A Discrete Modulo N Projective Radon Transform for $N \times N$ Images

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Abstract. This paper presents a Discrete Radon Transform (DRT) based on congruent mathematics that applies to $N \times N$ arrays where $N \in \mathbb{N}$. This definition incorporates and is a natural extension of the more restricted cases of the finite Radon transform [1] where N must be prime, the discrete periodic Radon transform [2] where N must be a power of 2, and the DRT over p^n [3], where N must be a power of a single prime. The DRT exactly and invertibly maps a 2-D image to a set of 1-D projections of length N. Projections are found as the sum of the pixels centred on a parallel set of discrete lines. The image is assumed to be periodic and these lines wrap around the array under modulo N arithmetic. Properties of the continuous Radon transform are preserved in the DRT; a discrete form of the Fourier slice theorem applies, as does the convolution property. A formula is given to find the projection set required to be exactly invertible for arrays with N any composite number, as well as a means to determine the level of redundancy in sampling that is introduced on such composite arrays.

1 Introduction

This paper presents a discrete projective transform for $N \times N$ arrays of image data where $N \in \mathbb{N}$. The motivation for this work is to remove the current restrictions on array size N of such transforms to be a power of a prime.

The Radon transform maps a 2-D function $f(x, y)$ to projection space $r(\rho, \theta)$ as the integral of f along the line $y = \tan\theta x + \rho/\cos\theta$. A projection is a set of all parallel line integrals at some angle θ. It was first defined by Johan Radon in 1917, however applications in tomography and image processing were not fully realised until the advent of computers. As the transform is predominantly implemented on discrete data sets, a discrete version of the Radon transform that minimises the need for interpolation is required. The Radon transform has no 1-D analogue, so defining discrete projection sets is not a trivial task and there have been many proposed formalisms.

A general algebraic definition for the discrete form of the Radon transform was presented by Beylkin in 1987 [4]. A specific class of Beylkin's definition, termed the Finite Radon Transform (FRT), was defined by Matus and Flusser in

E. Andres et al. (Eds.): DGCI 2005, LNCS 3429, pp. 136–147, 2005.

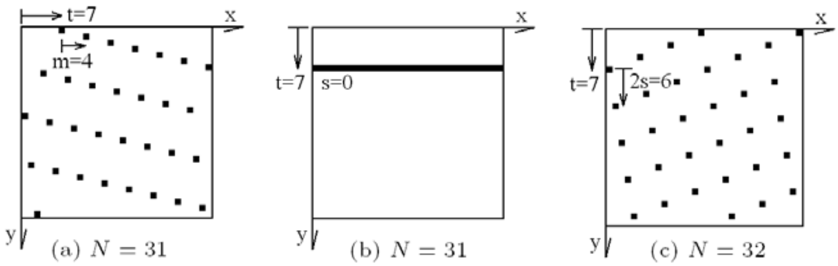

Fig. 1. Examples of discrete lines on $N \times N$ grids. Black pixels lie on the centre of the wrapped lines, the values of these pixels are summed to give the value of one element in the transform. (a) Pixels on $x \equiv 4y + 7 \pmod{31}$ sum to give $R_4(7)$. N is prime here so only one element on each row and column is sampled by a discrete wrapped line. (b) Pixels on $y \equiv 0x + 7 \pmod{31}$ sum to give $R_0^{\perp}(7)$. (c) Pixels on $y \equiv 6x + t \pmod{32}$ sum to give $R_3^{\perp}(7)$, i.e., $s = 3$. N is now composite and the gradient $2s$ has a common factor with N, each row and column is no longer uniquely sampled

1993 [1]. This transform is an exact and invertible projective mapping, requiring only additive operations. This contrasts with the projections of the general DRT in [4] which require an algebraic solution for the inversion process. The FRT applies to square arrays of prime size, $p \times p$. A discrete form of both the Fourier slice theorem and convolution property of the continuous transform hold [1]. Thus the transformation and inversion can also be achieved very efficiently via the 2-D discrete Fourier transform of the function. These properties of the FRT make it an attractive tool to transform and interpret discrete data [5, 6].

