On Extending Symmetry Sets for 2D Shapes^{*}

Arjan Kuijper and Ole Fogh Olsen

Image group, IT-University of Copenhagen
Glentevej 67, DK-2400 Copenhagen, Denmark

Abstract. Many attempts have been made to represent families of 2D shapes in a simpler way. These approaches lead to so-called structures as the Symmetry Set ($\mathcal{SS}$) and a subset of it, the Medial Axes ($\mathcal{MA}$). While the latter is commonly used, the former is still in the mathematical research stage. One reason for this is that in contrast to the $\mathcal{SS}$, the $\mathcal{MA}$ can be computed efficiently and fast, and yields one connected component for a closed shape.

In this paper a natural complement of the symmetry set, called the Anti-Symmetry Set ($\mathcal{ASS}$), is used to connect components bearing the full richness of the symmetry set. Secondly, new ways are presented to visualize these sets. One uses the radius of the describing circle as extra dimension, the other, the so-called pre-Symmetry Set (pre- $\mathcal{SS}$), uses the parameter space. Example shapes show the extra information carried in the $\mathcal{ASS}$ and the pre- $\mathcal{SS}$ in determining the special points on the $\mathcal{SS}$ as well as revealing the structure of the $\mathcal{SS}$ in more detail. They are also capable of distinguishing between different shapes where the $\mathcal{SS}$ and the $\mathcal{MA}$ in some cases fail.

1 Introduction

In 2D shape analysis the simplification of shapes into a skeleton-like structure is widely investigated. The Medial Axis ($\mathcal{MA}$) skeleton [4] is commonly used, since it can be calculated in a fast and robust way. Many results on simplification, reconstruction and database search are reported. The $\mathcal{MA}$ is a member of a larger family, the Symmetry Set ($\mathcal{SS}$), exhibiting nice mathematical properties, but difficult to compute and yielding distinct branches [6, 7]. To overcome the latter we introduce the anti-symmetry set ($\mathcal{ASS}$), resulting in a connected set.

Secondly, while the 2D visualization of the $\mathcal{MA}$ skeleton is unambiguous due to its limiting nature, the $\mathcal{SS}$ may give rise to intersecting curves that sometimes occur due to projections. Together with the need for augmenting the skeleton with information of the radius / scale / distance from the boundary at which it occurs, the need for a representation in a space with an extra dimension is evident - something that was already known for the skeleton [4] and used in the so-called Shock Graph method [15]. It has been mentioned by Wright et al. [17], but not been used afterwards. Another way of representation is obtained

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when the $\mathcal{SS}$ is considered in the parameter space, yielding the so-called Pre-Symmetry Set. A reason for using the $\mathcal{SS}$ instead of the $\mathcal{MA}$ is that the $\mathcal{SS}$ can be caught in a linear data structure, as presented elsewhere[12], in contrast to the graphs needed for representation of the $\mathcal{MA}$.

We will first give a short overview on the $\mathcal{MA}$ and the $\mathcal{SS}$ and then introduce the $\mathcal{ASS}$ and new ways for visualization.

2 Some Background Theory on Shapes

In this section we give the necessary background regarding properties of shapes, the Medial Axis, the Symmetry Set, labelling points on these sets and give an example to clarify the definitions.

Let $\mathcal{S}(x, y) = \{(x, y) | L(x, y) = 0\}$ denote a closed 2D shape. Then $\mathcal{N}(x, y) = (L_x, L_y)(L_x^2 + L_y^2)^{-1/2}$ denotes its unit normal vector, and $\kappa(x, y) = -(L_x^2 L_{yy} - 2L_x L_y L_{xy} + L_y^2 L_{xx})(L_x^2 + L_y^2)^{-3/2}$ its curvature. The evolute $\mathcal{E}(t)$ is given by the set $\mathcal{S} + \mathcal{N}/\kappa$. Even if the curve is smooth and differentiable, the evolute contains non-smooth and non-differentiable points, viz. those where the curvature is zero or takes a local extremum, respectively.

