Communication Leading to Nash Equilibrium without Acyclic Condition

– S4-Knowledge Model Case –

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Abstract. A pre-play communication-process is presented which leads to a Nash equilibrium of a strategic form game. In the communication process each player predicts the other players' actions, and he/she communicates privately his/her conjecture through message according to a protocol. All the players receiving the messages learn and revise their conjectures. We show that after a long round of the communication the profile of players' conjectures in the revision process leads a mixed strategy Nash equilibrium of the game.

Keywords: Knowledge, Information, Modal logic **S4**, Nash equilibrium, Communication, Mutual learning, Protocol, Conjecture, Non-corporative game.

1 Introduction

The purpose of this paper is to present the pre-play communication-process leading to a mixed strategy Nash equilibrium of a strategic form game.

The stage sets up as follows: The players start with the same prior distribution on a finite state-space. In addition they have the private information given by the non-partition structure corresponding to the modal logic **S4**. Each player communicates privately his/her conjecture about the other players' actions through messages according to a protocol, and the receiver of the message updates her/his conjecture. When a player communicates with another, the other players are not informed about the contents of the message. Suppose that all players are expected utility maximizers. Then

Main Theorem. *The players' predictions about the other players' actions regarding as the future conjectures converge in the long run, and those convergent conjectures constitute a mixed strategy Nash equilibrium of the game.*

The concept of Nash equilibrium has become central in game theory, economics and its related fields. Yet a little is known about the process by which

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players learn if they do. Recent papers by J. S. Jordan [4] and E. Kalai and E. Lehrer [5] indicate increasing interest in the mutual learning processes in Bayesian games which leads to Bayesian equilibrium.

As for as Nash's fundamental notion of strategic equilibrium (J. F. Nash [8]) is concerned, R.J. Aumann and A. Brandenburger [1] gives epistemic conditions for Nash equilibrium. However it is not clear just what learning process lead to Nash equilibrium. The present paper aims to fill this gap. The pre-play communication process according to a protocol is proposed which is a mutual learning process leading to a Nash equilibrium of a strategic form game as a cheap talk. The emphasis is on that any topological assumption on the communication graph is not required. T. Matsuhisa [6] proved the theorem under the assumption that the graph contains no cycle.

This paper organizes as follows. Section 2 presents the communication process for a game according a protocol. In Section 3 we give the explicit statement of the main theorem (Theorem 1) and we shall prove it.

2 The Model

Let Ω be a non-empty set called a *state-space*, N a set of finitely many *players* $1, 2, \dots, n$ ($n \geq 2$), and let 2^Ω be the family of all subsets of Ω . Each member of 2^Ω is called an *event* and each element of Ω called a *state*. Let μ be a probability measure on Ω which is common for all players.

2.1 Information and Knowledge¹

An *information structure* $(P_i)_{i \in N}$ is a class of mappings P_i of Ω into 2^Ω . It is called an *RT-information structure* if for every player i the two properties are true: For each ω of 2^Ω ,

Ref $\omega \in P_i(\omega);$

Trn $\xi \in P_i(\omega)$ implies $P_i(\xi) \subseteq P_i(\omega).$

Given our interpretation, player i for whom $P_i(\omega) \subseteq E$ knows, in the state ω , that some state in the event E has occurred. In this case we say that in the state ω the player i knows E . By i 's *knowledge operator* we mean the mapping $K_i : 2^\Omega \rightarrow 2^\Omega$ defined by $K_i E = \{\omega \in \Omega | P_i(\omega) \subseteq E\}$. This is the set of states of Ω in which i knows that E has occurred. We note that K_i satisfies the following properties²: For every E, F of 2^Ω ,

$$\begin{array}{ll} \mathbf{N} & K_i \Omega = \Omega \quad \text{and} \quad K_i \emptyset = \emptyset ; \\ \mathbf{T} & K_i F \subseteq F; \end{array} \qquad \begin{array}{ll} \mathbf{K} & K_i(E \cap F) = K_i E \cap K_i F; \\ \mathbf{4} & K_i F \subseteq K_i K_i F. \end{array}$$

¹ See Bacharach [2], Samet [10], Binmore [3]

² According to these we can say the structure $\langle \Omega, (K_i)_{i \in N} \rangle$ is a model for the multi-modal logic **S4**_n.