An image defined on the 2-D array, $I(x, y)$, is mapped to a set of 1-D projections of length p. The discrete FRT projections are found as a set of parallel discrete line sums with intercepts t. There are $p + 1$ projections in total, p projections $R_m(t)$ for $0 \leq m < p$, and one "perpendicular" projection, $R_0^{\perp}(t)$. $R_m(t)$ is defined as the sum of pixel values centred on the discrete lines $x \equiv my + t \pmod{p}$, while $R_0^{\perp}(t)$ is found as the sum of all pixel values centred on the line $y \equiv 0x + t \pmod{p}$. The transform is then defined as

$$\begin{aligned} R_m(t) &= \textstyle\sum_{y=0}^{p-1} I(\langle my + t\rangle_p, y) \quad \text{for} \quad 0 \leq m < p, \\ R_0^{\perp}(t) &= \textstyle\sum_{x=0}^{p-1} I(x, t), \end{aligned} \tag{1}$$

where $\langle x\rangle_\eta$ denotes $x \pmod{\eta}$. An example of these discrete lines for $p = 31$ is presented in Fig. 1a and 1b.

The FRT formalism was extended by Hsung, Lun and Siu in 1996 to apply to square arrays of size 2^n for any positive integer n [2]. This version, termed the Discrete Periodic Radon Transform (DPRT), makes the transform more conducive to image processing, which commonly utilises images of size 2^n for computational efficiency. The DPRT has projections $R_m(t)$, as defined above, for $0 \leq m < N = 2^n$. However, since the array size is not prime, the image size N now has factors other than $N \equiv 0 \pmod{N}$. Additional "perpendicular" projections are required as $R_s^{\perp}(t)$ for $0 \leq s < N/2$, which are discrete line sums along $y \equiv 2sx + t \pmod N$. An example of these discrete lines for $N = 32$ is presented in Fig. 1c. Including all these discrete projections represents $I(x, y)$ exactly in the transform space. The DPRT is then defined as

$$\begin{array}{ll} R_m(t) = \sum_{y=0}^{N-1} I(\langle my + t\rangle_N, y) & \text{for } 0 \leq m < N, \\ R_s^{\perp}(t) = \sum_{x=0}^{N-1} I(x, \langle 2sx + t\rangle_N) & \text{for } 0 \leq s < \frac{N}{2}. \end{array} \quad (2)$$

The DPRT is a redundant representation of $I(x, y)$ as it has $N(1 + 1/2)$ projections of length N. A non-redundant form of the DPRT with orthogonal bases was presented by Lun and Hsung and Shen in 2003 [7], it is termed the Orthogonal DPRT (ODPRT).

The formalism for the DRT applied to arrays of size $p^n \times p^n$ is a natural extension of the DPRT and was developed in [3]. Projections are defined as for the DPRT, with p replacing 2 in (2). This is a redundant transform but requires only $N(1 + 1/p)$ projections of length N. A non-redundant form of this DRT, with orthogonal bases, was also presented in [3].

The above DRTs are all restricted to apply to square arrays with dimensions based on a single prime. Should the DRT be required of an array of arbitrary size $M \times N$, the image must be padded with zeroes to a square array with minimum size being the smallest $p^n \geq \sup\{M, N\}$. Padding arrays adds redundant information and unwanted computational complexity. This paper extends the FRT (over p), the DPRT (over 2^n) and the DRT (over p^n) to apply to square arrays of any composite size $N = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots$, where p_i is prime and n_i is any positive integer giving the prime decomposition of N. This definition is referred to in this paper as the DRT over N. A DRT for non-square arrays is the subject of ongoing research.

Section 2 establishes the set of projections that are required for an exact and invertible DRT and provides a method to obtain that set. This leads to the definition for the DRT over $N \in \mathbb{N}$ in section 3. A discrete form of the Fourier slice theorem which applies to this DRT and the related convolution property is also explained in this section. The inversion process for exact image reconstruction from composite array projections is presented in section 4.

2 Projection Sets for Invertible Mappings

For the FRT, each projection, m, is generated by lines with gradient m, however $\tan^{-1}(1/m)$ is not necessarily the best way to define the angle of the projection. It is defined here as the angle, θ_m, which provides the smallest translation between adjacent sampled pixels of the discrete line. Due to the assumption that the image is periodic, The sample pattern for each discrete line $x \equiv my + t \pmod p$ produces an infinite 2-D lattice with basis vectors $\{m, 1\}$ and $\{0, p\}$ (example depicted in Fig. 2b). Identifying the angle of the shortest vector between elements of this lattice is equivalent to finding the projection angle. The design of efficient algorithms to find the shortest length vectors in lattices has received

