Building Blocks of Upward Planar Digraphs

Patrick Healy and Karol Lynch

CSIS Department, University of Limerick, Limerick, Ireland
`{patrick.healy,karol.lynch}@ul.ie`

Abstract. We show that a digraph is upward planar if and only if its biconnected components have certain properties.

1 Introduction

A drawing of a digraph is *planar* if no edges cross and *upward* if all edges are monotonically increasing in the vertical direction. A digraph is *upward planar* (UP) if it admits a drawing that is *both* upward and planar. The upward planarity of digraphs has been much studied and the area is surveyed by Di Battista et al. [1]. In this paper we show that a digraph is UP if and only if its blocks have UP drawings satisfying the conditions specified in Theorems 6–8. Following preliminary definitions in Section 2, lemmas concerned with sufficient conditions for upward planarity appear in Section 3, lemmas concerned with necessary conditions for upward planarity appear in Section 4, our main results appear in Section 5, and we conclude in Section 6.

2 Preliminaries

Let G be a digraph. We denote the *node set* of G by $V(G)$ and the *edge set* of G by $E(G)$. The *union* of two digraphs G and F , denoted $G \cup F$, is the digraph with node set $V(G \cup F) = V(G) \cup V(F)$ and edge set $E(G \cup F) = E(G) \cup E(F)$. A *null digraph* is a digraph whose node and edge sets are empty. A *strongly embedded digraph* G_φ is an equivalence class of planar drawings of a digraph G which belong to the same embedded digraph and which have the same outer face. Such a choice φ for an *embedding* and an outer face is called a *strong embedding* of G . A node v of G_φ is *bimodal* if the outgoing (or incoming) edges incident on v appear consecutively around v . If all the nodes of G_φ are bimodal then G_φ is *bimodal*. An *upward planar straight line* (UPSL) drawing of a digraph is an UP drawing in which each edge is represented by a straight line segment. Let G_ϕ be an embedded digraph with a node v . We use $\phi(v)$ to denote the clockwise ordering of the edges incident on v in G_ϕ . If e is an edge incident on v we use $\phi^e(v)$ to denote the linear sequence of edges with first edge e that is consistent with the circular sequence of edges $\phi(v)$. The *angles* of G_ϕ are ordered triples $\langle a, v, b \rangle$, where a and b are edges and v is a node incident on both a and b , such that either a directly precedes b in $\phi(v)$ or v is a node of degree 1. An angle $\langle a, v, b \rangle$ of G_ϕ is *incident on* the node v . An angle $\langle a, v, b \rangle$ is an *S-angle* (resp.,

T-angle) if both a and b leave (resp., enter) v ; and an *I-angle* if one of the edges a, b leaves v and the other enters v . Let f be a face of G_ϕ . We use the term *facial boundary* of f to refer to the circular sequence of nodes and edges W defined by traversing the boundary of f such that W contains a subsequence $\langle x, y, z \rangle$, where x and z are edges and y is a node if and only if $\langle x, y, z \rangle$ is an angle in G_ϕ . A face f is said to *contain* an edge, angle, or node x if x is a sub-sequence of the facial boundary of f . The angles of G_ϕ are mapped to geometric angles in an UPSL drawing Γ of G_ϕ . Let $\langle a, v, b \rangle$ be an angle of Γ . If $a \neq b$ the size of the corresponding geometric angle of $\langle a, v, b \rangle$ in Γ equals the number of radians one has to rotate a in the clockwise direction around v in order to reach b . If $a = b$ the size of the corresponding geometric angle of $\langle a, v, b \rangle$ is 2π . An angle of Γ is *large* (resp., *small*) if its corresponding geometric angle is greater (resp., smaller) than π . If f is a face we use $S_a(f)$ (resp., $L_a(f)$) to represent the number of *S-angles* (resp., (*large S-angles* + *large T-angles*)) contained by f . If v is a node we use $L_a(v)$ to represent the number of (*large S-angles* + *large T-angles*) incident on v . A *block* of a digraph G is a maximal connected subdigraph B of G such that no node of B is a cut-vertex of B . Let G be a digraph with a node v . A *component* of G with respect to v is formed from a connected component H of $G \setminus v$ by adding to H the node v and all edges between v and H . Let C be a component of G with respect to v . We use $G \setminus C$ to denote the digraph derived from G by deleting all the nodes in $C \setminus v$ from G . C is an *S-component*, *T-component*, or *I-component* of G with respect to (w.r.t.) v if v is a source node, sink node or internal node respectively in C . We use $S(v)$, $T(v)$, and $I(v)$ to refer to the subdigraph of G consisting of the union of all *S-components*, *T-components*, and *I-components* of G with respect to v , respectively. Definition 1 attempts to formalise the intuitive operation of adding two strongly embedded digraphs by identifying a node from each.