2.1 The Medial Axis and Symmetry Set

The Medial Axis ($\mathcal{MA}$) is defined as the closure of the set of centers of circles that are tangent to the shape at least two points and that contain no other tangent circles: the are so-called maximal circles. The Symmetry Set $\mathcal{SS}$ is defined as the closure of the set of centers of circles that are tangent to the shape at least two points [6, 5, 8, 7]. Obviously, the $\mathcal{MA}$ is a subset of the $\mathcal{SS}$ [7].

To calculate these sets from above definition, the following procedure can be used: Let a circle with unknown location be tangent to the shape at two points. Then its center can be found by using the normal vectors at these points: it is located at the position of each point minus the radius of the circle times the normal vector at each point. To find these two points, the location of the center and the radius, do the following: Given two vectors p_i and p_j (right, with $i = 1$ and $j = 2$) pointing at two locations at the shape, construct the difference vector $p_i - p_j$. Given the two unit normal vectors N_i and N_j at these locations, construct the vector $N_i + N_j$. If the two constructed vectors are non-zero and perpendicular,

$$(p_i - p_j) \cdot (N_i + N_j) = 0, \quad (1)$$

the two locations give rise to a tangent circle. The radius r and the center of the circle are given by

$$p_i - rN_i = p_j - rN_j. \quad (2)$$

2.2 Labelling Points

It is known that the $\mathcal{MA}$ in itself carries insufficient information for representing and reconstruction a shape, since different shapes can yield the same $\mathcal{MA}$.

Therefore additional information can be added to the $\mathcal{MA}$ and used, as proposed by various authors [13–16]. It has been shown by Bruce et al. [6] that only five distinct types of points can occur for the $\mathcal{SS}$, and by Giblin et al. [8, 7] that they are inherited by the $\mathcal{MA}$. An A_1^2 point is the “common” midpoint of a circle tangent at two distinct points of the shape. An A_1A_2 point is the midpoint of a circle tangent at two distinct points of the shape but located at the evolute. An $A_1^2A_1^2$ point is the midpoint of two circles tangent at two pairs of distinct points of the shape with different radii. An A_1^3 point is the midpoint of one circle tangent at three distinct points of the shape. An A_3 point is the midpoint of a circle located at the evolute and tangent at the point of the shape with the local extremal curvature.

Based on the behaviour of the radii one can impose vectors on the $\mathcal{MA}$ denoting increasing radius, for example. This has been done in the so-called Shock Graph approach by Siddiqi et al. [13–16]. The $\mathcal{MA}$ or the $\mathcal{SS}$ together with the radius function is sufficient to reconstruct the shape [7].

3 Beyond the Symmetry Set

In this section we present three ways to add extra information to the $\mathcal{SS}$ based on Eqs. (1-2). The first is straightforward: use the radius of tangent circle to each set, as described in the previous section as an extra dimension. The second method is an extension of the $\mathcal{SS}$, called the anti-symmetry set ($\mathcal{ASS}$). Thirdly, the pre-symmetry set (pre- $\mathcal{SS}$) will be introduced as analysis and visualization tool. We focus on the $\mathcal{SS}$, since the $\mathcal{MA}$ is only a subset of it.

3.1 The $\mathcal{SS}$ Radius Space

Having incorporated the radius function upon the $\mathcal{SS}$ one has extra information. This information can be exploited in much more detail when the radius is considered as an extra dimension. First attempts were reported in [17], albeit only on a spline approximation of the shape. Using this dimension, the 1D curves in the 2D plane become 1D curves in 3D space. In this 3D space the $\mathcal{SS}$ curves reveal information that does not appear in a trivial manner in the 2D plane. For example, at an $A_1^2A_1^2$ point of the $\mathcal{SS}$, two curves are intersecting in 2D, but obviously not in 3D, since two different radii were involved. At an A_1^3 point, in contrast, three curves still intersect in the 3D space.