The set $P_i(\omega)$ will be interpreted as the set of all the states of nature that i believes to be possible at ω , and $K_i E$ will be interpreted as the set of states of nature for which i believes E to be possible. We will therefore call P_i i 's *possibility operator* on Ω and also will call $P_i(\omega)$ i 's *possibility set* at ω . An event E is said to be i 's *truism* if $E \subseteq K_i E$. We should note that the RT -information structure P_i is uniquely determined by the knowledge operator K_i such that $P_i(\omega)$ is the minimal truism containing ω ; that is, $P_i(\omega) = \bigcap_{\omega \in K_i E} E = \bigcap_{\omega \in T = K_i T} T$.

2.2 Game and Knowledge³

By a *game* G we mean a *finite* strategic form game $\langle N, (A_i)_{i \in N}, (g_i)_{i \in N} \rangle$ with the following structure and interpretations: N is a finite set of players $\{1, 2, \dots, i, \dots, n\}$ with $n \geq 2$, A_i is a finite set of i 's *actions* (or i 's pure strategies) and g_i is an i 's *payoff-function* of A into $\mathbb{R}$, where A denotes the product $A_1 \times A_2 \times \dots \times A_n$, A_{-i} the product $A_1 \times A_2 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n$. We denote by g the n -tuple $(g_1, g_2, \dots, g_n)$ and denote by a_{-i} the $(n-1)$ -tuple $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ for a of A .

A probability distribution ϕ_i on A_{-i} is said to be i 's *overall conjecture* (or simply i 's *conjecture*). For each player j other than i , this induces the marginal on j 's actions; we call it i 's *individual conjecture* about j (or simply i 's *conjecture about* j .) Functions on Ω are viewed like random variables in a probability space (Ω, μ) . If $\mathbf{x}$ is a such function and x is a value of it, we denote by $[\mathbf{x} = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega \mid \mathbf{x}(\omega) = x\}$.

An RT -information structure $(P_i)_{i \in N}$ with a common-prior μ yields the distribution defined by $\mathbf{q}_i(a, \omega) = \mu([\mathbf{a} = a] \mid P_i(\omega))$; and i 's overall conjecture defined by the marginal $\mathbf{q}_i(a_{-i}, \omega) = \mu([\mathbf{a}_{-i} = a_{-i}] \mid P_i(\omega))$ which is viewed as a random variable of ϕ_i . Where we assume that $[a_i] := [\mathbf{a}_i = a_i]$ is i 's truism for every a_i of A_i . The pay-off functions $g = (g_1, g_2, \dots, g_n)$ is said to be *actually played* at a state ω if $\omega \in [\mathbf{g} = g] := \bigcap_{i \in N} [\mathbf{g}_i = g_i]$. i 's action a_i is said to be *actual* at a state ω if $\omega \in [\mathbf{a}_i = a_i]$.

Player i is said to be *rational* at ω if each i 's actual action a_i maximizes the expectation of his actually played pay-off function g_i at ω when the other players actions are distributed according to his conjecture $\mathbf{q}_i(*, \omega)$: Formally, letting $g_i = \mathbf{g}_i(\omega)$ and $a_i = \mathbf{a}_i(\omega)$, $\mathbf{Exp}(g_i(a_i, \mathbf{a}_{-i}); \omega) \geq \mathbf{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega)$ for every b_i in A_i .⁴ Let R_i denote the set of all the states at which player i is rational, and $R = \bigcap_{j \in N} R_j$.

2.3 Protocol⁵

We assume that the players in N communicate by sending *messages*. Let T be the time horizontal line $\{0, 1, 2, \dots, t, \dots\}$. A *protocol* among N is a mapping Pr of the set of non-negative integers into the Cartesian product $N \times N$ that assigns

³ See Aumann and Brandenburger [1]

⁴ We denote $\mathbf{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega) := \sum_{a_{-i} \in A_{-i}} g_i(b_i, a_{-i}) \mathbf{q}_i(a_{-i}, \omega)$.

⁵ See Parikh and Krasucki [9]

to each t a pair of players $(s(t), r(t))$ such that $s(t) \neq r(t)$. Here t stands for *time* and $s(t)$ and $r(t)$ are, respectively, the *sender* and the *receiver* of the communication which takes place at time t . We consider the protocol as the directed graph whose vertices are the set of all players M and such that there is an edge (or an arc) from i to j if and only if there are infinitely many t such that $s(t) = i$ and $r(t) = j$.