Definition 1. Let G_{φ_G} and F_{φ_F} be strongly embedded digraphs such that $V(G) \cap V(F) = \{u\}$, the outer face E_G of G_{φ_G} contains the angle $\langle g_1, u, g_2 \rangle$ and the face X_F of F_{φ_F} contains the angle $\langle f_1, u, f_2 \rangle$. The result of adding G_{φ_G} and F_{φ_F} by inserting $\langle g_1, u, g_2 \rangle$ within $\langle f_1, u, f_2 \rangle$ is the strongly embedded digraph H_φ where $H = G \cup F$, $\varphi(v) = \varphi_G(v), \forall v \in V(G) \setminus u, \varphi(v) = \varphi_F(v), \forall v \in V(F) \setminus u$, $\varphi(u) = \langle \varphi_G^{g_2}(u), \varphi_F^{f_2}(u) \rangle$, and the outer face is the face of H_φ whose facial boundary contains the facial boundary of the outer face of F_{φ_F} as a (not necessarily proper) subsequence.

Property 1. The set of faces of H_φ contains all the faces of G_{φ_G} except E_G , all the faces of F_{φ_F} except the face, X_F , and the “new” face X_H whose facial boundary is formed by concatenating the facial boundaries of E_G and X_F . Thus the facial boundary of X_H is $\langle u, g_2, \dots, g_1, u, f_2, \dots, f_1 \rangle$, where the list of nodes and edges from g_2 to g_1 concatenated with u is the facial boundary of E_G and the list of nodes and edges from f_2 to f_1 concatenated with u is the facial boundary of X_F . Therefore X_H contains all the angles contained by E_G except $\langle g_1, u, g_2 \rangle$, all the angles contained by X_F except $\langle f_1, u, f_2 \rangle$, and the new angles $\langle f_1, u, g_2 \rangle$ and $\langle g_1, u, f_2 \rangle$.

The following four preliminary lemmas have been proved by us previously [4].

Lemma 1. *If G and F are acyclic digraphs such that $V(G) \cap V(F) = \{u\}$, then $G \cup F$ is acyclic.*

Lemma 2. *Let $H_i, i = 1, \dots, c$ be the components of a digraph H with respect to a node u . Let $\mathcal{H}$ be a drawing of H and let $\mathcal{H}_i$ be the sub-drawing induced on $H_i, i = 1, \dots, c$. $\mathcal{H}$ is UP only if at least $c - 1$ of the sub-drawings $\mathcal{H}_i$ for $i = 1$ to c are UP drawings whose outer face contains u .*

Lemma 3. *Let H be a planar digraph with a node u such that there are exactly two components of H w.r.t. u , which we label G and F . Let $\mathcal{H}$ be a planar drawing of H and let $\mathcal{G}$ (resp., $\mathcal{F}$) be the sub-drawing induced on G (resp., F). Then all of $\mathcal{G}$ (resp., $\mathcal{F}$) lies in a single face of $\mathcal{F}$ (resp., $\mathcal{G}$) and at least one of $\mathcal{G}$ or $\mathcal{F}$ lies in the outer face of the other.*

Lemma 4. *Let H be a bimodal planar digraph with an internal node u such that there are exactly two components, G and F , of H with respect to u . Let $\mathcal{H}$ be a bimodal planar drawing of H and let $\mathcal{G}$ (resp., $\mathcal{F}$) be the subdrawing induced on G (resp., F). If G and F are I-components then all of $\mathcal{G}$ (resp., $\mathcal{F}$) lies in a single face of $\mathcal{F}$ (resp., $\mathcal{G}$) which contains an I-angle incident on u and at least one of $\mathcal{G}$ or $\mathcal{F}$ lies in the outer face of the other.*

We now present some previously published properties of UP digraphs. Lemma 5 and Theorem 1 are proved by Bertolazzi et al. [2]. Let G_φ be a strongly embedded digraph. Consider an *assignment* M that maps each source or sink v of G_φ to a face $M(v)$ of G_φ which contains v . Such an assignment M is *consistent* if the number of nodes assigned to the outer face h of G_φ equals $S_a(h) + 1$ and the number of nodes assigned to each internal face f of G_φ equals $S_a(f) - 1$. For each face z in G_φ we use $M^{-1}(z)$ to denote the set of nodes assigned to z by M .

