

Example 1. Figure 2 depicts a 26-presurface S , made of eight plates, and its assembly graph $G(S)$. Notice that the voxel $\tau_0 \in S$ belongs to exactly two plates $p_1, p_2 \in \mathbf{P}(O)$ for which $p_1 \cap p_2 = \{\tau_0, \tau_1, \tau_2\}$. Hence, $\mathbf{E}(O, \tau_0) = \{q_1, q_2\}$ and the assembly graph of S around τ_0 , $G(S, \tau_0)$, is the cycle defined by the vertices p_1, p_2, q_1 and q_2 in Fig. 2(b).

The main goal in this paper is to show that the assembly graph endows each 26-presurface with the combinatorial structure of a surface. More precisely, we will show in Theorem 3 that 26-presurfaces characterize the digital surfaces of a certain digital space $E^U = (R^3, f^U)$ defined in [6] within the approach to Digital Topology in [3]. In order to state and prove Theorem 3 we recall in the next section the basic elements of this framework and the definition of E^U .

Remark 1. The assembly graph of an object $O \subseteq \mathbb{Z}^3$ can be derived from a simpler graph whose nodes are the plates in O and two of them are adjacent if their intersection contains at least two voxels. Moreover, the five conditions defining 26-presurfaces can be rewritten in terms of this simpler graph.

3 A Framework for Digital Topology

In [3] we propose a framework for Digital Topology in which a notion of digital surface and, more generally, of digital manifold naturally arises. In this approach a *digital space* is a pair (K, f) , where K is a polyhedral complex, called *device model*, which represents the spatial layout of voxels and f is a *lighting function* from which we associate to each digital image an Euclidean polyhedron called its continuous analogue.

In this paper we will only consider digital spaces of the form (R^3, f) , where the device model R^3 , termed the *standard cubical decomposition* of the Euclidean space $\mathbb{R}^3$, is the polyhedral complex determined by the collection of unit cubes in $\mathbb{R}^3$ whose edges are parallel to the coordinate axes and whose centers are in the set $\mathbb{Z}^3$. Each 3-cell in R^3 is representing a voxel, and so the digital object displayed in a digital image is a subset of the set $\text{cell}_3(R^3)$ of 3-cells in R^3 ; while the other lower dimensional cells in R^3 (actually, k -cubes, $0 \leq k < 3$) are used to describe how the voxels could be linked to each other.

Remark 2. Each k -cell $\sigma \in R^3$ can be associated to its center $c(\sigma)$ which is a point in the set $\mathcal{Z}^3$, where $\mathcal{Z} = \frac{1}{2}\mathbb{Z} = \{x \in \mathbb{R}; x = z/2, z \in \mathbb{Z}\}$. If $\dim \sigma = 3$ then $c(\sigma) \in \mathbb{Z}^3$, so that every digital object O in R^3 can be naturally identified with a subset of the discrete space $\mathbb{Z}^3$. Henceforth we shall use this identification without further comment. Notice also that if C is a unit cube (square) of $\mathbb{Z}^3$ then its center is the point $c(\alpha)$, where $\alpha \in R^3$ is a 0-cell (1-cell, respectively). Thus, if $P = O \cap C$ is a plate in O we call α the *center* of P .

As it is usual, given two cells $\gamma, \sigma \in R^3$ we write $\gamma \leq \sigma$ if γ is a face of σ , and $\gamma < \sigma$ if in addition $\gamma \neq \sigma$. The interior of a cell σ is the set $\overset{\circ}{\sigma} = \sigma - \partial\sigma$, where $\partial\sigma = \cup\{\gamma; \gamma < \sigma\}$ stands for the boundary of σ . We refer to [11] for further notions on polyhedral topology.