A protocol is said to be *fair* if the graph is strongly-connected; in words, every player in this protocol communicates directly or indirectly with every other player infinitely often. It is said to contain a *cycle* if there are players $i_1, i_2, \dots, i_k$ with $k \geq 3$ such that for all $m < k$, i_m communicates directly with i_{m+1} , and such that i_k communicates directly with i_1 . The *period* of the protocol is the minimal number of all the natural number m such that $\Pr(t + m) = \Pr(t)$ for every t .

2.4 Pre-play Communication

By this we intuitively mean the learning process such that each player communicates privately his/her conjecture about the other players' actions through messages according to a protocol, and she/he updates her/his conjecture according to the message received. In addition, at every stage each player communicates privately not only his/her conjecture about the others' actions but also his/her rationality as messages, the receivers update their private information and revise their conjecture. When a player communicates with another, the other players are not informed about the contents of the message. Formally,

Definition 1. A *pre-play communication-process* according to a protocol among N for a game G with revisions of players' conjectures is a tuple

$$\langle \Pr, (P_i^t)_{i \in N}, (\phi_i^t)_{i \in N} \mid t \in T \rangle$$

with the following structures: the players have a common-prior μ on Ω , the protocol $\Pr$ among N , $\Pr(t) = (s(t), r(t))$, is fair and it satisfies the conditions that $r(t) = s(t+1)$ for every t and that the communications proceed in *rounds*.⁶ The information structure P_i^t at time t is the mapping of Ω into 2^Ω for player i that is defined inductively as follows: If $i = s(t)$ is a sender at t , he/she sends the message W_i^t defined as below to $j = r(t)$ at t .

- Set $P_i^0(\omega) = P_i(\omega)$.
- Assume that P_i^t is defined. It yields the overall conjecture $\mathbf{q}_i^t(a_{-i}, \omega) = \mu([\mathbf{a}_{-i} = a_{-i}] \mid P_i^t(\omega))$; whence
 - R_i^t denotes the set of all the state ω at which i is *rational* according to his conjecture $\mathbf{q}_i^t(\omega)$:⁷

⁶ That is, there exists a time m such that $\Pr(t) = \Pr(t + m)$ for all t .

⁷ That is, each i 's actual action a_i maximizes the expectation of his pay-off function g_i being actually played at ω when the other players actions are distributed according to his conjecture $\mathbf{q}_i^t(\omega)$ at time t . Formally, letting $g_i = \mathbf{g}_i(\omega)$, $a_i = \mathbf{a}_i(\omega)$, the

- $\mathcal{Q}_i^t$ denotes the partition induced by $\mathbf{q}_i^t$ on Ω , which is decomposed into the components $\mathcal{Q}_i^t(\omega)$ consisting of all the states ξ such that $\mathbf{q}_i^t(\xi) = \mathbf{q}_i^t(\omega)$.
 - $\mathcal{G}_i$ denotes the partition $\{[\mathbf{g}_i = g_i], \Omega \setminus [\mathbf{g}_i = g_i]\}$ of Ω , and $\mathcal{R}_i^t$ the partition $\{R_i^t, \Omega \setminus R_i^t\}$.
 - W_i^t denotes the join $\mathcal{G}_i \vee \mathcal{Q}_i^t \vee \mathcal{R}_i^t$ the partition of Ω generated by $\mathcal{G}_i$, $\mathcal{Q}_i^t$ and $\mathcal{R}_i^t$.⁸
- Then P_i^{t+1} is defined as follows:
- If i is a receiver of a message at time $t + 1$ then $P_i^{t+1}(\omega) = P_i^t(\omega) \cap W_{s(t)}^t(\omega)$.
 - If not, $P_i^{t+1}(\omega) = P_i^t(\omega)$. □

It is of worth noting that $(P_i^t)_{i \in N}$ is an RT -information structure for every $t \in T$. We require that the pre-play communication-process satisfies the following two conditions: Let K_i^t be the knowledge operator corresponding to P_i^t ;⁹

A-1 For each $i \in N$ and every $t \in T$, $[\phi_i^t] \subseteq K_i^t([\phi_i^t])$ and $R_i^t \subseteq K_i^t(R_i^t)$;

A-2 For every $t \in T$, $\bigcap_{i \in N} K_i^t([g_i] \cap [\phi_i^t] \cap R_i^t) \neq \emptyset$.

The specification of **A-1** is that each player's conjecture and his/her rationality are truism, and the specification of **A-2** is that each player knows his/her pay-off, rationality and conjecture at every time t .