Theorem 1 (Bertolazzi et al.). *A strongly embedded digraph G_φ is UP if and only if it is acyclic, bimodal and admits a consistent assignment of sources and sinks to its faces.*

Lemma 5 (Bertolazzi et al.). *The following properties hold for any UPSL drawing Γ of a digraph G . $L_a(f) = S_a(f) + 1$ if f is the outer face of Γ and $L_a(f) = S_a(f) - 1$ if f is an internal face of Γ . Also $L_a(v) = 0$ if v is an internal node of G and $L_a(v) = 1$ if v is a source or sink node of G .*

3 Sufficient Conditions for Upward Planarity

The same technique is used to prove each of the Lemmas 6–9. We present the proof of Lemma 6 here; proofs of Lemmas 7–9 can be found in [4]. Although the proofs of these lemmas are quite detailed the high level strategy is not. Given two UP strongly embedded digraphs G_{φ_G} and F_{φ_F} with certain properties we simply show that the strongly embedded digraph H_φ that results from adding G_{φ_G} and F_{φ_F} by inserting a certain angle from G_{φ_G} within a certain angle from the F_{φ_F} is acyclic, bimodal and has a consistent assignment of sinks and sources to its faces. It then follows from Theorem 1 that H_φ is UP.

Lemma 6. *Let G and F be UP digraphs such that $V(G) \cap V(F) = \{u\}$ and u is a source (resp., sink) node in both G and F . Let $H = G \cup F$. The following three statements hold. **1.** H is UP if at least one of G or F has an UP drawing whose outer face contains u . **2.** H has an UP drawing whose outer face contains u if G and F both have an UP drawing whose outer face contains u . **3.** H has an UPSL drawing whose outer face contains a large angle incident on u if G and F both have UPSL drawings whose outer face contains a large angle incident on u .*

Proof. Assume that u is a source node (the case when u is a sink node follows by symmetry) and that G has an UPSL drawing $\mathcal{G}$ whose outer face E_G contains the angle $\langle g_1, u, g_2 \rangle$. Let $\mathcal{F}$ be an UPSL drawing of F and let X_F be the face of $\mathcal{F}$ that contains $\langle f_1, u, f_2 \rangle$, the large angle incident on u in $\mathcal{F}$. Thus G (resp., F) has a strong embedding φ_G (resp., φ_F) such that G_{φ_G} (resp., F_{φ_F}) contains $\mathcal{G}$ (resp., $\mathcal{F}$) and has a consistent assignment, M_G (resp., M_F) of sinks and sources to faces, such that M_G (resp., M_F) assigns u to a face X_G (resp., the face X_F) of G_{φ_G} (resp., F_{φ_F}) (Theorem 1). Let H_φ be the strongly embedded digraph that results from adding G_{φ_G} and F_{φ_F} by inserting $\langle g_1, u, g_2 \rangle$ within $\langle f_1, u, f_2 \rangle$. We now show that H_φ is acyclic, bimodal and has a consistent assignment. That H is acyclic follows directly from Lemma 1. It follows from Definition 1 that $\varphi(v) = \varphi_G(v), \forall v \in V(G) \setminus u$ and that $\varphi(v) = \varphi_F(v), \forall v \in V(F) \setminus u$. But G_{φ_G} and F_{φ_F} are bimodal. Thus all nodes in $V(H) \setminus u$ are bimodal. As u is a source node it is also bimodal. Thus all nodes in H_φ are bimodal. The set of faces of H_φ consists of all the faces of G_{φ_G} except E_G , all the faces of F_{φ_F} except X_F , and the “new” face X_H (Property 1). We now show that H_φ has a consistent assignment M of sinks and sources to faces. M is defined as follows: **P1.** For each source or sink v of $V(H) \cap V(G)$, if $M_G(v) \neq E_G$ then $M(v) = M_G(v)$; otherwise $M(v) = X_H$. **P2.** For each source or sink v of $V(H) \cap V(F) \setminus u$, if $M_F(v) \neq X_F$ then $M(v) = M_F(v)$; otherwise $M(v) = X_H$. We now show that the assignment M is consistent. The number of S -angles in each face $z \neq X_H$ of H_φ is equal to the number of S -angles in its corresponding face in either G_{φ_G} or F_{φ_F} . It follows from **P1** that $|M_G^{-1}(z)| = |M^{-1}(z)|$ for each face $z \neq X_H$ of H_φ that is a face in both G_{φ_G} and H_φ . It follows from **P2** that $|M_F^{-1}(z)| = |M^{-1}(z)|$ for each face $z \neq X_H$ of H_φ that is a face in both F_{φ_F} and H_φ (because u was assigned to X_F by M_F). Thus from the consistency of M_G and M_F that the assignment M is consistent for each face $z \neq X_H$ of H_φ . We now consider the face X_H . All the S -angles contained by E_G (resp., X_F) except $\langle g_1, u, g_2 \rangle$ (resp., $\langle f_1, u, f_2 \rangle$) are also contained by X_H (Property 1). X_H also contains two “new” S -angles $\langle f_1, u, g_2 \rangle$ and $\langle g_1, u, f_2 \rangle$ (Property 1). Therefore $S_a(X_H) = S_a(E_G) - 1 + S_a(X_F) - 1 + 2 = S_a(E_G) + S_a(X_F)$. It follows from the consistency of M_G that the number of nodes in $V(G) \cap V(H)$ that are assigned to X_H by M equals $S_a(E_G) + 1$ (from **P1**). We separate the cases when X_F is an internal face of F_{φ_F} and X_F is the outer face of F_{φ_F} . **Case 1.** Assume X_F is an internal face of F_{φ_F} . Thus X_H is an internal face of H_φ . It follows from the consistency of M_F that the number of nodes in $V(F) \cap V(H) \setminus u$ that are assigned to X_H by M equals $S_a(X_F) - 2$ (from **P2**). Therefore $|M^{-1}(X_H)| = S_a(E_G) + 1 + S_a(X_F) - 2 = S_a(X_F) + S_a(E_G) - 1$. So M is consistent for X_H . **Case 2.** Assume X_F is the outer face of F_{φ_F} . Thus X_H