To recall the notion of lighting function we need the following definitions. Given a cell $\alpha \in R^3$ and a digital object $O \subseteq \text{cell}_3(R^3)$ the *star of α in O* is the set $\text{st}_3(\alpha; O) = \{\sigma \in O; \alpha \leq \sigma\}$ of 3-cells (voxels) in O having α as a face. Similarly, the *extended star of α in O* is the set $\text{st}_3^*(\alpha; O) = \{\sigma \in O; \alpha \cap \sigma \neq \emptyset\}$. Finally, the *support* of O is the set $\text{supp}(O)$ of cells of R^3 (not necessarily voxels) that are the intersection of 3-cells in O ; that is, $\alpha \in \text{supp}(O)$ if and only if $\alpha = \cap\{\sigma; \sigma \in \text{st}_3(\alpha; O)\}$. To ease the writing, we use the following notation: $\text{st}_3(\alpha; R^3) = \text{st}_3(\alpha; \text{cell}_3(R^3))$ and $\text{st}_3^*(\alpha; R^3) = \text{st}_3^*(\alpha; \text{cell}_3(R^3))$. Finally, we write $\mathcal{P}(A)$ for the family of all subsets of a given set A .

A *lighting function* on the device model R^3 is a map $f : \mathcal{P}(\text{cell}_3(R^3)) \times R^3 \rightarrow \{0, 1\}$ satisfying the following five axioms for all $O \in \mathcal{P}(\text{cell}_3(R^3))$ and $\alpha \in R^3$:

- (1) *object axiom*: if $\alpha \in O$ then $f(O, \alpha) = 1$;
- (2) *support axiom*: if $\alpha \notin \text{supp}(O)$ then $f(O, \alpha) = 0$;
- (3) *weak monotone axiom*: $f(O, \alpha) \leq f(\text{cell}_3(R^3), \alpha)$;
- (4) *weak local axiom*: $f(O, \alpha) = f(\text{st}_3^*(\alpha; O), \alpha)$; and,

- (5) *complement connectivity axiom*: if $O' \subseteq O \subseteq \text{cell}_3(R^3)$ and $\alpha \in R^3$ are such that $\text{st}_3(\alpha; O) = \text{st}_3(\alpha; O')$, $f(O', \alpha) = 0$ and $f(O, \alpha) = 1$, then the set $\alpha(O', O) = \cup\{\hat{\omega}; \omega < \alpha, f(O', \omega) = 0, f(O, \omega) = 1\} \subseteq \partial\alpha$ is non-empty and connected.

If $f(O, \alpha) = 1$ we say that f *lights* the cell α for the object O , otherwise f vanishes on α for O .

A digital space (R^3, f) is said to be *homogeneous* if for any spatial motion $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ preserving $\mathbb{Z}^3$ the equality $f(\varphi(O), \varphi(\alpha)) = f(O, \alpha)$ holds for all cells $\alpha \in R^3$ and digital objects $O \subseteq \text{cell}_3(R^3)$; see [6]. We refer to [3] for a definition of digital spaces on more general device models.

Given a lighting function f , we associate to each digital object $O \subseteq \text{cell}_3(R^3)$ a continuous analogue $|\mathcal{A}_O|$ which intends to be a “continuous interpretation” of O in the digital space (R^3, f) . Namely, $|\mathcal{A}_O|$ is the underlying polyhedron of the simplicial complex $\mathcal{A}_O$, whose k -simplexes are $\langle c(\alpha_0), c(\alpha_1), \dots, c(\alpha_k) \rangle$ where $\alpha_0 < \alpha_1 < \dots < \alpha_k$ are cells in R^3 such that $f(O, \alpha_i) = 1$, $0 \leq i \leq k$. The complex $\mathcal{A}_O$ is called the *simplicial analogue* of O . Notice that the center $c(\sigma)$ of a 3-cell $\sigma \in \text{cell}_3(R^3)$ is a 0-simplex of $\mathcal{A}_O^f$ if and only if $\sigma \in O$.

For the sake of simplicity we will write $\mathcal{A}_{R^3}$ instead of $\mathcal{A}_{\text{cell}_3(R^3)}$.

