Computing Radial Drawings on the Minimum Number of Circles^{*}

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Abstract. A radial drawing is a representation of a graph in which the vertices lie on concentric circles of finite radius. In this paper we study the problem of computing radial drawings of planar graphs by using the minimum number of concentric circles. We assume that the edges are drawn as straight-line segments and that co-circular vertices can be adjacent. It is proven that the problem can be solved in polynomial time.

1 Introduction

A radial drawing is a representation of a graph in which the vertices are constrained to lie on concentric circles of finite radius. Drawing graphs radially is relevant in situations where it is important to display a graph with the constraint that some vertices are drawn “more central” than others. Examples of such applications include social networks analysis (visualization of policy networks and co-citation graphs), operating systems (visualization of filesystems), cybergeography (visualization of Web maps and communities), and bioinformatics (visualization of protein-protein interaction diagrams); see e.g. [4, 8, 9].

This paper investigates crossing-free radial drawings of planar graphs. Let G be a planar graph. A crossing-free radial drawing of G induces a partition of its vertices into levels such that vertices in the same level are co-circular in the drawing; for each level, the planarity of the drawing induces a circular ordering of the vertices in the level. Conversely, in order to construct a radial drawing of G a partition of its vertices into levels and a circular ordering within each level must be found such that vertices of the same level are drawn co-circularly and the edges can be drawn without intersecting each other.

Bachmaier et al. [1, 2] investigate the radial planarity testing problem: Given a partition of the vertices of G into levels, they want to test whether there exists a crossing-free radial drawing of G consistent with the given leveling (i.e. vertices in the same level can be drawn on the same circle and the edges can be added without crossing). In [1] it is assumed that the edges are drawn as strictly monotone curves from inner to outer circles and that no two co-circular

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vertices are connected by an edge. The elegant linear-time algorithm presented by Bachmaier et al. tests radial planarity by using an extension of PQ -trees, called PQR -trees. In [2] the authors extend the algorithm to the case where edges between co-circular vertices are allowed.

In this paper we study radial drawings of planar graphs from a different perspective. We assume that the partition of the vertices of G is not given and our goal is to compute a partition that minimizes the number of levels, i.e. that corresponds to a crossing-free straight-line radial drawing of G on the minimum number of circles. We call such a drawing a *minimum radial drawing* of G . In contrast with the drawing conventions adopted in [1], we assume that the edges are straight-line segments and that vertices on the same level can be adjacent. These choices are justified by different application-oriented examples of radial drawings that adopt the straight-line standard (see e.g. [12, 13]) and by the observation that allowing edges among co-circular vertices appears to be a natural approach for the reduction of the number of levels.

The contribution of the present paper is to characterize those graphs that can be drawn on a given number of concentric circles and to use this characterization to solve the above described optimization problem. More precisely:

- We show that every 2-outerplanar graph admits a crossing-free straight-line radial drawing on two circles. The proof is constructive and the radial drawing can be computed in linear time. Preliminary results on computing radial drawing of 2-outerplanar graphs appear in [6].
- We generalize this results and characterize the family of graphs that admit a crossing-free straight-line radial drawing on at most $k \geq 2$ circles. We recall that similar characterization problems for straight-line k -layered drawings are studied for the case of $k \leq 3$; see, e.g. [5]. We also recall that a planar graph admits a drawing on one circle if each edge can bend at most once [7].
- Based on the characterization above, we show that there exists a polynomial time algorithm to compute a minimum radial drawing of a planar graph. The drawing has the additional property of being “proper”, i.e. an edge always connects either co-circular vertices or vertices on consecutive circles.

For reasons of space some proofs are sketched or omitted.

2 Preliminaries

A *1-outerplanar embedded graph* (also called *1-outerplane graph*) is an embedded planar graph where all vertices are on the external face. An embedded graph is a *k-outerplanar embedded graph* (also called *k-outerplane graph*) ($k > 1$) if the embedded graph obtained by removing all vertices of the external face is a $(k - 1)$ -outerplane graph. The planar embedding of a k -outerplane graph is called a *k-outerplanar embedding*. A graph is *k-outerplanar* if it admits a k -outerplanar embedding. A planar graph G has *outerplanarity* k (for an integer $k > 0$) if it is k -outerplanar and it is not j -outerplanar for $0 < j < k$. In other words, the outerplanarity of G denotes the minimum value of k for which G is k -outerplanar.