is the outer face of H_φ . It follows from the consistency of M_F that the number of nodes in $V(F) \cap V(H) \setminus u$ that are assigned to X_H by M equals $S_a(X_F)$ (from **P2**). So M is consistent for X_H . Thus M is a consistent assignment of nodes to the faces of H_φ for case 1 and for case 2. But H_φ is also acyclic and bimodal. Therefore H_φ is UP (Theorem 1). Thus H is UP. So Statement 1 is true. If the outer face of F_{φ_F} contains u then the outer face of H_{φ_H} also contains u (Definition 1). Thus Statement 2 holds. If $M_G(E_G) = u$ and X_F is the outer face of F_{φ_F} then X_H is the outer face of H_φ (Definition 1) and $M(u) = X_H$ (from **P1**). Therefore M assigns u to the outer face of H_φ . Thus H has an UPSL drawing whose outer face contains the large angle incident on u and so Statement 3 holds.

Lemma 7. *Let G and F be UP digraphs such that $V(G) \cap V(F) = \{u\}$ and u is an internal node in both G and F . Let $H = G \cup F$. Then the following four statements hold. 1. H is UP if at least one of G or F has an UP drawing whose outer face contains an I-angle incident on u . 2. H has an UP drawing whose outer face contains an I-angle incident on u if G and F both have an UP drawing whose outer face contains an I-angle incident on u . 3. H has an UP drawing whose outer face contains u , but does not contain an edge which enters (resp., leaves) u if at least one of G or F has an UP drawing whose outer face contains u , but does not contain an edge which enters (resp., leaves) u and the other has an UP drawing whose outer face contains an I-angle incident on u . 4. H has an UP drawing whose outer face contains u if at least one of G or F has an UP drawing whose outer face contains u and the other has an UP drawing whose outer face contains an I-angle incident on u .*

Lemma 8. *Let G and F be UP digraphs such that $V(G) \cap V(F) = \{u\}$ and u is a source node in G and a sink node in F . If $H = G \cup F$ then H is UP if at least one of G or F has an UPSL drawing whose outer face contains a large angle incident on u and H has an UP drawing whose outer face contains an I-angle incident on u if G and F both have an UPSL drawing whose outer face contains a large angle incident on u .*

Lemma 9. *Let G and F be UP digraphs such that $V(G) \cap V(F) = \{u\}$, u is a source (resp., sink) node in G , and u is an internal node in F . If $H = G \cup F$ then H is UP if G has an UPSL drawing whose outer face contains the large angle incident on u or F has an UP drawing whose outer face contains an edge which leaves (resp., enters) u . H has an UP drawing whose outer face contains an edge which enters (resp., leaves) u if G has an UPSL drawing whose outer face contains a large angle incident on u and F has an UP drawing whose outer face contains an edge which enters (resp., leaves) u .*

Lemmas 10 and 11 follow by induction from Lemma 6, whilst Lemma 12 follows by induction from Lemma 7.

Lemma 10. *Let G be a digraph with a source or sink node u . G has an UP drawing whose outer face contains u if every component of G w.r.t. u has an UP drawing whose outer face contains u .*

Lemma 11. *Let G be a digraph with a source (resp., sink) node u . G has an UPSL drawing whose outer face contains the large angle incident on u if every component of G w.r.t. u has an UPSL drawing whose outer face contains the large angle incident on u .*

Lemma 12. *Let G be a digraph with an internal node u such that all components of G w.r.t. u are I -components. G has an UP drawing whose outer face contains an I -angle incident on u if every component of G w.r.t. u has an UP drawing whose outer face contains an I -angle incident on u .*