Continuous analogues allow us, in a natural way, to introduce digital notions in terms the corresponding continuous ones. For example, we will say that an object O is *connected* in a digital space (R^3, f) if its continuous analogue $|\mathcal{A}_O|$ is a connected polyhedron. And, in the same way, the complement $\text{cell}_3(R^3) - O$ of O is said to be connected if $|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$ is connected. Moreover, we call $C \subseteq \text{cell}_3(R^3)$ a component of O ($\text{cell}_3(R^3) - O$) if it consists of all the voxels σ whose centers $c(\sigma)$ belong to a component of $|\mathcal{A}_O|$ ($|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$, respectively). See Section 4 in [3] for more details on these notions of connectedness defined in a much more general context.

For certain digital spaces (R^3, f) , these notions of connectedness can be related with the usual ones given on $\mathbb{Z}^3$ by mean of adjacency pairs. More precisely, given an adjacency pair $(k, \bar{k})$ on $\mathbb{Z}^3$ we say that the digital space (R^3, f) is $(k, \bar{k})$ -*connected* if the two following properties hold for any digital object $O \subseteq \text{cell}_3(R^3)$:

1. C is a component of O if and only if it is a k -component of O ; and,
2. C is a component of the complement $\text{cell}_3(R^3) - O$ of O if and only if it is a $\bar{k}$ -component.

Several examples of (homogeneous) $(26, 6)$ -connected digital spaces (R^3, f) can be found in [1, 3, 5] as well as examples of $(k, \bar{k})$ -connected spaces for $k, \bar{k} \in \{6, 18, 26\}$.

3.1 A Maximum Set of Digital Surfaces

Similarly to the previous definition of connectedness, we use continuous analogues to introduce a notion of digital surface. Namely, an object S in a digital space (R^3, f) is called a *digital surface* if $|\mathcal{A}_S|$ is a combinatorial surface without

Fig. 3. Some surfaces in (R^3, f^U) which are not simplicity 26-surfaces. Both black and grey dots are voxels in the surface. In (a) and (b) the pattern (i.e., the intersection of the surface with a unit cube C of $\mathbb{Z}^3$) consisting of the grey dots is not allowed in a simplicity 26-surface. In (c), the configuration made of the four square plates containing σ cannot appear in a simplicity 26-surface

boundary; that is, if for each vertex $v \in \mathcal{A}_S$ its link $\text{lk}(v; \mathcal{A}_S) = \{A \in \mathcal{A}_S; v, A < B \in \mathcal{A}_S \text{ and } v \notin A\}$ is a 1-sphere.

This notion of digital surface is closely related to the families of 26-connected digital surfaces quoted in the Introduction. Actually, each of these families agrees or is contained in the set of digital surfaces of a particular homogeneous (26, 6)-connected digital space; see [1, 3, 5]. Moreover, in [6] we give a homogeneous (26, 6)-connected digital space $E^U = (R^3, f^U)$ whose set of digital surfaces is the largest in this class of digital spaces. More precisely, we prove

Theorem 1 (Th. 4 in [6]). *Any digital surface S in an arbitrary homogeneous (26, 6)-connected digital space (R^3, f) is also a digital surface in E^U .*

The lighting function $f^U : \mathcal{P}(\text{cell}_3(R^3)) \times R^3 \rightarrow \{0, 1\}$ is defined as follows. Given a digital object $O \subseteq \text{cell}_3(R^3)$ and a cell $\delta \in R^3$, $f^U(O, \delta) = 1$ if and only if one of the following conditions holds:

1. $\dim \delta = 0$ and the intersection $O \cap C$, where δ is the center of the unit cube C of $\mathbb{Z}^3$, corresponds (up to rotations and symmetries) with one of the patterns in the set $\mathfrak{B} \cup \mathfrak{C}$, with $\mathfrak{B} = \{B_2^c, B_5^c, B_6^b, B_8\}$ and $\mathfrak{C} = \{C_3^c, C_4^c, C_4^e, C_4^f, C_5^b, C_6^c\}$; see Fig. 1.
2. $\dim \delta > 1$ and $\delta \in \text{supp}(O)$.
3. $\dim \delta = 1$ and either $\text{st}_3(\delta; R^3) \subseteq O$ (i.e., δ is the center of a square plate in O) or $\text{st}_3(\delta; O) = \{\sigma, \tau\}$, with $\delta = \sigma \cap \tau$, and $f^U(O, \alpha_1) = f^U(O, \alpha_2)$ for the vertices α_1, α_2 of δ .