Let G be a k -outerplane graph with $k > 1$. We associate a *level* with each vertex v of G , denoted as $lev(v)$, according to the following definition: $lev(v) = 0$ if v is on the external face of G and $lev(v) = i$ ($i = 1, \dots, k-1$) if v is on the external face after the removal of every vertex u with $lev(u) < i$. If $lev(v) = i$, we say that v is a vertex of level i . Let V_i be the set of vertices v with $lev(v) = i$. The subgraph induced by V_i is denoted by $G_i = (V_i, E_i)$. Notice that G_i is a graph of outerplanarity 1. Let $V_{i,i+1} = V_i \cup V_{i+1}$. The subgraph induced by $V_{i,i+1}$ is denoted by $G_{i,i+1} = (V_{i,i+1}, E_{i,i+1})$.

We use $C_0, C_1, \dots, C_{k-1}$ to denote a set of k concentric circles in the plane, where the radius of C_i is greater than the radius of C_{i+1} ($i = 0, \dots, k-2$). Let G be a planar graph and let Γ be a crossing-free straight-line drawing of G . The drawing Γ is a *radial drawing* if the vertices of G are placed on a set of concentric circles. Γ will be called a *k -radial drawing* of G if it is a radial drawing on $C_0, C_1, \dots, C_{k-1}$. Γ is a *minimum radial drawing* if it uses the minimum number of circles. An edge (u, v) with u and v on C_i is called an *intra-level edge*. An edge (u, v) with u and v on C_i and C_j with $i \neq j$ is called an *inter-level edge*. If all inter-level edges of a radial drawing Γ connect vertices on consecutive circles, Γ is called a *proper radial drawing*.

Let G be a k -outerplane graph. A radial drawing of G is *level-preserving* if it is a k -radial drawing and every vertex v with $lev(v) = i$ is drawn on circle C_i . A level-preserving k -radial drawing of a k -outerplane graph is proper.

3 Overview of the Approach

We study the problem of computing a radial drawing of a planar graph G on the minimum number of circles. We show that a minimum radial drawing of G can be computed in polynomial time. Namely: (a) We prove that if a graph has outerplanarity k then it admits a k -radial drawing; Also if a graph has a radial drawing on k -circles then it has outerplanarity at most k . (b) We use the above characterization and a result by Bienstock and Monma [3] to show that there exists an $O(n^5 \log n)$ -time algorithm that computes a minimum radial drawing of G .

The trickiest part is to show that a graph with outerplanarity k has a k -radial drawing. We provide a linear-time algorithm that receives as input a k -outerplane graph G and computes a level-preserving k -radial drawing of G . Our approach can be summarised as follows. We start with G_0 , draw the vertices in V_0 on C_0 while maintaining their circular ordering in G_0 . After placing V_i on C_i we compute the radius of C_{i+1} and draw V_{i+1} on C_{i+1} without moving any vertex from V_j with $0 \leq j \leq i$. For ease of presentation, we will define *canonical k -outerplanar graphs* and show how each k -outerplane graph can be transformed into a canonical k -outerplane graph. We will also show that a k -outerplane graph has a k -radial drawing if and only if its canonical form has a k -radial drawing.

4 Canonical Graphs and Equipped *BC*-Trees

Let G be a k -outerplane graph. A *mixed face* of G is a face containing vertices of two consecutive levels. G is called *inter-triangulated* if all its mixed faces are

three-cycles. Assume G is inter-triangulated. Let c be a cut-vertex of G_{i+1} . Let B and B' be two blocks (i.e. biconnected components) of G_{i+1} that are consecutive when going around c in clockwise direction.