Lemma 13. *Let G be a digraph with a node u such that all components of G w.r.t. u are I -components. G has an UP drawing whose outer face contains u , but no edge which enters (resp., leaves) u if some component G_x of G w.r.t. u has an UP drawing whose outer face contains u , but does not contain an edge which enters (resp., leaves) u and all other components of G w.r.t. u have an UP drawing whose outer face contains an I -angle incident on u .*

Proof. Let $G_{\overline{x}}$ be the union of all components of G w.r.t. u which are distinct from G_x . $G_{\overline{x}}$ has an UP drawing whose outer face contains an I -angle incident on u (Lemma 12). Therefore $G = G_x \cup G_{\overline{x}}$ has an UP drawing whose outer face contains u , but no edge which enters (resp., leaves) u (Lemma 7).

Lemma 14. *Let G be a digraph with a node u such that all components of G w.r.t. u are I -components. G has an UP drawing whose outer face contains u if all components of G with respect to u have an UP drawing whose outer face contains u and at most one component of G w.r.t. u does not have an UP drawing whose outer face contains an I -angle incident on u .*

Proof. If all components of G w.r.t. u have an UP drawing whose outer face contains an I -angle incident on u then G has an UP drawing whose outer face contains an I -angle incident on u (Lemma 12). Suppose some component G_y of G w.r.t. u has no UP drawing whose outer face contains an I -angle incident on u , but does have an UP drawing whose external face contains u . Let $G_{\overline{y}}$ be the union of the all components of G with respect to u which are distinct from G_y . If all the components of $G_{\overline{y}}$ w.r.t. u have an UP drawing whose external face contains an I -angle incident on u then $G_{\overline{y}}$ has an UP drawing whose outer face contains an I -angle incident on u (Lemma 12). It follows from Lemma 7 that $G = G_y \cup G_{\overline{y}}$ has an UP drawing whose external face contains u .

4 Necessary Conditions for Upward Planarity

Lemma 15. *Let G and F be DAGs such that $V(G) \cap V(F) = \{u\}$ and u is a source node in G and a sink node in F . If $H = G \cup F$ then H is UP only if at least one of G or F has an UPSL drawing whose outer face contains a large angle incident on u and the other is UP.*

Proof. If neither F nor G admit an UPSL drawing whose external contains a large angle incident on u then some components F' of F and G' of G w.r.t. u do

not admit an UPSL drawing whose external face contains a large angle incident on u (Lemma 11). Let $H' = F' \cup G'$, let $\mathcal{H}'$ be an UPSL drawing of H' and let $\mathcal{G}'$ and $\mathcal{F}'$ be the sub-drawings induced on G' and F' , respectively by $\mathcal{H}'$. At least one of $\mathcal{G}'$ or $\mathcal{F}'$ lies in the outer face of the other (Lemma 3). As u is an internal node in H' it follows from Lemma 5 that both $\mathcal{G}'$ and $\mathcal{F}'$ must lie in the face of the other that contains the large angle incident on u . This is a contradiction.

Lemma 16. *Let $H = G \cup F$ where G is an UP digraph with a source node u , F is an UP digraph with a node u such that all components of F w.r.t. u are I -components, and $V(G) \cap V(F) = \{u\}$. H is UP only if G has an UPSL drawing whose outer face contains a large angle incident on u or F has an UP drawing whose outer face contains an edge which leaves u .*

Proof. (outline) As in the proof of Lemma 15 we show that if both G and F do not have drawings with the stated properties then some component G' of G w.r.t. u does not admit an UPSL drawing whose outer face contains a large angle incident on u , and some component F' of F w.r.t. u does admit an UP drawing whose external face contains an edge which leaves u . We then use Lemmas 5 and 3 to show that $F' \cup G'$ is not upward planar.

Lemma 17. *Let $G_i, i = 1, \dots, c$ be the components of a digraph G with respect to a node u such that all components of G with respect to u are I -components. Let $\mathcal{G}$ be a drawing of G and let $\mathcal{G}_i$ be the sub-drawing induced on G_i , for $i = 1, \dots, c$. Then $\mathcal{G}$ is an UP drawing whose outer face contains an I -angle incident on u only if each $\mathcal{G}_i$ for $i = 1, \dots, c$ is an UP drawing whose external face contains an I -angle incident on u .*