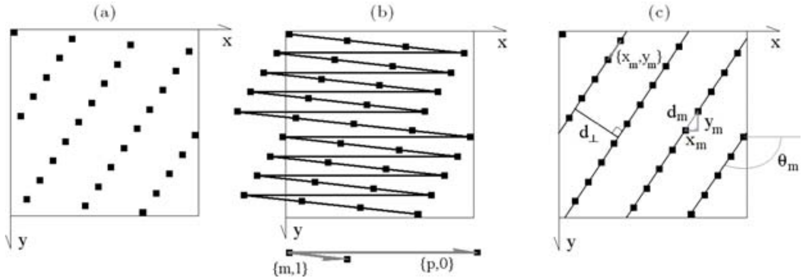

Fig. 2. Example of discrete line sums for $p = 29$, $m = 9$ and $t = 0$. Black pixels are those summed as a periodic discrete line (a) $x \equiv 9y + 0 \pmod{29}$. (b) Lattice basis $\{9, 1\}$ and $\{29, 0\}$ generates the sampling pattern of this periodic discrete line. (c) Shortest vector in lattice, $\{x_m, y_m\} = \{-2, 3\}$ at rational projection angle, $\theta_m = \tan^{-1}(y_m/x_m) = 123.7°$

considerable attention. For example, Rote showed that any shortest vector can be determined very efficiently through finding the reduced basis for 2D lattices [8]. This shortest vector is denoted $\{x_m, y_m\}$. The projection angle is then defined as $\theta_m = \tan^{-1}(y_m/x_m)$ (see Fig. 2c). Each projection, m, has a unique rational angle. These fractions are irreducible and form a subset of the Farey sequence [9].

The lattice can be generated by $\{0, p\}$ and/or $\{p, 0\}$, along with any other vector in the lattice $\{m, 1\}$, $\{2m, 2\}$, ..., $\{(p-1)m, p-1\}$, reduced modulo p. These vectors are equivalent for all parallel lines in a projection, regardless of the translate, t. Therefore each vector specifies an entire projection. Let Θ_p be the set of these vectors that define the projection set for a $p \times p$ array. Θ_p contains $\{m, 1\}$ (or any equivalent basis vector) for $0 \leq m < p$ and $\{1, 0\}$.

The pixels sampled by each discrete line with $t = 0$ correspond to the $p + 1$ unique cyclic subgroups of order p in the group $\mathbb{Z}_p \times \mathbb{Z}_p$ under addition. Each subgroup contains the identity $(0, 0)$ and, since p is prime, each subgroup contains $p - 1$ unique elements. These elements also uniquely sample each row and column of the image, (see the example in Fig. 1a). Therefore there must be $p+1$ subgroups to contain all p^2 elements, as $(p+1)(p-1)+1 = p^2$. Let $\Upsilon(N)$ represent the number of discrete lines in the projection set for the DRT of an $N \times N$ array. It can be seen $\Upsilon(p) = p + 1 = p(1 + 1/p)$ where p is prime. $\Upsilon(N)$ also gives the number of unique cyclic subgroups of order N in the group $\mathbb{Z}_N \times \mathbb{Z}_N$ under addition.

For the DPRT, the additional "perpendicular" discrete lines $y \equiv 2sx + t \pmod{2^n}$ are necessary, since $N = 2^n$ has factors so each pixel is not necessarily contained in a unique cyclic subgroup. Here Θ_{2^n} contains $\{m, 1\}$ for $0 \leq m < 2^n$ and $\{1, 2s\}$ for $0 \leq s < 2^{n-1}$. There are $N + N/2$ cyclic subgroups of order N in the group $\mathbb{Z}_N \times \mathbb{Z}_N$ under addition when $N = 2^n$. There are also cyclic subgroups of order less than N, however it is unnecessary to include discrete lines resulting from these, as each one is entirely contained within a cyclic subgroup of order

N. Therefore $\Upsilon(2^n) = 2^n + 2^{n-1} = 2^n(1+1/2)$. Similarly for the DRT over p^n, Θ_{p^n} contains $\{m,1\}$ for $0 \le m < p^n$ and $\{1,ps\}$ for $0 \le s < p^{n-1}$ with $\Upsilon(p^n) = p^n + p^{n-1} = p^n(1+1/p)$.

Define the discrete lines for the DRT over square arrays of composite size as $ax \equiv by + t \pmod{N}$. Θ_N then contains a number of vectors $\{b,a\}$. How many of these are there to ensure the transform is invertible and how are they found?

A function $f(n)$ is said to be *multiplicative* if $\gcd(m,n) = 1$ implies that $f(mn) = f(m)f(n)$ [10]. We can show that $\Upsilon(N)$ is multiplicative and can then define

$$\Upsilon(N) = N \prod_{p|N} (1+1/p). \tag{3}$$

Suppose that $\gcd(m,n) = 1$ and Θ_m contains $\{b_i,a_i\}_m$ for $0 \le i < \Upsilon(m)$ which gives the complete projection set for an $m \times m$ array and Θ_n contains $\{d_i,c_i\}_n$ for $0 \le i < \Upsilon(n)$ which gives the complete projection set for an $n \times n$ array. Then $n\Theta_m + m\Theta_n$ gives the projection set for the $mn \times mn$ array which will have $\Upsilon(m)\Upsilon(n)$ discrete lines.