Since G is inter-triangulated, there exists at least one edge of $E_{i,i+1}$ incident on c that is encountered between B and B' when going around c in the clockwise direction. Such an edge of $E_{i,i+1}$ is called a *separating edge* because it separates blocks B and B' around c . G is said to be *canonical* if it is inter-triangulated and for any i ($i = 0, \dots, k-2$) and for any two clockwise consecutive blocks B, B' of G_{i+1} around a cut-vertex, there is exactly one separating edge.

Every connected k -outerplane graph can be made canonical as stated by the following lemma.

Lemma 1. *Let G be a connected k -outerplane graph with n vertices. There exists an $O(n)$ -time algorithm that computes an augmented graph G' such that: (1) G' is k -outerplane, (2) G' is canonical, and (3) the levels of the vertices of G are preserved in G' .*

We now introduce equipped BC -trees. Let G be a k -outerplane graph with $k > 1$. We extend the block cut-vertex tree data structure [11] to identify specific subgraphs of G . Because of Lemma 1, we can (and will) restrict our attention to canonical graphs. Let K be a connected component of G_i with $i > 1$. An *equipped BC -tree* T of K is an embedded rooted tree such that (for an illustration see Figures 1(a), 1(b), 1(c)):

- T has three types of nodes: (a) A *B-node* for each block B_K of K , referred to as the *B-node of B_K* . (b) A *C-node* for each cut-vertex c of K , referred to as the *C-node of c* . (c) A *D-node* for each separating edge e of G that is incident on a cut-vertex of K , referred to as the *D-node of e* .
- If K is biconnected, T consist of a single *B-node*. If K is not biconnected, we choose an arbitrary *C-node* as the root of T .
- The edges of T are of two types: (a) Edges connecting a *C-node* of a cut-vertex c to a *B-node* of a block that contains c . (b) Edges connecting a *C-node* of a cut-vertex c to a *D-node* of a separating edge incident on c .
- The planar embedding of T reflects the embedding of G : if e is a separating edge incident on a cut-vertex c and e is between blocks B and B' in clockwise ordering around c , then the *D-node* of e is between the *B-nodes* of B and B' in clockwise ordering around the *C-node* of c .

For example cut vertex 1 is chosen as the root of the tree shown in Figure 1(c). Separating edge $(2, b)$ in Figure 1(b) separates blocks A and D . Correspondingly, the *D-node* of $(2, b)$ appears between the *B-nodes* of A and D in the circular clockwise ordering around the *C-node* of cut-vertex 2 in the equipped BC -tree of Figure 1(c).

Let μ be the *C-node* of a cut-vertex c of K , if μ is not the root of T , the parent of μ is a *B-node* and the leftmost child and the rightmost child of μ are *D-nodes*. If μ is the root of T we arbitrarily choose the leftmost child of μ as a *D-node*; as a consequence the rightmost child of the root is a *B-node*. See for example Figure 1(c) where the rightmost child of the root is the *B-node* of B , while the leftmost child is the *D-node* of $(1, e)$.

Fig. 1. (a) A 2-outerplane graph G . The blocks of G_1 are highlighted and labeled with capital letters. The cut-vertices of G_1 are numbered squares, and their separating edges are bold. (b) A schematic representation of the structure of G . The skeleton is highlighted with thick edges. (c) An equipped BC -tree of G rooted at the C -node of cut-vertex 1. (d) A drawing of the skeleton of G . The labels of the regions reflect those of the corresponding blocks. (e) A level-preserving 2-radial drawing of G .

Let ν be a C -node of a cut-vertex c of K that is not the root of T . Let (c, v) be a separating edge of K . Vertex v is called a *separating vertex* of ν . For example, in Figure 1(c), vertices b, c, d are separating vertices of the C -node 2. Let ν_l and ν_r be the leftmost and the rightmost D -node children (also called D -children) of ν , and let (c, v_l) and (c, v_r) be their associated separating edges; v_l and v_r are called the *leftmost separating vertex* and the *rightmost separating vertex* of ν , respectively. For example, vertices d and b are the leftmost and rightmost separating vertices of the C -node 2 in Figure 1(c).

