Extended Associative Memories for Recalling Gray Level Patterns

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Abstract. We show how the binary $\alpha\beta$ associative memories recently proposed by Yáñez in [1] can be extended to work now in the gray-level case. To get the desired extension we take the operators α and β , foundation of the $\alpha\beta$ memories, and propose a more general family of operators among them the original operators α and β are a subset. For this we formulate a set of functional equations, solve this system and find a family of solutions. We show that the α and β originally proposed in [1] are just a particular case of this new family. We give the properties of the new operators. We then use these operators to build the extended memories. We provide the conditions under which the proposed extended memories are able to recall a pattern either from the pattern's fundamental set or from altered versions of them. We provide real examples with images where the proposed memories show their efficiency.

1 Introduction

An associative memory $\mathbf{M}$ is a system that relates input vectors, and outputs vectors as follows: $\mathbf{x} \rightarrow \mathbf{M} \rightarrow \mathbf{y}$ with $\mathbf{x}$ and $\mathbf{y}$, respectively the input and output patterns vectors. Each input vector forms an association with a corresponding output vector.

An associative memory, $\mathbf{M}$ is represented by a matrix whose ij -th component is m_{ij} . $\mathbf{M}$ is generated from a finite a priori set of known associations, known as the *fundamental set of associations*, or simply the *fundamental set* (FS). If ξ is an index, the fundamental set is represented as: $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) \mid \xi = 1, 2, \dots, p\}$ with p the cardinality of the set. The patterns that form the fundamental set are called *fundamental patterns*.

If it holds that $\mathbf{x}^\xi = \mathbf{y}^\xi \forall \xi \in \{1, 2, \dots, p\}$, $\mathbf{M}$ is auto-associative, otherwise it is hetero-associative. A distorted version of a pattern $\mathbf{x}$ to be recuperated will be denoted as $\mathbf{X}$. If when feeding a distorted version of $\mathbf{x}^w$ with $w \in \{1, 2, \dots, p\}$ to an associative memory $\mathbf{M}$, it happens that the output corresponds exactly to the associated pattern $\mathbf{y}^w$, we say that recuperation is perfect.

Many ways to build an associative memory have been reported in the literature. Recently, Yáñez in [1], the authors describe a new class of associative memories, the so-called $\alpha\beta$ memories. Their functioning is founded in two binary operators: α and

β . In the properties of these two important operators lays the power of the above-mentioned memories. These memories work very well in the presence of additive or subtractive noise; its domain is however restricted to the binary one. In this work we extend this class of associative memories to work but with gray-level patterns. For this we solve a systems of functional equations by using some of the central properties of the original α and β .

2 Development of the Extended Memories

In this section the extended memories are developed. First we derive the extended operators **A** and **B** operators. The new memories are then developed. We also investigate the conditions under which the extended memories are able to recall patterns either from the fundamental set or from altered versions of them. Due to space limitations we only memories of type **M** are described.

2.1 Operators A and B

When the $\alpha\beta$ memories make use of the α and β operators described in [10] its range of usability is $[0,1]$; when they make use of the extended operators **A** and **B**, introduced in this section, its range as we will see is the gray-level: $[0, L-1]$, with L the number of gray levels. We will use the symbols, **A** and **B** to denote the extended operators, the capitals of α and β . Taking into account the properties of the original binary operators α and β described in Section 2.1, one way to find a generalization consists on formulating a system of functional equations [7]. It can be shown that in this case the system should have the following form:

$$\begin{aligned} \mathbf{B}(\mathbf{A}(x, y), y) &= x \\ \mathbf{B}((x \vee y), z) &= \mathbf{B}(x, z) \vee \mathbf{B}(y, z) \end{aligned} \quad (1)$$

As we will later see, these two properties of **A** and **B** are all that we need to characterize most of the solutions of interest. Once a family of solutions is found, one should show that the α and β originally proposed in [6] are just a particular case of this new family. By taking into account Cauchy's functional equation [8], we propose as initial solution the following:

$$\begin{aligned} \mathbf{A}(x, y) &= f(x) + g(y) \\ \mathbf{B}(x, y) &= p(x) + q(y) \end{aligned} \quad (2)$$