Proof. Let a be any integer such that $1 \leq a \leq c$. Suppose that $\mathcal{G}_a$ is not an UP drawing whose outer face contains an I -angle incident on u . If $\mathcal{G}_a = \mathcal{G}$ then it is trivially true that $\mathcal{G}$ is not an UP drawing whose outer face contains an I -angle incident on u . Otherwise let b be any integer such that $(1 \leq b \leq c) \wedge (b \neq a)$ and let $\mathcal{G}_a \cup \mathcal{G}_b$ be the sub-drawing of $\mathcal{G}$ induced on $G_a \cup G_b$. Because the outer face of $\mathcal{G}_a$ does not contain an I -angle incident on u it follows from Lemma 4 that $\mathcal{G}_a \cup \mathcal{G}_b$ is bimodal planar only if all of $\mathcal{G}_b$ is drawn within an internal face of $\mathcal{G}_a$. Therefore $\mathcal{G}_a \cup \mathcal{G}_b$ is UP only if the outer face of $\mathcal{G}_a$ is also the outer face of $\mathcal{G}_a \cup \mathcal{G}_b$. Therefore $\mathcal{G}$ is UP only if the outer face of $\mathcal{G}_a$ is also the outer face of $\mathcal{G}$. Therefore $\mathcal{G}$ is an UP drawing whose outer face contains an I -angle incident on u only if each $\mathcal{G}_i$ for $i = 1, \dots, c$ is an UP drawing whose outer face contains an I -angle incident on u .

Lemma 18. *Let $G_i, i = 1, \dots, c$ be the components of a digraph G with respect to a node u . Let $\mathcal{G}$ be a drawing of G and let $\mathcal{G}_i$ be the sub-drawing induced on $G_i, i = 1, \dots, c$. $\mathcal{G}$ is an UP drawing whose outer face contains u only if each $\mathcal{G}_i$ for $i = 1$ to c is an UP drawing whose outer face contains u .*

Proof. Let x be any integer such that $1 \leq x \leq c$. Suppose that $\mathcal{G}_x$ is an UP drawing whose outer face $\mathcal{E}_x$ does not contain u . Therefore u is a point in the interior of some closed curve $\mathcal{E}'_x$ that is a sub-drawing of $\mathcal{E}_x$. But $\mathcal{E}'_x$ is a sub-drawing of $\mathcal{G}$. Therefore u is not contained by the outer face of $\mathcal{G}$. Therefore $\mathcal{G}$

is an UP drawing whose outer face contains u only if each $\mathcal{G}_i$, $i = 1, \dots, c$, is an UP drawing whose outer face contains u .

Proofs of the next two lemmas appear previously [4].

Lemma 19. *Let G_i , $i = 1, \dots, c$ be the components of a digraph G with respect to a source (resp., sink) node u . Let $\mathcal{G}$ be a drawing of G and let $\mathcal{G}_i$ be the sub-drawing induced on G_i , for $i = 1, \dots, c$. Then $\mathcal{G}$ is an UPSL drawing whose outer face contains a large angle incident on u only if each $\mathcal{G}_i$, $i = 1, \dots, c$, is an UPSL drawing whose outer face contains a large angle incident on u .*

Lemma 20. *Let G_i , $i = 1, \dots, c$ be the components of a digraph G w.r.t. a node u such that all components of G with respect to u are I -components. Let $\mathcal{G}$ be a drawing of G and let $\mathcal{G}_i$ be the sub-drawing induced on G_i , for $i = 1, \dots, c$. Then $\mathcal{G}$ is an UP drawing whose outer face contains u , but does not contain an edge which enters (resp., leaves) u only if exactly one of the drawings $\mathcal{G}_i$, $i = 1, \dots, c$ is an UP drawing whose outer face contains u , but does not contain an edge which enters (resp., leaves) u and exactly $c - 1$ of the drawings $\mathcal{G}_i$, $i = 1, \dots, c$ are UP drawings whose outer face contains an I -angle incident on u .*

5 Main Results

Theorem 2. *Let G be a digraph with a source or sink node u . G is UP if and only if all components of G w.r.t. u are UP and at most one component of G w.r.t. u does not have an UP drawing whose outer face contains u .*

Proof. Suf. If all components of G w.r.t. u have an UP drawing whose outer face contains u then G is UP (Lemma 10). Suppose some UP component X of G w.r.t. u does not have an UP drawing whose outer face contains u , but that all components of $G \setminus X$ w.r.t. u do have an UP drawing whose outer face contains u . Therefore $G \setminus X$ has an UP drawing whose outer face contains u (Lemma 10). Thus $G = X \cup G \setminus X$ is UP (Lemma 6). **Nec.** Clearly if some component Y of G w.r.t. u is not UP then G is not UP because Y is a subdigraph of G . Also if two components Y_1 and Y_2 of G with respect to u do not have an UP drawing whose outer face contains u then it follows from Lemma 2 that G is not UP.

Theorem 3. *Let G be a digraph with a internal node u such that all components of G w.r.t. u are either S -components or T -components. G is UP if and only if all components of G with respect to u are UP, at most one component of G w.r.t. u does not have an UP drawing whose outer face contains u and at least one of $S(u)$ or $T(u)$ has an UP drawing whose outer face contains a large angle incident on u .*

Proof. (outline) The sufficiency follows from Theorem 2 and Lemma 8 whilst the necessity follows from Lemma 2 and Lemma 15.