Remark 3. From the definition of the lighting function f^U it is not difficult to check that digital surfaces in E^U can contain each of the patterns in $\mathfrak{C}$, while those in $\mathfrak{B}$ are forbidden for a surface (see Lemma 3 below).

It is not difficult to check that each of the objects in Fig. 3 is (a piece of) a digital surface in E^U and, however, none of them is a simplicity 26-surface. Therefore, the family of simplicity 26-surfaces is strictly contained in the set of digital surfaces of E^U . At this point, it is worth mentioning that Malandain et al. [9] suggest to consider the object depicted in Fig. 3(a) as a surface.

We get the following separation theorem for digital surfaces in E^U as a corollary of the Jordan–Brouwer Theorem in [3] for almost arbitrary digital spaces.

Theorem 2. *Each 26-connected digital surface S in E^U separates its complement $\text{cell}_3(R^3) - S$ into two 6-components.*

In addition the digital surfaces in E^U also satisfy the general Index Theorem in [2] which provides us with a criterion to determining the components given by Theorem 2.

4 Local Characterization of Digital Surfaces in E^U

Despite its good properties, digital surfaces in the space E^U might not be considered completely digital since their definition relies on the construction of a continuous analogue. Our goal in this section is to characterize these surfaces as the set of 26-presurfaces. This way, we get a procedure to determine whether a digital object S is a digital surface in E^U which does not depend on the lighting function f^U . Moreover, such procedure is local since the five conditions defining 26-surfaces can be checked in the 26-neighbourhood of each voxel $\sigma \in S$. More precisely, we will prove

Theorem 3. *A digital object $S \subseteq \mathbb{Z}^3$ is a 26-presurface if and only if S is a digital surface in the space (R^3, f^U) .*

Next lemmas, which we will use later in the proof of Theorem 3, state some elementary properties of the lighting function f^U in relation to the conditions defining 26-presurfaces. We do not include their proofs (which are not too hard anyway) in order to keep the paper within the required length.

Lemma 1. *Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying condition (1) in Definition 2. The two following properties hold for any cell $\delta \in R^3$.*

- a) *If $\dim \delta = 0$ then $f^U(O, \delta) = 1$ if and only if δ is the center of a plate in O .*
- b) *If $\dim \delta = 1$ and δ is the center of a plate in O then $f^U(O, \delta) = 1$ and $f^U(O, \alpha_i) = 0$ for the two vertices $\alpha_1, \alpha_2 < \delta$.*

Lemma 2. *Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying conditions (1)–(3) in Definition 2. If a cell $\beta \in R^3$, with $\dim \beta \leq 2$ and $f^U(O, \beta) = 1$, is not the center of a plate in O then $\text{st}_3(\beta; O) = \{\sigma_1, \sigma_2\}$ and the set $A = \{\alpha < \beta; f^U(O, \alpha) = 1\}$ consists of exactly two cells, which are actually the centers of plates in O . Moreover, if $\dim \beta = 1$ then $f^U(O, \gamma) = 0$ for any 2-cell $\gamma > \beta$.*

Lemma 3. *Let $O \subseteq \mathbb{Z}^3$ be a digital object. If there exists a unit cube C of $\mathbb{Z}^3$ such that $O \cap C \in \mathfrak{B} = \{B_2^c, B_5^c, B_6^b, B_8\}$ then O is not a digital surface in (R^3, f^U) .*

The lemmas above are also used in the proof of the following result which is a crucial step in the proof of Theorem 3.