If	$n\{b_1,a_1\}_m + m\{d_1,c_1\}_n \equiv n\{b_2,a_2\}_m + m\{d_2,c_2\}_n$	$\pmod{mn}$
then	$n\{b_1,a_1\}_m \equiv n\{b_2,a_2\}_m$	$\pmod{m}$
and so	$\{b_1,a_1\}_m \equiv \{b_2,a_2\}_m$	$\pmod{m}$,
similarly	$\{d_1,c_1\}_n \equiv \{d_2,c_2\}_n$	$\pmod{n}$.

So all the new discrete lines defined are incongruent (mod mn) and form the complete projection set for $mn \times mn$. The projection set for a composite N can be found via the projection sets for all the primes making up N. Below is an example demonstrating how to obtain the projection set for $N = 6$ from the sets for the two primes that make up 6, $N = 2$ and $N = 3$:

Given $\Theta_2 = \{1,0\}, \{0,1\}, \{1,1\}$.
$\Theta_3 = \{1,0\}, \{0,1\}, \{1,1\}, \{2,1\}$.

$\Theta_6 = 2\Theta_3 + 3\Theta_2$, i.e.,

+	$\{1,0\}$	$\{0,1\}$	$\{1,1\}$	$\{2,1\}$	$\times 2$
$\{1,0\}$	$\{5,0\}$	$\{3,2\}$	$\{5,2\}$	$\{1,2\}$	
$\{0,1\}$	$\{2,3\}$	$\{0,5\}$	$\{2,5\}$	$\{4,5\}$	
$\{1,1\}$	$\{5,3\}$	$\{3,5\}$	$\{5,5\}$	$\{1,5\}$	
$\times 3$					

This property is the underlying basis for generating the DRT for an arbitrary composite array size N.

3 DRT Formalism

3.1 Definition

Denote each element of the DRT as $R_{b,a}(t)$ which is found as the sum of all pixels in $I(x,y)$ such that $ax \equiv by + t \pmod{N}$. Here for the FRT and DPRT $R_m(t) =$

$R_{m,1}(t)$ and $R_s^{\perp}(t) = R_{1,2s}(t)$. The DRT over square arrays of composite size, $N \times N$, can be defined as

$$R_{b,a}(t) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} I(x,y)\delta\langle ax - by - t\rangle_N \quad \text{for} \quad \{b,a\} \in \Theta_N. \tag{4}$$

where $\delta\langle x\rangle_\eta$ is 1 when $x \equiv 0 \pmod{\eta}$ and 0 otherwise and Θ_N is as defined in section 2.

3.2 Properties

An important property of the continuous RT is the Fourier slice theorem which states that the 1-D Fourier transform (FT) of a continuous projection at angle θ is equivalent to a central radial slice through the 2-D FT of the original object/function at the angle $\theta^{\perp} = \theta + \pi/2$ [11]. A discrete form of this Fourier slice theorem can be demonstrated for the DRT. Denote $\widehat{R}_{b,a}(u)$ as the 1-D DFT of $R_{b,a}(t)$ for $\{b,a\} \in \Theta_N$, then

$$\begin{aligned}
\widehat{R}_{b,a}(u) &= \sum_{t=0}^{N-1} R_{b,a}(t) e^{-i2\pi ut/N} \\
&= \sum_{t=0}^{N-1} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} I(x,y) e^{-i2\pi ut/N} \delta\langle ax - by - t\rangle_N \\
&= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} I(x,y) e^{-i2\pi(axu - byu)/N} \\
&= \widehat{I}(au, -bu),
\end{aligned} \tag{5}$$

where $\widehat{I}(u,v)$ is the 2-D discrete FT (DFT) of $I(x,y)$. This gives a wrapped discrete line through the origin of Fourier space, $-bu \equiv av \pmod{N}$, which is perpendicular to the discrete line of projection as the product of the gradients is -1.

The DRT of an image can be obtained in $O(N\Upsilon(N)\log N)$ operations by utilising this property. $\widehat{I}(u,v)$, the 2-D DFT of the image $I(x,y)$, can be obtained in $O(N^2 \log N)$ operations. Each of the $\Upsilon(N)$ projections of length N can be obtained as the inverse 1-D DFT of the corresponding discrete slices in $O(N \log N)$ operations.