Let μ be a B -node whose parent is a node ν where ν is the C -node of cut-vertex c . Let ν_r, ν_l be the D -children of ν that precede and follow μ in the clockwise ordering around ν . Let $e_l = (c, v_l)$ be the separating edge associated with the D -node ν_l and $e_r = (c, v_r)$ be the separating edge associated with the D -node ν_r . Edges e_l and e_r are called the *left separating edge* and the *right separating edge* of μ , respectively. Also, v_l and v_r are called the *left separating vertex* and the *right separating vertex* of μ , respectively. For example, $(2, d)$ and $(2, c)$ are the left and right separating edges for the B -node E in Figure 1(c). A *separating edge* (*separating vertex*) of μ is either its left or right separating edge (vertex). The following lemma can be proven by using standard techniques for BC -trees [10].

Lemma 2. *Let G be a canonical 2-outerplane graph with n vertices such that the subgraph G_1 induced by the vertices of level 1 is connected. There exists a $O(n)$ -time algorithm that computes an equipped BC -tree of G_1 .*

The equipped block-cut-vertex tree T of K is used by the drawing algorithm described in the next section to split G into small subgraphs each of which is drawn independently of the others. Note that if K is biconnected the equipped BC -tree of K has only one node, which is a B -node. However, in order to simplify the description of the drawing algorithm in the following sections, we assume that K has at least one cut-vertex. The assumption is not restrictive, since we can always find a triangular mixed face f consisting of one vertex c of K and two vertices u, w of G_0 , attach to c a new dummy vertex v in f , and triangulate the face again by adding two dummy edges $(v, u), (v, w)$. Vertex v will be temporary added to V_1 and removed at the end of the drawing algorithm. The augmented graph is still canonical 2-outerplane and c is now a cut-vertex.

5 Radial Drawings of 2-Outerplanar Graphs

Let G be a 2-outerplane graph. In this section we show how to compute a 2-radial drawing of G . This result will be a basic building block for the drawing techniques and the characterization of Section 6.

By Lemma 1 we can assume that G is canonical. Also, from the observation at the end of the previous section, we may assume that each connected component of G_1 has at least one cut-vertex. Let K be a connected component of G_1 . The subgraph of G consisting of the separating edges of K is called the *skeleton* of K and is denoted as $skel(K)$. The *skeleton* of G is the union of all $skel(K)$, for

every connected component K of G_1 . We denote it by $skel(G)$. For example in Figure 1(a) and 1(b) the bold edges highlight the skeleton of the graph.

In order to use the algorithm as the basic tool to compute a k -radial drawing of a graph with outerplanarity k (see Section 3), we assume that it receives as input a drawing Γ_0 of G_0 on a circle C_0 and that it computes a drawing Γ of G without changing Γ_0 , i.e. $\Gamma_0 \subset \Gamma$. We do not put any restrictions on the drawing Γ_0 ; the only hypothesis is that it preserves the planar embedding of G_0 . The algorithm consists of four main steps:

1. **Choice of C_1 :** The radius of circle C_1 is determined.
2. **Drawing the Skeleton of G :** For each connected component K of G_1 , the drawing of $skel(K)$ is computed. Figure 1(d) shows a drawing of the skeleton of the graph of Figure 1(a).
3. **Associating Blocks with Regions:** Let Σ_K be the drawing of $skel(K)$. Σ_K induces a set of connected regions in the plane; each region is bounded by C_0 and by two separating edges. Each block is associated with a region and it will be drawn inside its region. Let B_K be a block of K , let T be an equipped BC -tree of K , and let μ be the B -node of B_K in T . Let (c, v_l) and (c, v_r) be the left and right separating edges of μ , respectively; let $\overline{cv_l}$, $\overline{cv_r}$ be the segments representing the separating edges of μ in Σ_K . Block B_K is associated with the region bounded by $\overline{cv_l}$, $\overline{cv_r}$ and by the arc of C_0 from v_r to v_l in clockwise direction. For example, let Σ_K be the drawing of Figure 1(d) and consider block D of Figure 1(b). Block D is associated with the region bounded in Figure 1(d) by segments $\overline{2c}$, $\overline{2b}$, and by the arc of C_0 from b to c in clockwise direction.
4. **Drawing the Blocks of Each Connected Component:** The drawing of G is computed by defining the coordinates of the vertices of G_1 that are not cut-vertices. To do that, the algorithm draws each connected component K of G_1 independently. For each block B_K of K it draws B_K inside its corresponding region, as defined in the previous step. Figure 1(e) shows a level-preserving 2-radial drawing of the graph of Figure 1(a).