On the other hand, points at the $\mathcal{SS}$ that arise from locally minimal or maximal circles are clearly visible as local extrema of the 3D curve with respect to the radius.

3.2 The Anti-symmetry Set

Another extension is the Anti-Symmetry Set ($\mathcal{ASS}$). It is defined as the set of points satisfying Eq. 1, but not being part of the symmetry set. Figure 1b clarifies this.

The points p_1 to p_4 have in common that they give rise to points on the $\mathcal{MA}$ or the $\mathcal{SS}$ in specific pairs. For example, p_1 and p_2 define a $\mathcal{MA}$ (and $\mathcal{SS}$) point and p_1 and p_4 a $\mathcal{SS}$ point. The other combination p_1 and p_3 satisfies Eq. 1, but is not part of the $\mathcal{SS}$: they are part of the $\mathcal{ASS}$.

The anti-symmetry set appeared in the early 1990's due to Blake et al. [3, 2] in the field of robotics. There they considered this set in order to find an optimal finger position for a two finger grasp. Another – perhaps less striking – name is due to Giblin [11]: the Mid Parallel Tangent Locus, describing exactly what it is.

3.3 The Pre-symmetry Set

One way to visualize the locations the sign changes of Eq. 1 is by taking all points on the shape pair wise and plot these sign changes in a diagram. This was used by Holtom [11] and Giblin and Sapiro [9, 10, 1] in a different context (affine symmetry sets). Following their line of reasoning this diagram should be called the pre- $\mathcal{SS}$. Since it consists of non-intersecting lines for the $\mathcal{SS}$ it can act as a linear data structure describing the shape. In combination with the $\mathcal{ASS}$, same intersections occur as with the $\mathcal{SS}$ [12]. In the next section we will discuss the pre- $\mathcal{SS}$ and its properties in more detail.

4 Example Shapes

In this section we deal with some example shapes, starting with the simplest one: an ellipse. Since the ellipse is highly symmetric, some results may seem trivial, while others give a false intuition. We therefore continue with two other examples: a “cubic oval” [5] and a concave shape.

4.1 Ellipse

The ellipse $L(x, y) = x^2 + 4y^2 - 4 = 0$ has $\mathcal{N}(x, y) = (x, 4y)(x^2 + 16y^2)^{-1/2}$, and $\kappa(x, y) = -32(x^2 + 4y^2)(4x^2 + 64y^2)^{-3/2}$, with extremal values $1/4$ and 2 at the four locations $(0, \pm 1)$ and $(\pm 2, 0)$, respectively. The evolute consists of four parts, joined at cusps at the locations $(0, \pm 3)$ and $(\pm 3/2, 0)$.

To find the $\mathcal{MA}$ with Eq. 2 we have two points on the shape with $y_1 = -y_2$ and $x_1 = x_2$ and $\tilde{y}_1 = \tilde{y}_2 = 0$ and $\tilde{x}_1 = \tilde{x}_2$ for $(\tilde{x}, \tilde{y})$ a point on the $\mathcal{MA}$. Then $y_1 - 4ry_1/\sqrt{x^2 + 16y_1^2} = y_2 - 4ry_2/\sqrt{x^2 + 16y_2^2}$ yields $r = -\sqrt{x^2 + 16y_2^2}/4$. Together with $y_{1,2} = \pm\sqrt{4 - x^2}$ this gives $r = -\sqrt{16 - 3x^2}/4$. Then the points on the $\mathcal{MA}$ are given by $(\tilde{x}, \tilde{y}, r) = (x + rN, y + rN, r) = (x - x/4, y - 4y/4, -\sqrt{16 - 3x^2}/4) = (3x/4, 0, -\sqrt{16 - 3x^2}/4)$. Note that $x \in [-2, 2]$, so $\tilde{x} \in [-3/2, 3/2]$. The radius varies from $-1/2$ at the endpoints to -1 at the origin. So it has a $\mathcal{MA}$ formed by a straight line along the x -axis with its endpoints at the cusps of the evolute within the shape: $(\pm 3/2, 0)$.