Another useful property of the continuous RT, which results from the Fourier slice theorem, is the convolution property. A discrete form of this property is also conserved in the DRT. Assume the function $F(x,y)$ on an $N \times N$ array is to be determined from the $N \times N$ function $G(x,y)$ and $L \times M$ function H(x,y), $0 < L, M \leq N$, by the 2D convolution

$$F(x,y) = \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{M-1} G(\langle x - \alpha\rangle_N, \langle y - \beta\rangle_N) H(\alpha,\beta). \tag{6}$$

This operation can be performed on one projection at a time in the DRT. Denote the DRT over N of $\eta(x,y)$ as $R^{\eta}_{b,a}(t)$. The value of $F(x,y)$ can be found through the DRT of $G(x,y)$ and $H(x,y)$ as

$$R^{F}_{b,a}(t) = \sum_{k=0}^{N-1} R^{G}_{b,a}(\langle t-k\rangle_N) R^{H}_{b,a}(k). \tag{7}$$

This shows the 2-D convolution of arrays can be performed in DRT space as a set of 1-D circular convolutions of each discrete projection. This reduces the computational complexity of 2-D problems such as filtering or matching, analogous to the applications for the FRT and DPRT outlined in [2].

4 DRT Inversion

The inverse transform (or image reconstruction from projections) for the DRT is achieved via a form of back-projection similar to the process of projection (4). To recover the value of the original function at pixel (i,j), the discrete line sums in each of the $\Upsilon(N)$ projections that contain this pixel are summed. This incorporates the value of the pixel (i,j) $\Upsilon(N)$ times and all other pixels at least once (depending on the degree of compositeness of N). The next section investigates how to determine the degree of over-representation, and the following section establishes a method to correct for it, allowing the exact value for the pixel (i,j) to be recovered.

4.1 Sampling Function, $\upsilon_N(x,y)$

Let $\upsilon_N(x,y)$ represent the number of times a pixel (x,y) is sampled by all discrete lines in the $N \times N$ DRT that include the origin. This function shows the over-representation of each pixel after back-projection when reconstructing the origin. This over-representation must be accounted for in reconstructing the original function. Since the array is congruent (mod N), the over-representation for any pixel (x,y) from the back-projection to reconstruct a pixel (i,j) can be found as $\upsilon_N(x-i,y-j)$, so it is sufficient to investigate reconstructing the origin only.

For the FRT case, where $N=p$, a prime, $\upsilon_p(x,y)=1$ for all pixels except the origin which is $\Upsilon(N)$ or $p+1$. An example of this for $p=7$ is shown in Fig. 3a. This can be written as $\upsilon_p(x,y) = d\prod_{d=p}(1+1/p)$ where $d=\gcd(x,y,p)$. For the DPRT case, where $N=2^n$, it was shown in [2] that $\upsilon_{2^n}(x,y)=\gcd(x,y,2^n)$ for all pixels except the origin which is $\Upsilon(N)=2^n(1+1/2)$. An example of this for $N=8$ is shown in Fig. 3b. This can be written as $\upsilon_{2^n}(x,y) = d\prod_{d=2^n}(1+1/2)$ where $d=\gcd(x,y,2^n)$. Similarly, for the DRT over p^n, it was shown in [3] that $\upsilon_{p^n}(x,y) = d\prod_{d=p^n}(1+1/p)$ where $d=\gcd(x,y,p^n)$. An example of this for $N=9$ is shown in Fig. 3c.

Suppose that $\gcd(m,n)=1$ where $\{a_i,b_i\}$ for $0 \le i < \upsilon_m(x,y)$ is the set of solutions $ax+by \equiv 0 \pmod{m}$ and $\{c_i,d_i\}$ for $0 \le i < \upsilon_n(x,y)$ is the set of solutions $cx+dy \equiv 0 \pmod{n}$. Then $n\{a,b\}+m\{c,d\}$ gives the solution

(a)

8	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

(b)

12	1	2	1	4	1	2	1
1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1
1	1	1	1	1	1	1	1
4	1	2	1	4	1	2	1
1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1
1	1	1	1	1	1	1	1

(c)

12	1	1	3	1	1	3	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
3	1	1	3	1	1	3	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
3	1	1	3	1	1	3	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