Proposition 1. *Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying conditions (1)-(3) in Definition 2. Then, for each $\sigma \in O$, there exists a simplicial isomorphism $\varphi_\sigma : G(O, \sigma) \rightarrow \text{lk}(c(\sigma); \mathcal{A}_O)$.*

Proof (Sketch). Firstly notice that $\text{lk}(c(\sigma); \mathcal{A}_O)$ is 1-dimensional since no tetrahedra can appear in the simplicial analogue of any object satisfying condition (1) in Def. 2. Furthermore, by Lemma 1 there is a bijection $p \mapsto \varphi_\sigma(p)$ between $\mathbf{P}(O, \sigma)$ and the set X of centers of cells $c(\alpha) \in L = \text{lk}(c(\sigma), \mathcal{A}_O)$ such that α is also the center of a plate in O . Moreover, Lemmas 1 and 2 and the definition of f^U allow us to describe L as a bipartite graph generated by vertices in X on one side and the set Y of those $c(\alpha) \in L$ such that α is not the center of a plate on the other. Moreover, by using the same lemmas one can show that there is a bijection between $\mathbf{E}(O, \sigma)$ and Y . Therefore one gets a bijection between the vertex sets of L and $G(O, \sigma)$. Finally, Lemmas 1 and 2 are used one more time to show that this bijection extends to the required simplicial isomorphism.

Proof (of Theorem 3). Assume $S \subseteq \mathbb{Z}^3$ is a 26-presurface. It will suffice to check that the link $L = \text{lk}(c(\delta); \mathcal{A}_S)$ is a 1-sphere for each cell $\delta \in R^3$ such that $f^U(S, \delta) = 1$. For any voxel $\delta \in S$ Proposition 1 yields that L can be identified with the assembly graph $G(S, \delta)$ around δ , and hence L is a 1-sphere by Condition (4) in Definition 2.

If $\dim \delta = 2$ then $\text{st}_3(\delta; S) = \{\sigma_1, \sigma_2\}$ by Lemma 2 and, moreover, exactly two faces $\alpha_1, \alpha_2 < \delta$ are lighted by f^U . Therefore, L coincides with the cycle defined by the centers $\{c(\alpha_1), c(\alpha_2), c(\sigma_1), c(\sigma_2)\}$.

If δ is an edge of R^3 which is the center of a square plate in S then L is obviously a 1-sphere by Lemma 1 and the definition of f^U . Otherwise, if δ is not the center of a plate, the result follows by Lemma 2 as in the case $\dim \delta = 2$.

Finally, if δ is a vertex then it is the center of a plate p by Lemma 1. Moreover the plate p belongs to the set $\mathfrak{C}$ in Definition 1. Assume p does not corresponds to pattern C_4^f in Fig. 1. If $\sigma_1, \sigma_2 \in p$ are strictly 18-adjacent voxels, we derive from the fact that $c(\delta) \in \text{lk}(c(\sigma_i); \mathcal{A}_S)$, $i = 1, 2$, which has been proved to be a 1-sphere, that $f^U(S, \sigma_1 \cap \sigma_2) = 1$. Therefore L is necessarily a 1-sphere by the definition of f^U .

If $p = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ corresponds to C_4^f we have some choices to make. For this we set $L_i = \text{lk}(c(\sigma_i); \mathcal{A}_S)$, $1 \leq i \leq 4$. As $c(\delta) \in L_1$ is a 1-sphere, it contains exactly two of the centers $c_i^1 = c(\sigma_1 \cap \sigma_i)$, $i = 2, 3, 4$. Assume they are c_2^1 and c_3^1 . Similarly, by looking at L_2 we have that either $c_3^2 = c(\sigma_2 \cap \sigma_3)$ or $c_4^2 = c(\sigma_2 \cap \sigma_4)$ belongs to L_2 . But c_3^2 is ruled out by the fact that L_4 is a cycle since otherwise we would get a one-point union of three edges in L_i for some $1 \leq i \leq 3$. Hence $c_4^2 \in L_2$, and the only choice for L_4 is to have $c_3^4 = c(\sigma_3 \cap \sigma_4) \in L_4$ since

$c_1^4 = c(\sigma_1 \cap \sigma_4)$ would imply that L_4 contains a one-point union of three edges. Therefore $\mathcal{A}_S$ is a combinatorial surface and hence S is a digital surface.

Conversely, assume that S is a digital surface. It will be enough to check conditions (1)-(3) in Definition 2 for S since, under the assumption of these properties, we get (4) as an immediate consequence of Proposition 1.