Remark 1. For $i \in N$, the sequence of correspondences $\{P_i^t \mid t = 0, 1, 2, \dots\}$ is stationary in finitely many rounds. Furthermore so is the sequence of i 's conjectures $\{\mathbf{q}_i^t \mid t = 0, 1, 2, \dots\}$ in finitely many rounds. That is, there is a sufficiently large time $\tau \in T$ such that for every i , for all $\omega \in \Omega$ and for all $t \geq \tau$, $P_i^t(\omega) = P_i^\tau(\omega)$, and therefore $\mathbf{q}_i^t = \mathbf{q}_i^\tau$.

Notation: For sufficient large $t \geq \tau$ we denote τ by ∞ . Hence we can write $\mathbf{q}_i^\tau$ by $\mathbf{q}_i^\infty$ and ϕ_i^τ by ϕ_i^∞ .

3 Proof of Main Theorem

We now state the main theorem in Introduction as below and we shall prove it:

Theorem 1. *Suppose that the players in a strategic form game have a common-prior. In a pre-play communication process according to a protocol among all players in the game with revisions of their conjectures $\{(\phi_i^t)_{i \in N} \mid t = 0, 1, 2, \dots\}$, there exists a time τ such that for each $t \geq \tau$, the n -tuple $(\phi_i^t)_{i \in N}$ induces a mixed strategy Nash equilibrium of the game.*

expectation at time t , $\mathbf{Exp}^t$, is defined by

$$\mathbf{Exp}^t(g_i(b_i, \mathbf{a}_{-i}); \omega) := \sum_{a_{-i} \in A_{-i}} g_i(b_i, a_{-i}) \mathbf{q}_i^t(\omega)(a_{-i}).$$

Player i is said to be rational according to his conjecture $\mathbf{q}_i^t(\omega)$ at ω if for all b_i in A_i , $\mathbf{Exp}^t(g_i(a_i, \mathbf{a}_{-i}); \omega) \geq \mathbf{Exp}^t(g_i(b_i, \mathbf{a}_{-i}); \omega)$.

⁸ Therefore the component $W_i^t(\omega) = [g_i] \cap [\phi_i^t] \cap R_i^t$ if $\omega \in [g_i] \cap [\phi_i^t] \cap R_i^t$.

⁹ That is, K_i^t is defined by $K_i^t E = \{\omega \in \Omega \mid P_i^t(\omega) \subseteq E\}$.

A non-empty event H is said to be P_i -invariant if for every ξ of H , $P_i(\xi)$ is contained in H .

Fundamental lemma Let $(P_i)_{i \in N}$ be an RT-information structure with μ a common-prior. Let X be an event and q_i i 's posterior of X ; that is, $q_i = \mu(X|P_i(\omega))$. If there is an event H such that the following two conditions (a), (b) are true:

- (a) H is non-empty and it is P_i -invariant,
- (b) H is contained in $[q_i] := \{\omega \in \Omega \mid \mu(X|P_i(\omega)) = q_i\}$,

then we obtain that $\mu(X|H) = q_i$.

Proof. See Matsuhisa and Kamiyama [7].

Let $\omega_\infty \in \bigcap_{i \in N} K_i^\infty([g_i] \cap [\phi_i^\infty] \cap R_i^\infty) \subseteq \bigcap_{i \in N} ([g_i] \cap [\phi_i^\infty] \cap R_i^\infty)$. The following result is the another key to proving Theorem 1:

Proposition 1. In a pre-play communication-process among all the players in a game with revisions of their conjectures $\{(\phi_i^t)_{i \in N} \mid t = 0, 1, 2, \dots\}$, both the marginals of the conjectures ϕ_i^∞ and ϕ_j^∞ on A_{-i-j} must coincide for all $i, j \in N$; that is, $\phi_i^\infty(a_{-i-j}) = \phi_j^\infty(a_{-i-j})$ for all $a \in A$.