Theorem 4. *Let G be a digraph with a node u such that all components of G w.r.t. u are I -components. G is UP if and only if all components of G w.r.t. u are UP and at most one component of G w.r.t. u does not have an UP drawing whose outer face contains an I -angle incident on u .*

Proof. Suf. Let X be a component of G w.r.t. u such that all components of $G \setminus X$ w.r.t. u admit an UP drawing whose outer face contains an I -angle incident on u . Therefore $G \setminus X$ has an UP drawing whose outer face contains an I -angle incident on u (Lemma 12). Therefore $G = X \cup G \setminus X$ is UP if X is UP (Lemma 7). **Nec.** Suppose that two components G^1 and G^2 of G w.r.t. u have no UP drawing whose outer face contains an I -angle incident on u . Let $H = G_1 \cup G_2$. Let $\mathcal{H}$ be a planar drawing of H and let $\mathcal{G}^1$ (resp., $\mathcal{G}^2$) be the sub-drawing induced on G^1 (resp., G^2). It follows from Lemma 4 that $\mathcal{H}$ is bimodal planar only if $\mathcal{G}^1$ (resp., $\mathcal{G}^2$) lies in a face of $\mathcal{G}^1$ (resp., $\mathcal{G}^2$) which contains an I -angle incident on u and at least one of G^1 or G^2 lies in the external face of the other. Therefore H has no UP drawing. But H is a subgraph of G .

A proof of Theorem 5 appears previously [4].

Theorem 5. *Let G be a digraph with an internal node u such that there are no T -components (resp., S -components) of G w.r.t. u . G is UP if and only if all components of G w.r.t. u are UP, at most one I -component of G w.r.t. u does not have an UP drawing whose outer face contains an I -angle incident on u , and at least one of the following two statements is true:*

1. *All S -components (resp., T -components) of G w.r.t. u have an UPSL drawing whose outer face contains the large angle incident on u .*
2. *All I -components of G w.r.t. u have an UP drawing whose outer face contains an edge which leaves (resp., enters) u and at most one S -component (resp., T -component) of G with respect to u does not have an UP drawing whose outer face contains u .*

Theorem 6. *Let G be a digraph and let u be any node in G . G is UP if and only if all components of G w.r.t. u are UP, at most one I -component of G w.r.t. u does not have an UP drawing whose outer face contains an I -angle incident on u , at most one S -component or T -component of G w.r.t. u does not have an UP drawing whose outer face contains u , and at least one of the following three statements is true:*

1. *There are no S -components or T -components of G w.r.t. u which do not admit an UP drawing whose outer face contains a large angle incident on u .*
2. *There are no S -components of G w.r.t. u which do not admit an UP drawing whose outer face contains a large angle incident on u and there are no I -components of G w.r.t. u which do not admit an UP drawing whose outer face contains an edge which enters u .*
3. *There are no T -components of G w.r.t. u which do not admit an UP drawing whose outer face contains a large angle incident on u and there are no I -components of G w.r.t. u which do not admit an UP drawing whose outer face contains an edge which leaves u .*

Proof. Suf. Assume that all components of G w.r.t. u are UP and that at most one I -component of G w.r.t. u does not have an UP drawing whose outer face contains an I -angle incident on u . Thus $I(u)$ is UP (Theorem 4). Also assume that at most one S -component or T -component does not have an UP drawing