Similarly, one can find the expression for the $\mathcal{SS}$ to be the curve above combined with the curve $(\tilde{x}, \tilde{y}, r) = (0, 3y, -\sqrt{4 + 12y_2^2})$, which varies from $(0, -3, -4)$ via $(0, 0, -2)$ to $(0, 3, -4)$, since $y \in [-1, 1]$, as shown in Figure 1c.

Fig. 1. a) The ellipse, its evolute and its MA (horizontal line) and SS (both lines) in 2D. b) Combinations of points contribute to the MA , SS , or ASS . c) 3D representation of the SS . d) The SS and ASS in 3D.

Fig. 2. The pre- SS . Left to right: all zerocrossings, the ASS part and the SS part.

It therefore has a SS formed by the MA , and a straight line along the y -axis with its endpoints at the cusps of the evolute outside the shape: $(0, \pm 3)$. So at the origin in 2D we have an $A_1^2 A_1^2$ point, with radii 2 and 1, see Figure 1a.

The ASS points are found as those points with $(x_1, y_1) = (-x_2, -y_2)$ and with radii $r = \sqrt{x_1^2 + y_1^2}$, varying between 1 and 2. This is identical to the two radii of the SS at the planar origin. The pre- SS and the parts of it determining the SS and the ASS are shown in Figure 2 from left to right. The diagrams are symmetric in the diagonal p_i, p_i . The dark lines represent the zero crossings of Eq. 1. At intersections of the diagonal and a zero crossing, a branch of the SS starts in an A_3 point. Since the shape is closed, the lines continue through the boundaries: the square represents a torus, since the starting point is arbitrary. Therefore the SS part contains four intersections with the diagonal, corresponding to the four cusps of the evolute. The two lines forming the SS of the square in the left image constitute one branch of the SS . The long line together with the point at the origin (being a A_3 point) forms the second part of the SS . The ASS image connects the curves. The intersections of the SS and the ASS are local extrema w.r.t. radius of the SS , see Figure 1d.

4.2 The Cubic Oval

Firstly the closed part of a cubic oval given by $y^2 = 2bxy + a^2(x - x^3)$, with $a = 1.025$ and $b = 0.09$ is taken. Then six extrema of the curvature occur, while the curvature doesn't change sign and the shape is thus convex [6]. The

Fig. 3. Evolute of the cubic oval with the SS (a) and the ASS and the SS (b) in 2D. Radius space with the SS (c) and the ASS and the SS (d).

Fig. 4. The pre- SS of the cubic oval. Left to right: all zerocrossings, the ASS part and the SS part.

extra extrema of the curvature arise from a perturbation of the shape [5]. Since they alternate along the shape, a maximum and a minimum are created. As a consequence, the evolute is self-intersecting. Furthermore, the evolute consists six cusps. A direct consequence of this in its turn is that a new branch of the SS is created, since these branches always start in the cusps. This branch must be essentially different from the two other branches, since the original branches start in cusps that both arise from either local maxima of the curvature, or minima. These SS curves essentially need to have a local extremum with respect to the radius in 3D, as in Figure 1c. The newly created branch, however, has a minimum and a maximum as endpoints, so its behaviour in 3D must be different. The behaviour of the pre- SS is also different: a new branch implies a new “line” in the pre- SS . Since the perturbation is a local effect, not all points on the shape are involved in creation the new branch. The “line” in the pre- SS thus must be a closed loop. This is indeed what occurs in Figure 4.