5.1 Choice of C_1

Let K be a connected component of G_1 . The radius r_1 of C_1 depends on the drawing of G_0 . Radius r_1 is chosen so that when a drawing of $skel(G)$ is computed the region associated with each block contains an arc of C_1 . This will be useful when drawing the vertices of the blocks inside their associated regions.

Let $K_0, K_1, \dots, K_h$ be the connected components of G_1 and let T_j be the equipped BC -tree of K_j ($j = 0, \dots, h$). For each B -node of T_j with separating vertices v_l, v_r , compute the distance between the point representing v_l and the point representing v_r in Γ_0 . Let δ_j be the minimum of these distances over all B -nodes of T_j and let $\delta = \min\{\delta_j : j = 0, \dots, h\}$. We define the radius of C_1 to be such that C_1 intersects the chords of C_0 with length δ . Computing the radius of C_1 can be performed in a time that is linear in the number of blocks of G , and therefore linear in the number of vertices of G , since the graph is planar.

5.2 Drawing the Skeleton of G

In this section we assume that the root of the equipped BC -tree T of K has at least three B -children. The case when the root has only two B -children can be handled similarly. For reasons of space we do not discuss this case.

The algorithm computes a drawing Σ_K of $skel(K)$ such that Σ_K is inside the polygon of Γ_0 representing the face of G_0 that contains K in G . The drawing algorithm computes Σ_K so that for each block B_K , the separating edges that define the region associated with B_K form a convex angle inside this region (the angle is called *corner of the region* in the following). This invariant will be used in the next steps of the algorithm and is important to prove the planarity of the computed drawing of G . The algorithm performs a top-down left-to-right visit of T . When a C -node ν is visited, the associated cut-vertex c is drawn on C_1 together with all its incident separating edges. To better describe this algorithm we need to introduce some more terminology.

Let Γ be a 2-radial drawing of G on two circles C_0 and C_1 and let p be a point on C_1 . A *free arc* of p is a maximal arc of C_1 having p as one end-point and containing neither vertices of Γ nor crossings between an edge of Γ and C_1 . Point p has always two free arcs, one moving from p clockwise (the *left free arc of p*) and the other moving from p counterclockwise (the *right free arc of p*). Given any circle C , and two points a and b on C , the arc of C traversed when moving from a to b clockwise will be denoted as $\langle a, b \rangle$. Points a and b will be called the *first point* and the *last point* of the arc, respectively. Each point of the arc distinct from a and b will be referred to as an internal point of $\langle a, b \rangle$. Finally, let q be a point outside C . A point p of C is *visible* from q if the segment $\overline{pq}$ does not cross C . The set of points of C that are visible from q is an arc called the *visible region of q on C* . Note that, the first and the last points of the visible region of q on C are the intersection points between C and the straight lines through q tangent to C . The algorithm distinguishes among two cases:

- *Node ν Is the Root of T .* Let $\mu_1, \dots, \mu_h$ be the B -children of ν and let $\mathcal{P}$ be the polygon defined by their separating vertices. From the choice of C_1 , every side of $\mathcal{P}$ crosses C_1 in two distinct points. This implies that $\mathcal{P}$ contains a set of arcs of C_1 . Draw c as a point of one of these arcs. See for example the cut-vertex 1 in Figure 1(d).
- *Node ν Is Not the Root of T .* Let u_l and u_r be the leftmost and the rightmost separating vertices of ν , respectively. By the choice of C_1 (Subsection 5.1), segment $\overline{u_l u_r}$ crosses circle C_1 in two distinct points; let p be the intersection point that is closer to u_r , and denote by γ the intersection of the left free arc of p and the visible region of u_r . Draw c as a point of γ . It can be proven that any point in γ guarantees that the corners of the regions of the B -children of ν are convex. However, in order to correctly complete the drawing of the blocks of K without changing the drawing of the skeleton (Subsection 5.4), the algorithm may need to make γ smaller for some cases. Details about how to reduce γ are omitted.