Proof: It suffices to verify that $\mathbf{q}_i^\infty(a; \omega) = \mathbf{q}_j^\infty(a; \omega)$ for all $(a; \omega) \in A \times \Omega$. Let us first consider the case that $(i, j) = (s(\infty), t(\infty))$. We denote by Π the partition on Ω with which each component is defined by $\Pi_i^\infty(\omega) = \{\xi \in \Omega \mid P_i^\infty(\xi) = P_i^\infty(\omega)\}$. In view of the construction of $\{P_i^t\}_{t \in T}$ we can observe that $P_i^\infty(\omega)$ is P_j^∞ -invariant; i.e., $P_j^\infty(\xi) \subseteq P_i^\infty(\omega)$ for all $\xi \in P_i^\infty(\omega)$. It immediately follows that $P_i^\infty(\omega)$ is decomposed into a disjoint union of components $\Pi_j^\infty(\xi)$ for $\xi \in P_i^\infty(\omega)$; $P_i^\infty(\omega) = \bigcup_{k=1,2,\dots,m} \Pi_j^\infty(\xi_k)$ where $\xi_k \in P_i^\infty(\omega)$. It can be observed that $\mu([\mathbf{a} = a] \mid P_i^\infty(\omega)) = \sum_{k=1}^m \lambda_k \mu([\mathbf{a} = a] \mid \Pi_j^\infty(\xi_k))$ for some $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$.¹⁰ By Fundamental lemma we note that $\mu([\mathbf{a} = a] \mid \Pi_j^\infty(\xi_k)) = \mathbf{q}_i^\infty(a; \xi_k)$ and thus by the above equation it can be observed that for all $\omega \in \Omega$ there is some $\xi_\omega \in P_i^\infty(\omega)$ such that $\mathbf{q}_i^\infty(a; \omega) \leq \mathbf{q}_j^\infty(a; \xi_\omega)$. Continuing this process according to the fair protocol the below facts can be plainly verified: For each $\omega \in \Omega$

- 1. For every $i \neq j$, $\mathbf{q}_i^\infty(a; \omega) \leq \mathbf{q}_j^\infty(a; \xi)$ for some $\xi \in \Omega$; and
- 2. $\mathbf{q}_i^\infty(a; \omega) \leq \mathbf{q}_i^\infty(a; \xi) \leq \mathbf{q}_i^\infty(a; \zeta) \leq \dots$ for some $\xi, \zeta, \dots \in \Omega$.

Since Ω is finite the equation $\mathbf{q}_i^\infty(a; \omega) = \mathbf{q}_j^\infty(a; \omega)$ can be obtained for every a, ω and for all i, j , in completing the proof.

Proof of Theorem 1. We denote by $\Gamma(i)$ the set of all the players who directly receive the message from i on N ; i.e., $\Gamma(i) = \{j \in N \mid (i, j) = \text{Pr}(t) \text{ for some } t \in T\}$. For any subset I of N denote $a_{-I} := (a_i)_{i \in N \setminus I}$.

For each $i \in N$, we denote $[g_i] \cap [\phi_i^\infty] \cap R_i^\infty$ by F_i . It is noted that F_i is a non-empty P_i -invariant set because $\emptyset \neq \bigcap_{i \in N} ([g_i] \cap [\phi_i^\infty] \cap R_i^\infty) \subseteq F_i$ and

¹⁰ This property is called *convexity* in Parikh and Krasucki [9].

because $P_i^\infty(\omega) \subseteq F_i$ for every $\omega \in F_i$ by the definition. We observe the first point that for each $i \in N$, $j \in \Gamma(i)$ and for every $a \in A$,

$$\mu([\mathbf{a}_{-j} = a_{-j}] | F_i \cap F_j) = \phi_j^\infty(a_{-j}) : \quad (1)$$

For, we note that $F_i \cap F_j \subseteq [\phi_j^\infty(a_{-j})]$ and $F_i \cap F_j$ is P_j^∞ -invariant because $j \in \Gamma(i)$. Hence by Fundamental lemma (3), we plainly obtain (1) as required. Then summing over a_{-i} , we can observe that

$$\mu([\mathbf{a}_i = a_i] | F_i \cap F_j) = \phi_j^\infty(a_i) \text{ for any } a \in A. \quad (2)$$

In view of Proposition 1 it can be observed that $\phi_j^\infty(a_i)$ is independent of the choices of every $j \in \Gamma(i)$.

We set the probability distribution σ_i on A_i by $\sigma_i(a_i) := \phi_i^\infty(a_i)$, and the profile $\sigma = (\sigma_i)$. We observe the second point that for every $a \in \prod_{i \in N} \text{Supp}(\sigma_i)$,

$$\phi_i^\infty(a_{-i}) = \sigma_1(a_1) \cdots \sigma_{i-1}(a_{i-1}) \sigma_{i+1}(a_{i+1}) \cdots \sigma_n(a_n) : \quad (3)$$