The projection of the SS and the ASS with the SS , Figure 3, shows that again three curves joining pair wise in cusps for the ASS . The newly created branch shows some extra behaviour, as shown in Figure 5a. As clearly visible, the new branch bounces twice to the evolute, thus containing two A_1A_2 points. In the pre- SS (Figure 5b) these points are visible as the local extrema in vertical or horizontal direction of the closed loop (which has four, but two are due to the symmetry along the diagonal) due to the same bouncing.

Taking a closer look at the pre- SS , one can see that the curve starting top-left also contains two pairs of local extrema. Consequently, one of the two original curves also contains two A_1A_2 points. This could also have been seen from the SS shown in Figure 5a: one curve traverses the evolute, which can only occur

Fig. 5. Close up of the cubic oval with its evolute and symmetry set. Pre-SS of the cubic oval, with special points (see text).

by a double A_1A_2 combination. A way to get a grip on this is by the following: Consider the manifold making the swallowtail, with the A_3 's on the corners. The symmetry sets walks along the manifold, starting from one A_3 . Then it comes to the “end” of the manifold and continues its walk along the down-under part until it comes again to an end, where it goes on along the upper part until it reaches the second A_3 corner point. The same holds for the part of the symmetry set from which this new part originated¹.

As a result of the creation the new and old curve intersect in a A_1^3 point, where a circle is located tangent to three different points of the shape. Here three branches of the SS intersect. This A_1^3 point is also visible, albeit a bit hidden, in the pre-SS. Figure 5b shows the pre-SS with all the special points: the A_1A_2 points as the local extrema in horizontal or vertical direction, the A_3 points at the intersection of a curve with the diagonal, the local extrema on the 3D SS curves as the intersections of the SS and ASS zero crossing curves, and the A_1^3 as the set of points linked by the lines. Only at a A_1^3 point there are three points in the pre-SS with the combination (p_1, p_2) , (p_1, p_3) , and (p_2, p_3) (and, of course, its diagonal symmetric counterpart). The 3D visualization of the ASS and the SS, Figure 3d, shows again this all in one plot.

4.3 The Concave Case

As a more complicated concave shapes the one given by the equation $(x^2 + y^2 + a^2)^2 = b^2 + 4a^2x^2$, with $a = 1.99$ and $b = 4$ is taken. However, this example shows that the pre-SS for solely the SS part fails in determining the shape, see Figure 7: it has the same structure as the ellipse. This difference becomes clearer in Figure 6: The SS contains two straight lines, just as for the ellipse, albeit in this case one line “goes via infinity”, since $\kappa = 0$, causing the two crosses. In 3D the distinction between the two branches on each side going to infinity is clearly visible (Figure 6c). The upper one corresponds to a ‘negative’ radius, i.e.

¹ This is the result of unfolding [5] the A_4 that appeared when the perturbation held one double (non-generic) extremum, i.e. the transition from 0 to 2 new extremal values of the curvature of the shape.

Fig. 6. Evolute of the concave shape with the SS (a) and the ASS and the SS (b). Radius space with the SS (c) and the ASS and the SS (d).

Fig. 7. Left to right: The pre- SS , the ASS part and the SS part.

the radius related to a normal pointing outwardly. So again in the SS and ASS combination, the local extrema of the positive radii branches are connected. Note that the part along the x -axis contains three extrema.

5 Summary

In order to gain sufficient information from the MA or the SS additional information is needed. In this paper we extended the SS with information of the ASS , the pre- SS and the visualization using the describing circle as an extra dimension. The representations in parameter space and in 3D space carry more information than the commonly used 2D visualization. In 3D possible ambiguities are avoided, while the use of the ASS guarantees a connection between the local extrema of the main positive radii branches, which are the main curves of symmetry. Next, we showed that special points along the SS can also be found in the pre- SS , that can be used as a good indicator of the complexity of the SS and as localization tool for determining special points on it. Using these extensions of the SS , one is able to catch the SS in a linear data structure [12], dependent on the 1D curves in the pre- SS , in contrast to the graphs needed for representation of the MA .

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