Fig. 3. Examples of the sampling function for (a) FRT over p, $\upsilon_7(x,y)$ (b) DPRT over 2^n, $\upsilon_8(x,y)$ (c) DRT over p^n, $\upsilon_9(x,y)$

set (mod mn) and there are $\upsilon_m(x,y)\upsilon_n(x,y)$ discrete lines. The proof that each new discrete line is unique is identical to that for $\Upsilon(N)$ in section 2, so all the new solutions defined are incongruent (mod mn) and form the complete set of solutions. Therefore we can say $\upsilon_N(x,y)$ is multiplicative and define

$$\upsilon_N(x,y) = d \prod_{p|d\nmid \frac{N}{d}} (1 + 1/p) \quad \text{where } d = \gcd(x,y,N). \tag{8}$$

where $p \mid d \nmid \frac{N}{d}$ denotes some prime p that divides d but does not divide N/d. Below is an example for finding $\upsilon_6(2,4)$ as $\upsilon_3(2,4)\upsilon_2(2,4)$, a graphical representation of the discrete lines is presented in Fig. 4.

$$\text{Given} \quad \begin{cases} \upsilon_2(2,4) = \upsilon_2(0,0) = 3 \;\; \text{as} \;\; \{1,0\}, \{0,1\}, \{1,1\}. \\ \upsilon_3(2,4) = \upsilon_3(2,1) = 1 \;\; \text{as} \;\; \{2,1\}. \end{cases}$$

$$\upsilon_6(2,4) = 3 \text{ found as } \begin{array}{c|l} + & \{1,0\}\ \{0,1\}\ \{1,1\}\ \times 3 \\ \hline \{2,1\} & \{1,2\}\ \{4,5\}\ \{1,5\} \\ \times 2 & \end{array}$$

$$\text{so the lines} \quad \begin{cases} 2x \equiv 1y \pmod 6 \\ 5x \equiv 4y \pmod 6 \\ 5x \equiv y \pmod 6 \end{cases} \quad \text{all include pixel } (i,j)$$

From the discrete Fourier slice theorem, given in section 3.2, υ_n defined about the origin of the image also gives the over-representation of spatial frequencies in the 2-D discrete Fourier transform, $\widehat{I}(u,v)$ as $\upsilon_N(v,-u) = \upsilon_N(u,v)$.

4.2 Correcting for $\upsilon_N(x,y)$ in Back-Projection

For the FRT, $\upsilon_p(x,y)$ is $\Upsilon(p) = p+1$ at the origin and 1 at all other (x,y). The sum of all discrete line sums containing a specific pixel (i,j),[i.e., $R_{1,0}(j)$ and $R_{m,1}\langle i - mj\rangle_p$ for $0 \le m < p$] gives the sum of the entire image, I_{sum}, with $I(i,j)$ an additional p times, (see example in Fig. 3b). The over-representation can be corrected for by subtracting the sum of the image, I_{sum} and dividing the result by p. So the image is recovered from its FRT as

$$I(x,y) = \frac{1}{p}\left(\sum_{m=0}^{p-1} R_{m,1}\langle x - ym\rangle_p + R_{1,0}(y) - I_{\text{sum}}\right), \tag{9}$$

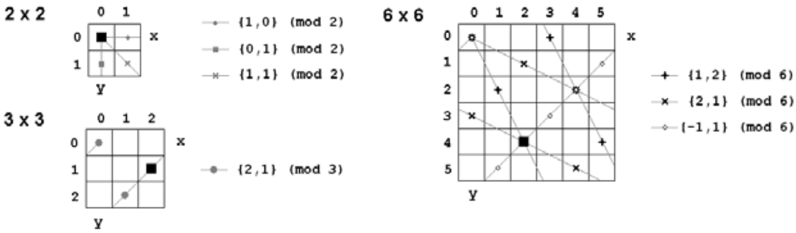

Fig. 4. Depiction of discrete lines presented in $\upsilon_6(2,4)$ example. Note in 6×6 case the basis vector $\{2,1\}$ generates the same lattice as $\{4,5\}$ and the basis vector $\{-1,1\}$ generates the same lattice as $\{1,5\}$, those used in this figure are simply the reduced bases

(a)

3	1
1	1

(b)

3	1	3	1	3	1	3	1
1	1	1	1	1	1	1	1
3	1	3	1	3	1	3	1
1	1	1	1	1	1	1	1
3	1	3	1	3	1	3	1
1	1	1	1	1	1	1	1
3	1	3	1	3	1	3	1
1	1	1	1	1	1	1	1

(c)

6	1	2	1
1	1	1	1
2	1	2	1
1	1	1	1

(d)

6	1	2	1	6	1	2	1
1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1
1	1	1	1	1	1	1	1
6	1	2	1	6	1	2	1
1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1
1	1	1	1	1	1	1	1

Fig. 5. (a) $\upsilon_2(x,y)$ (b) All discrete lines in Θ_2 on the 8×8 array, with intercept 0 (mod 2) yields $\upsilon_2(x,y)$ replicated (mod 2) (c) $\upsilon_4(x,y)$ (d) All discrete lines in Θ_4 on the 8×8 array, with intercept 0 (mod 4) yields $\upsilon_4(x,y)$ replicated (mod 4)

where I_{sum} can be obtained as the sum of all discrete line sums from any one projection, i.e., $\sum_{t=0}^{p-1} R_{q,1}(t)$ for any $0 \leq q < p$.