If **A** and **B** have this form, the matter of being increasing or decreasing with respect to x or y reduces to the fact that f , g , p , q , be also increasing or decreasing. As we will also see, distributivity by the left with respect to the **max** operator is equivalent to the fact that **B** is increasing with respect to its first element. As can be seen, the problem is already solved by means of the second equation. With this, we can see that we only need to focus on the inverse left relation between **A** and **B**. By taking into account the proposed structure for **A** and **B** we have:

$$B(\alpha(x, y)) = p(f(x) + g(y)) + q(y) = x \quad (3)$$

p distributes in its argument only if it is a solution of Cauchy's equation [8], this is p is a homotecy of the form $p(x) = cx$ where c is an arbitrary constant. Then

$$pf(x) + pg(y) + q(y) = x. \quad (4)$$

This implies that

$$pf(x) = x, (pg + q)(y) = 0. \quad (5)$$

Thus, $p = f^{-1}$, $q = -f^{-1}g$

As a first observation, we have that once established the parameters of **A**, those of **B** are determined. Also as p is a homotecy, f , its inverse, also is. Taking f as the most simple homotecy (the identity) we have:

$$f(x) = x, p(x) = x, q = -g. \quad (6)$$

If also g is minus the identity, **A** and **B** are as follows:

$$\begin{aligned} A(x, y) &= x - y \\ B(x, y) &= x + y \end{aligned} \quad (7)$$

On the other hand, it can be shown that when adding a constant, let say k to **A**, generates a non-linearity on **B**, this is:

$$A(x, y) = x - y + k. \quad (8)$$

This implies that $B(x, y) = \phi(x + y)$ where ϕ is non-linear function.

The family of binary operators generated this way allows us to obtain, on the one hand, the expression for binary operators α and β , where ϕ (Figure 1) is the sep function centered on 1:

$$\begin{aligned} \alpha(x, y) &= x - y + 1 \\ \beta(x, y) &= \phi(x + y) \end{aligned} \quad (9)$$

$$\phi(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases}. \quad (10)$$

Fig. 1. Graph of ϕ .

Fig. 2. Graph of **B**.

On the other hand, the same family of operators allows us to obtain an expression for patterns with L gray levels, whose discrete domain is, for example $A = \{0, 1, 2, \dots, L-1\}$, in this case **A** and **B** take the form

$$\begin{aligned} A(x, y) &= x - y + L - 1 \\ B(x, y) &= \phi(x + y) \end{aligned} \quad (11)$$

where now ϕ is in this case the staircase function (Figure 2) defined as follows:

$$B(x, y) = \begin{cases} 0 & \text{if } (x + y) \leq L - 1 \\ x + y - (L - 1) & \text{if } L \leq x + y < 2(L - 1) \\ L - 1 & \text{if } (x + y) \geq 2(L - 1) \end{cases} \quad (12)$$

Functions **A** and **B** have the same form as in the binary case, this is: $A: A \times A \rightarrow B$ and $B: B \times A \rightarrow A$ where now $A = \{0, 1, 2, \dots, L-1\}$ and $B = \{0, 1, 2, \dots, 2(L-1)\}$. The properties of these two important operations are much alike those of the binary operators. They are not here enlisted due to space limitations.

2.2 Extended AB Memories

In this section we introduce the extended memories able to recall gray-level patterns. The proposed memories have the same structure as the $\alpha\beta$ memories presented in [10]. The difference between the extended memories and the standard $\alpha\beta$ memories is of course the use of operators **A** and **B** instead of operators α and β . We provide the conditions under which the extended memories are able to recall patterns either from the fundamental set or from altered versions of them. Due to space limitations we only analyze extended auto-associative memories of type **M** (EAS **M** memories). We provide several examples to better illustrate the ideas here presented. To operate an EAS **M** memory we again first use the operator $\otimes$, then the **max** operator $\vee$. During training equation (4) changes as follows:

$$m_{ij} = \bigvee_{\xi=1}^p A(x_i^{\xi}, x_j^{\xi}). \quad (13)$$

In the same way, during recuperation, equations (5) and (6) change to:

$$(\mathbf{M} \wedge_{\mathbf{B}} \mathbf{x}^{\omega})_i = \bigwedge_{j=1}^n B(m_{ij}, x_j^{\omega}). \quad (14)$$

$$(\mathbf{M} \wedge_{\mathbf{B}} \mathbf{x})_i = \bigwedge_{j=1}^n B(m_{ij}, \tilde{x}_j). \quad (15)$$