whose outer face contains u . Thus $S(u)$ and $T(u)$ are both UP (Theorem 2). We consider three cases. **Case 1.** Assume Statement 1 is true. Therefore $S(u)$ and $T(u)$ both have an UPSL drawing whose outer face contains a large angle incident on u (Lemma 11). Thus $S(u) \cup T(u)$ has an UP drawing whose outer face contains an I -angle incident on u (Lemma 8). But $I(u)$ is UP. Therefore $G = I(u) \cup (S(u) \cup T(u))$ is UP (Lemma 7). **Case 2.** Assume Statement 2 is true. Thus $S(u)$ has an UP drawing whose outer face contains a large angle incident on u (Lemma 11) and $I(u)$ has an UP drawing whose external face contains an edge which enters u (Lemmas 14 and 20). It follows from Lemma 9 that $S(u) \cup I(u)$ has an UP drawing whose outer face contains an edge which enters u . But $T(u)$ is UP. Thus $G = (S(u) \cup I(u)) \cup T(u)$ is UP (Lemma 9). **Case 3.** Assume Statement 3 is true. It follows by symmetry from the sufficiency of case 2 that G is UP. **Nec.** Every component of G w.r.t. u is a subgraph of G . Therefore G is UP only if all components of G with respect to u are UP. It follows from Theorem 4 that $I(u)$ is UP only if at most one I -component of G with respect to u does not have an UP drawing whose outer face contains an I -angle incident on u . It follows from Lemma 2 that $S(u) \cup T(u)$ is UP only if at most one S -component or T -component of G w.r.t. u does not have an UP drawing whose outer face contains u . We now prove the necessity of at least one of Statements 1, 2 or 3 of Theorem 6 being true. We will need the following four statements. Statement $\mathcal{A}$ (resp., Statement $\mathcal{B}$) is that all components of $S(u)$ (resp., $T(u)$) w.r.t. u have an UPSL drawing whose outer face contains a large angle incident on u . Statement $\mathcal{C}$ (resp., Statement $\mathcal{D}$) is that all components of $I(u)$ w.r.t. u have an UP drawing whose outer face contains an edge which leaves (resp., enters) u . $S(u) \cup T(u)$ is UP only if at least one of Statement $\mathcal{A}$ or Statement $\mathcal{B}$ is true (Theorem 3). $I(u) \cup S(u)$ is UP only if at least one of Statement $\mathcal{A}$ or Statement $\mathcal{C}$ is true (Theorem 5). $I(u) \cup T(u)$ is UP only if at least one of Statement $\mathcal{B}$ or Statement $\mathcal{D}$ is true (Theorem 5). Therefore $G = I(u) \cup S(u) \cup T(u)$ is UP only if Statements $\mathcal{A}$ and $\mathcal{B}$ are both true, and/or Statements $\mathcal{A}$ and $\mathcal{D}$ are both true, and/or Statements $\mathcal{B}$ and $\mathcal{C}$. But (Statement $\mathcal{A} \wedge$ Statement $\mathcal{B}$) is equivalent to Statement 1 of Theorem 6; (Statement $\mathcal{A} \wedge$ Statement $\mathcal{D}$) is equivalent to Statement 2 of Theorem 6; (Statement $\mathcal{B} \wedge$ Statement $\mathcal{C}$) is equivalent to Statement 3 of Theorem 6. Thus G is UP only if at least one of Statements 1, 2 or 3 of Theorem 6 is true.

Proofs of the following two theorems are similar to that of Theorem 6.

Theorem 7. *Let G be a digraph with a node u , such that C is an S -component (resp., T -component) of G w.r.t. u that contains a node w . G has an UP drawing whose outer face contains a certain type of angle incident on w if and only if C has an UP drawing whose outer face contains the same type of angle incident on w , there are no S -components (resp., T -components) of $G \setminus C$ w.r.t. u which do not admit an UP drawing whose outer face contains u , there are no T -components (resp., S -components) of $G \setminus C$ with respect to u which do not admit an UP drawing whose outer face contains a large angle incident on u , there are no I -components of $G \setminus C$ w.r.t. u which do not admit an UP drawing whose outer face contains an edge which leaves (resp., enters) u and at most one*

I-component of $G \setminus C$ w.r.t. u does not have an UP drawing whose outer face contains an *I*-angle incident on u .

Theorem 8. *Let G be a digraph with a node u , such that C is an *I*-component of G w.r.t. u that contains a node w . G has an UP drawing whose outer face contains a certain type of angle incident on w if and only if C admits an UP drawing whose outer face contains the same type of angle incident on w , there are no *S*-components or *T*-components of $G \setminus C$ with respect to u which do not admit an UP drawing whose outer face contains a large angle incident on u , and all *I*-components of $G \setminus C$ w.r.t. u have an UP drawing whose external face contains an *I*-angle incident on u .*

6 Conclusions

Bertolazzi et al.'s algorithm for testing an embedded digraph for upward planarity [2] can be modified to detect for the properties listed in Theorems 6–8. A divide and conquer approach to upward planarity testing follows that involves recursively splitting a digraph at its cut-vertices and testing the individual blocks for the given properties. Bertolazzi et al. also describe a branch-and-bound algorithm that tests the upward planarity of biconnected digraphs [3]. An interesting question is could their algorithm also be tailored to detect the properties listed in Theorems 6–8.

References

1. G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs*. Prentice-Hall, 1999.
2. P. Bertolazzi, G. Di Battista, G. Liotta, and C. Mannino. Upward drawings of triconnected digraphs. *Algorithmica*, 6(12):476–497, 1994.
3. P. Bertolazzi, G. Di Battista, and W. Didimo. Quasi-upward planarity. *Algorithmica*, 32:474–506, 2002.
4. P. Healy and W. K. Lynch. Investigations into upward planar digraphs. Technical Report TR-04-02, Dept. of CSIS, University of Limerick, 2004.
<http://www.csis.ul.ie/Research/TechRpts.htm>