For the DPRT and DRT over p^n, $\upsilon_p(x,y)$ is $\Upsilon(p^n)$ at the origin and $\gcd(x,y,p^n)$ at all other (x,y). This gives an over-representation at each resolution $p \times p$, $p^2 \times p^2$, …, $p^{n-1} \times p^{n-1}$. To correct for this, an important property to note is that the sampling achieved by taking the projection set for a $p^k \times p^k$ array, Θ_{p^k}, for all discrete line sums t (mod p^k), (i.e., $\sum_{j=0}^{p^{n-k-1}} R_{b,a}(t + jp^k)$ for all $\{b,a\}$ in Θ_{p^k}) yields the sampling pattern for p^k, i.e., $\upsilon_{p^k}(x,y)$ replicated (mod p^k) in the x and y directions. An example of this for $p^k = 2$ and $p^k = 4$ within an 8×8 array is depicted in Fig. 5b and 5d.

This can be used to correct for $\upsilon_{p^n}(x,y)$ at resolution $p^k \times p^k$. $\upsilon_{p^n}(x,y)$ can therefore be corrected through a multi-resolutional process; each step, ξ_i, corrects for the over-representation described above at resolution $N/p^i \times N/p^i$. The inversion process is given as

$$I(x,y) = \xi_0(x,y) - \sum_{i=1}^{n-1} \frac{(p-1)}{p^i} \xi_i(x,y) - \frac{p}{N^2} I_{\text{sum}}, \tag{10}$$

where $\xi_i = \frac{1}{N} \left[\begin{array}{c} \sum_{m=0}^{\frac{N}{p^i}-1} \sum_{j=0}^{p^i-1} R_{m,1}\left(\langle x - my \rangle_{\frac{N}{p^i}} + j\frac{N}{p^i} \right) \\ + \sum_{s=0}^{\frac{N}{p^{i+1}}-1} \sum_{j=0}^{p^i-1} R_{1,s}\left(\langle y - psx \rangle_{\frac{N}{p^i}} + j\frac{N}{p^i} \right) \end{array} \right]$.

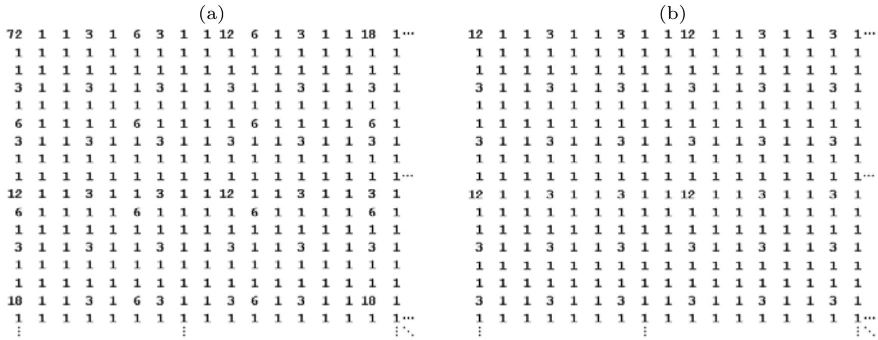

Fig. 6. Top left corner of (a) $\upsilon_{45}(x, y)$ (b) All discrete lines in Θ_9 on the 45×45 array, with intercept 0 (mod 9) yields $\upsilon_9(x, y)$ replicated (mod 9)

The inversion process for composite N is a natural extension of that for $N = p^n$. It is also undertaken by correcting for the over-representation at each resolution. The resolutions requiring correction correspond to all the factors, F, of N $(F|N)$. For the example in Fig. 6a, where $N = 3^2 5 = 45$, the oversampling must be corrected for at scales $15 \times 15, 9 \times 9, 5 \times 5$ and 3×3. Subtracting a certain fraction of I_{sum} corrects at resolution 1×1 leaving only a multiple (N) of $I(x, y)$.

The sampling achieved by taking the projection set for some $F|N$ for all projections t (mod F) yields the sampling pattern $\upsilon_F(x, y)$ repeated (mod F) in the x and y directions. An example of this is depicted in Fig. 6b for $N = 45$ and $F = 9$. This sampling pattern can be used to correct the over-representation from back-projection at a resolution of 9×9. This technique is used to correct the back-projection at each resolution.