Next we give the conditions under which an EAS **M** memory provides perfect recall. We first give the results concerning the perfect recall of a pattern of the fundamental set. We do then the same thing but an altered version of a pattern of the fundamental set is presented to the memory. The results here presented are essentially the

same as given in [10]. Instead of using operators α and β , they take into account operators $\mathbf{A}$ and $\mathbf{B}$. Their proof is not included due to space limitations.

Example 3.1. Suppose we want to first memorize and then recuperate the following fundamental set, with $L=8$:

$$\mathbf{x}^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{x}^2 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{x}^3 = \begin{pmatrix} 6 \\ 5 \\ 5 \end{pmatrix}.$$

TRAINING PHASE:

$$\mathbf{x}^1 \vee_{\mathbf{A}} (\mathbf{x}^1)^T = \begin{pmatrix} 7 & 8 & 7 \\ 6 & 7 & 6 \\ 7 & 8 & 7 \end{pmatrix}, \mathbf{x}^2 \vee_{\mathbf{A}} (\mathbf{x}^2)^T = \begin{pmatrix} 7 & 9 & 8 \\ 5 & 7 & 6 \\ 6 & 8 & 7 \end{pmatrix}, \mathbf{x}^3 \vee_{\mathbf{A}} (\mathbf{x}^3)^T = \begin{pmatrix} 7 & 8 & 8 \\ 6 & 7 & 7 \\ 6 & 7 & 7 \end{pmatrix}.$$

Thus

$$\mathbf{M} = \begin{pmatrix} 7 & 9 & 8 \\ 6 & 7 & 6 \\ 7 & 8 & 7 \end{pmatrix}.$$

RECALLING PHASE:

$$\mathbf{M} \wedge_{\mathbf{B}} \mathbf{x}^1 = \begin{pmatrix} 7 & 9 & 8 \\ 6 & 7 & 6 \\ 7 & 8 & 7 \end{pmatrix} \wedge_{\mathbf{B}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{B}(7,1) \wedge \mathbf{B}(9,0) \wedge \mathbf{B}(8,1) \\ \mathbf{B}(6,1) \wedge \mathbf{B}(7,0) \wedge \mathbf{B}(6,1) \\ \mathbf{B}(7,1) \wedge \mathbf{B}(8,0) \wedge \mathbf{B}(7,1) \end{pmatrix} = \begin{pmatrix} 1 \wedge 2 \wedge 2 \\ 0 \wedge 0 \wedge 0 \\ 1 \wedge 1 \wedge 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

You can easily verify that the other two patterns are also perfectly recalled.

The following proposition provides conditions for perfect recall of a pattern of the fundamental set when an altered version of it is presented to EAS $\mathbf{M}$ memory.

Theorem 3.2. Let $\{(\mathbf{x}^\xi, \mathbf{x}^\xi) \mid \xi = 1, 2, \dots, p\}$ the fundamental set of an EAS $\mathbf{M}$ memory and L the number of levels the elements of each $\mathbf{x}^\xi$ can take. Let $\tilde{\mathbf{X}}$ an altered version with additive noise. If $\tilde{\mathbf{X}}$ is presented as input to the EAS $\mathbf{M}$ memory and if besides for each $i \in \{1, \dots, n\}$ it holds that $\exists j = j_0 \in \{1, \dots, n\}$, which depends on w and i such as $m_{ij_0} \leq A(x_i^w, \tilde{x}_{j_0})$, then we have perfect recall, this is $\mathbf{M} \wedge_{\mathbf{B}} \tilde{\mathbf{X}} = \mathbf{x}^w$.