Lemma 3. *Let G be a 2-outerplane graph with n vertices, let $G_0 \subset G$ be the subgraph of level 0, and let $\text{skel}(G)$ be the skeleton of G . Let Γ_0 be an embedding preserving 1-radial drawing of G_0 . There exists an $O(n)$ -time algorithm that computes an embedding preserving 2-radial drawing Γ of $G_0 \cup \text{skel}(G)$ such that $\Gamma_0 \subset \Gamma$. Also, Γ is a level-preserving drawing.*

5.3 Associating Blocks with Regions

Since G is canonical, the interior of each face f of G_0 contains exactly one connected component K of G_1 . K is drawn inside the region f of the plane and is denoted by $\mathcal{R}_f$. Note that $\mathcal{R}_f$ is defined by Γ_0 . Let T be the BC -tree of K ; $\mathcal{R}_f$ is recursively subdivided into connected sub-regions $\mathcal{R}_\mu$, one for each B -node μ of T . As explained in Subsection 5.4, block μ is drawn inside $\mathcal{R}_\mu$. We formally define the regions induced by Σ_K in the following.

For each B -node μ of T let (c, v_l) and (c, v_r) be the separating edges of μ . Denote by $\mathcal{R}_\mu^*$ the region of the plane delimited by the segments $\overline{cv_l}$, $\overline{cv_r}$, and by the arc $\angle v_r, v_l$ of C_0 . The drawing technique of Subsection 5.4 will draw the blocks of the subtree of T rooted at μ inside $\mathcal{R}_\mu^*$. The region $\mathcal{R}_\mu$ containing the single block associated with μ is as follows: (a) If μ is a leaf, let $\mathcal{R}_\mu = \mathcal{R}_\mu^*$. (b) If μ is an internal node with grandchildren $\mu_1, \dots, \mu_h$, let $\mathcal{R}_\mu = \mathcal{R}_\mu^* \setminus (\mathcal{R}_{\mu_1}^* \cup \dots \cup \mathcal{R}_{\mu_h}^*)$.

5.4 Drawing the Blocks

Let B_K be a block of a connected component K of G_1 , and let μ be the B -node of T representing B_K . As in Section 5.2, we assume in the following that the root of T has at least three B -children.

A vertex of G_1 is an *internal joint vertex* if it is adjacent to at least two vertices of G_0 . A vertex u of G_0 is an *external joint vertex* if it is adjacent to at least two vertices of G_1 . The algorithm that draws B_K inside its associated region $\mathcal{R}_\mu$ distinguishes among two cases:

Case 1: μ Is a Leaf of T . Let c be the cut-vertex associated with the parent of μ and let $c = a_0, a_1, \dots, a_t$ be the internal joint vertices of B_K in the clockwise order they appear on the external face of B_K . Since G is inter-triangulated then the internal joint vertices a_l and a_{l+1} ($l = 0, \dots, t-1$) are adjacent to an external joint vertex, which is denoted by u_{l+1} . Also a_t and a_0 are adjacent to an external joint vertex, which is denoted as u_{t+1} . Since G is canonical, edges (c, u_1) and (c, u_{t+1}) are the separating edges of μ .

The algorithm first places the internal joint vertices $a_0, \dots, a_t$ in this order. At step l ($l = 1, \dots, t$) vertex a_l and its incident edges are added to the drawing. Each vertex a_l is placed on C_1 as follows. If edge (u_l, a_{l-1}) crosses C_1 then let p be its crossing, otherwise let p be coincident with a_{l-1} . Vertex a_l is drawn as an internal point of the intersection between the right free arc of p and the visible region of u_l .