In fact, viewing the definition of σ_i we shall show that $\phi_i^\infty(a_{-i}) = \prod_{k \in N \setminus \{i\}} \phi_i^\infty(a_k)$. To verify this it suffices to show that for every $k = 1, 2, \dots, n$, $\phi_i^\infty(a_{-i}) = \phi_i^\infty(a_{-I_k}) \prod_{k \in I_k \setminus \{i\}} \phi_i^\infty(a_k)$: We prove by induction on k . For $k = 1$ the result is immediate. Suppose it is true for $k \geq 1$. On noting the protocol is fair, we can take the sequence of sets of players $\{I_k\}_{1 \leq k \leq n}$ with the following properties:

- (a) $I_1 = \{i\} \subset I_2 \subset \cdots \subset I_k \subset I_{k+1} \subset \cdots \subset I_m = N$:
- (b) For every $k \in N$ there is a player $i_{k+1} \in \bigcup_{j \in I_k} \Gamma(j)$ with $I_{k+1} \setminus I_k = \{i_{k+1}\}$.

We let take $j \in I_k$ such that $i_{k+1} \in \Gamma(j)$. Set $H_{i_{k+1}} := [\mathbf{a}_{i_{k+1}} = a_{i_{k+1}}] \cap F_j \cap F_{i_{k+1}}$. We note that $H_{i_{k+1}}$ is not empty because $\sigma_i(a_i) = \phi_j^\infty(a_i) = \mu([\mathbf{a}_i = a_i] | F_i \cap F_j) > 0$ in viewing of (2), and we note that $H_{i_{k+1}}$ is $P_{i_{k+1}}^t$ -invariant which is included in $[\phi_{i_{k+1}}^\infty(a_{-j-i_{k+1}})]$. It immediately follows from Fundamental lemma (3) that $\mu([\mathbf{a}_{-j-i_{k+1}} = a_{-j-i_{k+1}}] | H_{i_{k+1}}) = \phi_{-j-i_{k+1}}^\infty(a_{-j})$. Dividing $\mu(F_j \cap F_{i_{k+1}})$ yields that $\mu([\mathbf{a}_{-j} = a_{-j}] | F_j \cap F_{i_{k+1}}) = \phi_{i_{k+1}}^\infty(a_{-j}) \mu([\mathbf{a}_{i_{k+1}} = a_{i_{k+1}}] | F_j \cap F_{i_{k+1}})$. In viewing of (1) and (2) it follows $\phi_j^\infty(a_{-j}) = \phi_{i_{k+1}}^\infty(a_{-j-i_{k+1}}) \phi_j^t(a_{i_{k+1}})$; then summing over a_{I_k} it immediately follows from Proposition 1 that $\phi_i^\infty(a_{-I_k}) = \phi_i^\infty(a_{-I_k-i_{k+1}}) \phi_i^\infty(a_{i_{k+1}})$. Viewing (4) we have just observed that $\phi_i^\infty(a_{-i}) = \phi_i^\infty(a_{-I_{k+1}}) \prod_{k \in I_{k+1} \setminus \{i\}} \phi_i^\infty(a_k)$, as required. Furthermore we can observe from (4) that all the other players i than j agree on the same conjecture $\sigma_j(a_j) = \phi_i^\infty(a_j)$ about j .

Each action a_i with $\phi_i^\infty(a_i) > 0$ for some $j \in \Gamma(i)$ maximizes g_i against ϕ_i^∞ because $a_i = \mathbf{a}_i(\omega_i)$, $g_i = \mathbf{g}_i(\omega_i)$ and $\phi_i^\infty = \mathbf{q}_i^\infty(\omega_i)$ at some state ω_i of $H_i = [a_i] \cap F_i \cap F_j$. Viewing (3) we conclude that each action a_i appearing with positive probability in σ_i maximizes g_i against the product of the distributions σ_l with $l \neq i$. This implies that the profile $\sigma = (\sigma_i)_{i \in N}$ is a Nash equilibrium of G . This establishes the theorem.

4 Concluding Remarks

Our real concern is with what learning process leads to a mixed strategy Nash equilibrium of a finite strategic form game from the epistemic point view. As we have observed, in the pre-play communication process with revisions of players' conjectures about the other actions, their predictions induces a Nash equilibrium of the game in the long run. Where the players privately communicate each other through message according to any non-acyclic graph, and they are required neither to have the common-knowledge assumption about their conjectures nor to have a partition information structure.

The communication process treated in this article will give a new aspect of the algorithms converging to Nash equilibrium from the epistemic point of view. This issue needs to be sorted out at a more fundamental level, and it has not been discussed at all. There is a research agenda of potential interest which we hope to pursue further.

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