Example 3.3. Let us take an altered version by additive noise of pattern $\mathbf{x}^2 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$,

$$\text{for example } \tilde{\mathbf{x}}^2 = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}:$$

$$\mathbf{M} \wedge_{\mathbf{B}} \mathbf{x}^2 = \begin{pmatrix} 7 & 9 & 8 \\ 6 & 7 & 6 \\ 7 & 8 & 7 \end{pmatrix} \wedge_{\mathbf{B}} \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} \mathbf{B}(7,4) \wedge \mathbf{B}(9,3) \wedge \mathbf{B}(8,3) \\ \mathbf{B}(6,4) \wedge \mathbf{B}(7,3) \wedge \mathbf{B}(6,3) \\ \mathbf{B}(7,4) \wedge \mathbf{B}(8,3) \wedge \mathbf{B}(7,3) \end{pmatrix} = \begin{pmatrix} 4 \wedge 5 \wedge 4 \\ 3 \wedge 3 \wedge 2 \\ 4 \wedge 4 \wedge 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}.$$

The reader can easily verify that this example satisfies the conditions given by Theorem 3.2 for perfect recall in the presence of additive noise.

3 Experimental Results

In this section the proposed extended associative memories are tested with more realistic patterns. Images of five well-known personalities (Descarte, Einstein, Euler, Galileo and Newton) were used are shown in Figure 3. The images are 32 by 29 pixels and 256 gray levels. Only EAS associative memories of type $\mathbf{M}$ were tested.

3.1 Construction of the Association Matrix

The images shown in Figure 3 were first converted to vectors of 968 elements (32 times 29) each one. These vectors were then used to construct the corresponding $\mathbf{M}$ matrices, by using the techniques described in Section 3.

Fig. 3. Images of the five objects used in to test the proposed associative memories.

3.2 Recalling of the Fundamental Set

In this first experiment, the five images were fed to $\mathbf{M}$ matrices already built. To all of them, the procedures described in Section 3 were applied. In all cases, of course, the five patterns were perfectly recalled.

3.3 Recalling of a Pattern by a Corrupted Version of It

Three groups of images were generated: The first one with additive noise, the second one with saturated noise of type salt, and the third one with manually added saturated noise. In the first case, to the gray-value $f(x, y)$ of pixel with coordinates (x, y) , an integer v was added, such that $f(x, y) + v \leq (L - 1)$. In the second case, again to the gray-value of a pixel an integer v was added, such that $f(x, y) + v = (L - 1)$. The value $f(x, y)$ of the gray-level of the pixel was changed if $s < t$. $s \in [0, 1]$ is an uniformly randomly distributed random variable, t is the parameter controlling how much of the image is corrupted. In the third case Microsoft PAINT utility was used. Due to space, only the result for the first set are presented.

Case of additive noise. Twenty five images were obtained as explained. Parameter s was varied from 0.99 to 0.95 in steps of 0.01. Figure 4(a) shows the obtained images. In some cases the original images were perfectly recalled, in others they were not. The recalled versions are shown in Figure 4(b). Under each recalled image it is indicated if it matches the original image or not. Notice how despite the level of noise introduced, recalled versions match very well with the originals.

Fig. 4. (a) Altered versions with additive noise. (b) Recalled versions.

We want it to investigate the performance of the proposed model in the presence of huge quantities of additive noise. For this all the pixels of the five images of Fig. 3 were contaminated with additive noise. Figure 5(a) shows the obtained altered versions. Figure 5(b) shows the recalled images. Notice how despite the level of noise introduced, recalled versions match very well with the originals.

Fig. 5. (a) Completely altered versions of the original images. (b) Recalled versions.

It is worth mentioning that the average time to recall an image in all cases, when using the proposed model, is 0.4 seconds in a Pentium 4 at 1.3 GHz.

4 Conclusions

We have proposed a new set of associative memories able to work with gray-level patterns as an extension of the associative binary $\alpha\beta$ memories recently introduced in [10]. To derive the set of extended memories, we first we take operators α and β , and solve a set of functional equations to get the extended operators, **A** and **B**. It is shown that the operators α and β are a special case if the general operators **A** and **B**. We give conditions under which the proposed set of memories is able to recall first the funda-

mental set of patterns, and second a pattern from an altered version of it when additive or subtractive noise is added to the pattern.

The proposed extension was tested with several real patterns (images of five famous people) with very satisfactory results. Even in the case of severe noise, the proposed extended memories are able to recall patterns from distorted versions of them.

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