Once all the internal joint vertices are placed, the algorithm draws the remaining vertices of B_K and their incident edges. More precisely, let $v_1, \dots, v_{r_l}$

be the vertices that are between vertices a_l and a_{l+1} ($l = 0, \dots, t-1$) in clockwise ordering on the external face of B_K . All these vertices are adjacent to the external joint vertex u_{l+1} . If edge (a_l, u_{l+1}) crosses C_1 then let q_1 be its crossing, otherwise let q_1 be coincident with a_l . Analogously, if edge (a_{l+1}, u_{l+1}) crosses C_1 then let q_2 be its crossing, otherwise let q_2 be coincident with a_{l+1} . Vertices $v_1, \dots, v_{r_l}$ are drawn in this order as points of the arc $\langle q_1, q_2 \rangle$.

Case 2: μ Is an Internal Node of T . This case can be handled with techniques similar to those used for the previous case. We omit the description of them for reasons of space.

Theorem 1. *Let G be a 2-outerplane graph with n vertices. G admits a level-preserving 2-radial drawing that preserves the embedding of G . Also there exists an $O(n)$ -time algorithm that computes such a drawing.*

6 Minimum Radial Drawings of Planar Graphs

In this section we first characterize the family of graphs that admit a radial drawing on at most k -concentric circles and then use the characterization to solve in polynomial time the problem of computing a minimum radial drawing of a planar graph.

Theorem 2. *Let G be a graph with outerplanarity k and n vertices. Then G admits a proper k -radial drawing. Also, there exists an $O(n)$ -time algorithm that computes such a drawing.*

Proof. Since G has outerplanarity k then it has a k -outerplanar embedding. We show how to compute a level-preserving k -radial drawing Γ of G that preserves this embedding. This implies that Γ is proper. An algorithm to compute Γ is based on first drawing the subgraph induced by the vertices of level 0 on a circle C_0 and then by adding at each step the vertices of level i on a circle C_i ($i = 1, \dots, k-1$). At Step i the subgraph $G_{i-1,i}$ is drawn by using the algorithm described in Section 5. Since $G = \bigcup_{i=0}^{k-2} G_{i,i+1}$ the computed drawing is a radial drawing of G . The fact that no two edges cross is a consequence of Theorem 1. It follows that the above described algorithm computes a level-preserving k -radial drawing of G . As for the time complexity, it follows from Theorem 1 that the computation of drawing $G_{i-1,i}$ requires $O(n_i)$ time where n_i is the number of vertices in $G_{i-1,i}$. Therefore the overall time complexity is $O(n)$. $\square$

Lemma 4. *If a graph G admits a k -radial drawing then it has outerplanarity at most k .*

Proof. A k -radial drawing Γ of G defines an embedding of G . All vertices on the outerface of this embedding are drawn on C_0 . Removal of all vertices on C_0 results in a $(k-1)$ -radial drawing. So we can use induction to show that Γ is a k -outerplane graph. It follows that the outerplanarity of G is at most k . $\square$

Theorem 3. *Let G be a planar graph. G admits a radial drawing on at most k -circles if and only if the outerplanarity of G is at most k .*

Proof. Assume that G has a k -radial drawing. Then by Lemma 4 G has outerplanarity at most k . Conversely, if G has outerplanarity $j \leq k$, by Theorem 2 it admits a j -radial drawing with $j \leq k$. $\square$

Theorem 4. *Let G be a planar graph with n vertices. There exists an $O(n^5 \log n)$ -time algorithm that computes a radial drawing of G on the minimum number of concentric circles. Furthermore the computed drawing is proper.*

Proof. Bienstock and Monma [3] describe an algorithm to compute the outerplanarity k of G and to determine a k -outerplanar embedding of G . This algorithm takes $O(n^5 \log n)$ time. The result in [3] together with Theorem 3 imply that k is the minimum number of circles for which there exists a radial drawing of G . The fact that such a drawing is proper is a consequence of Theorem 2. Again by Theorem 2 it follows that the time complexity of the whole algorithm is dominated by the technique in [3]. $\square$

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