Back projection for pixel (x, y) is defined as $\left(\sum_{\{a,b\} \in P_N} R_{a,b}\langle ay - bx\rangle_N\right)/N$. Correction for the over-representation at each resolution $F \times F$, for $F|P$, is found as $A_F \sum_{\{b,a\} \in \theta_F} \sum_{j=0}^{N/F-1} R_{b,a}\langle ay - bx + jF\rangle_N$ where the scaling factors, A_F, required are found as

$$A_F = \frac{F}{N} \prod_{p|\frac{N}{F}} (1 - p) \prod_{p|\frac{N}{F} \nmid F} \frac{p}{(1 - p)}. \tag{11}$$

where $p \mid \frac{N}{F} \nmid F$ denotes some prime, p, that divides N/F but does not divide F. Therefore the entire inversion process can be written as

$$I(x, y) = \frac{1}{N} \begin{bmatrix} \sum_{\{a,b\} \in P_N} R_{b,a}\langle ay - bx\rangle_N \\ - \sum_{F|N} A_F \sum_{\{b,a\} \in \Theta_F} \sum_{j=0}^{N/F-1} R_{b,a}\langle ax - by + jF\rangle_N \\ - A_1 I_{\text{sum}} \end{bmatrix}. \tag{12}$$

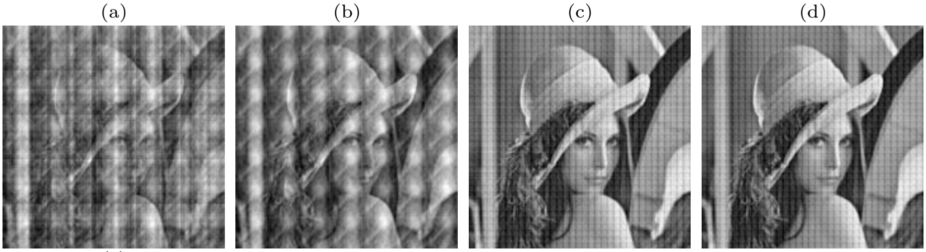

Fig. 7. Depiction of the correction made by each resolution of the reconstruction of the $\Upsilon(175) = 240 \times 175$ DRT of a 175 Lena image. (a) Uncorrected back-projection. (b) Corrected at resolution 35×35. (c) Corrected at resolution 25×25. (d) Corrected at resolution 7×7. (e) Correcting at resolution 5×5 and subtracting $A_1 I_{\text{sum}}$ completes all required corrections and exactly reproduces the original image

The result of the correction process at each resolution for the inversion of the $\Upsilon(175) \times 175$ DRT of a 175×175 Lena image is depicted in Fig. 4.2.

The reconstruction can also be performed via Fourier space by taking the 1-D DFT of each projection as $\mathrm{O}(N\Upsilon(N) \log N)$ and mapping it onto the 2-D DFT of the image, using the discrete Fourier slice theorem given in section 3.2. This will over-represent some spatial frequencies according to $\upsilon_N(u, v)$. Dividing the value at each spatial frequency by $\upsilon_N(u, v)$ and applying the inverse 2-D DFT to this data in $\mathrm{O}(N^2 \log N)$ operations recovers the original image.

5 Conclusion

A DRT based on modulo arithmetic which applies to $N \times N$ arrays for $N \in \mathbb{N}$ has been presented. It projects the 2-D image into a set of $N \prod_{p|N}(1 + 1/p)$ projections of length N. It is a redundant transform as with the DPRT. The multi-resolutional nature of the transform may prove useful in image analysis, particularly for textures and patterns. Research into this aspect and the development of an orthogonal DRT over N and $M \times N$ are the subject of ongoing work. An investigation into the distribution of the discrete angles required for $N \times N$ as compared to the $p \times p$ set and the uniformly distributed angle set for the continuous RT is also the subject of ongoing research.

Important properties of the continuous Radon transform are preserved in this discrete formalism. A discrete form of the Fourier slice theorem and the convolution property hold for this DRT over N. These properties allow the DRT to be obtained in $\mathrm{O}(N^2 \log N \prod_{p|N}(1 + 1/p))$ operations and make it a useful image processing tool, reducing 2-D problems to a set of 1-D problems.

Acknowledgements

Thanks to Fred Ninio, School of Physics and Materials Engineering, Monash University, for discussions on Group theory applied to give the DRT. AK is a Monash University postgraduate student in receipt of an Australian Postgraduate Award scholarship, provided through the Australian Government.

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