Condition (1) follows immediately from Lemma 3 above. To check condition (2), let $\sigma, \tau \in S$ be two strictly 18-adjacent 3-cells and assume that they are not 6-connected by a third voxel in S . For the edge $\beta = \langle \alpha_1, \alpha_2 \rangle = \sigma \cap \tau$ we consider the two possible cases:

Case $f^U(S, \beta) = 0$. The definition of f^U shows that $f^U(S, \alpha) = 1$ for a vertex $\alpha < \beta$ (f^U vanishes on the other one necessarily). Then by condition (1), already proved, and Lemma 1 it follows that α is the center of a plate in $\mathbf{P}(S, \sigma) \cap \mathbf{P}(S, \tau)$.

Case $f^U(S, \beta) = 1$. Then it can be readily checked that $f^U(S, \alpha_i) = 1$ for the two vertices α_1, α_2 of β , since S is a digital surface in E^U . Therefore the vertices α_i are centers of plates in $\mathbf{P}(S, \sigma) \cap \mathbf{P}(S, \tau)$ by Lemma 1.

Finally we prove condition (3). For this let $\sigma, \tau \in S$ be two 6-adjacent voxels $\gamma = \sigma \cap \tau$. Then $f^U(S, \gamma) = 1$ by definition of f^U . As $\text{lk}(c(\gamma); \mathcal{A}_S)$ is a 1-sphere there exist exactly two faces $\alpha_1, \alpha_2 < \gamma$ with $f^U(S, \alpha_i) = 1$. If $\dim \alpha_i = 0$ then α_i is the center of a plate in $\mathbf{P}(S, \sigma) \cap \mathbf{P}(S, \tau)$ by Lemma 1. Similarly, if $\dim \alpha_i = 1$ then the definition of f^U yields that $\text{st}_3(\alpha_i; S) = \text{st}_3(\alpha_i; R^3)$ since it contains the two 6-adjacent voxels σ and τ , and hence α_i is the center of a square plate.

5 Final Results

As a consequence of the characterization above and Theorem 2 we get that each 26-connected 26-presurface S separates its complement $\mathbb{Z}^3 - S$ into two 6-components; that is, S is a *Jordan* object. However, we find in this set examples (as the 26-presurfaces S_1 and S_2 shown in Fig. 2(a) and 3(c), respectively) which might be considered as pathological surfaces. More precisely, one readily checks that the voxel $\tau_0 \in S_1$ in Fig. 2(a) is not 6-adjacent to both 6-components of $\mathbb{Z}^3 - S_1$; that is, S_1 is not a *strongly separating* object as defined in [4, 7]. Hence the deletion of τ_0 from S_1 yields an object which is still Jordan (actually, $S_1 - \{\tau_0\}$ is a 26-presurface). We next state without proof a local characterization for the set of strongly separating 26-presurfaces.

Theorem 4. *Let $S \subseteq \mathbb{Z}^3$ a 26-connected 26-presurface. Then S is strongly separating if and only if S is 6-thin; that is, for each $\sigma \in S$, the complement of S in the 26-neighbourhood of σ , $\text{st}_3^*(\sigma; R^3) - S$, contains exactly two 6-components which are 6-adjacent to S .*

The proof of this result makes use of the difference $\text{lk}(c(\sigma); \mathcal{A}_{R^3}) - \text{lk}(c(\sigma); \mathcal{A}_S)$ of links in the continuous analogue to prove that the 6-components of $\text{st}_3^*(\sigma; R^3) - S$ which are 6-adjacent to σ characterize the 6-components of $\mathbb{Z}^3 - S$, as in the proof of the digital Index Theorem given in [2].

This result suggests the following

Definition 3. A strongly separating 26-surface is any 26-presurface which is also 6-thin.

Finally, we point out that the set of strongly separating 26-surfaces also contains strictly the set of simplicity 26-surfaces since each one of them is a strongly separating object [7] as well as the 26-presurfaces pictured in Fig. 3(a) and (b).

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