On Approximation of Average Expectation Prices for Path Dependent Options in Fractional Models^{*}

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Abstract. Using integral representation of a fractional Brownian motion (fBm) we propose a new method of approximation of its trajectories as well as trajectories of a geometric fBm. We give the rate of convergence in $\mathbb{L}^p$ -norm of the approximation. Applications to pricing of path dependent options in a fractional model are considered. Lookback and Asian options are examined in detail.

1 Introduction

Let $(B_t^{\mathbb{H}})_{t \in [0, T]}$ be a fractional Brownian motion (fBm) with Hurst index $\mathbb{H} \in (1/2, 1)$, i.e. $B^{\mathbb{H}}$ is a centered Gaussian process with covariance $EB_t^{\mathbb{H}}B_s^{\mathbb{H}} = \frac{1}{2}(t^{2\mathbb{H}} + s^{2\mathbb{H}} - |t - s|^{2\mathbb{H}})$ defined on some probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t \in [0, T]}$ the filtration generated by $(B_t^{\mathbb{H}})$. We consider a continuous-time market model with two assets, the riskless bond $(B_t)_{t \in [0, T]}$, and the risky stock $(S_t)_{t \in [0, T]}$. The dynamics for the prices are

$$B_t = e^{rt}, \quad S_t = S_0 e^{\mu t + \sigma B_t^{\mathbb{H}}},$$

where $S_0 > 0$ and $\mu \in \mathbb{R}$, $r, \sigma > 0$. It is convenient to make the assumption that there are no dividends, and no transaction costs. An *option* will be any non-negative $\mathcal{F}_T$ -measurable random variable f_T .

Note that if $\mathbb{H} = \frac{1}{2}$, the above model coincides with the well known Black-Scholes model. In this case a *fair price* of the option is the discounted expectation of f_T with respect to the martingale measure of this model.

If $\mathbb{H} \neq \frac{1}{2}$, then $B^{\mathbb{H}}$ is not a semimartingale, and martingale measures do not exist (see e.g. [4]), so we need a different method for option pricing. Instead of martingale measures we will use the so-called *average risk neutral measures*.

Definition 1 ([6]). *If the measure Q is equivalent to the measure P , and*

$$E^Q(S_t) = e^{rt}$$

for all $t \in [0, T]$ then Q is called the average risk neutral measure.

Existence and uniqueness of Q was proved by Valkeila [6].

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Definition 2. The average expectation price $C(f_T)$ of option f_T is defined by

$$C(f_T) = e^{-rT} E^Q f_T . \quad (1)$$

If $\mathbb{H} = \frac{1}{2}$, the price $C(f_T)$ coincides with the classical one for the Black-Scholes model. Definition 2 extends the one given by Valkeila [6], where $C(f_T)$ is defined for options depending only on the price of S at the expiration time T , i.e. $f_T = h(S_T)$ for some $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. In this case we have the following explicit formula

$$C(f_T) = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h \left(S_0 \exp\{\sigma y T^{\mathbb{H}} + rT - \frac{\sigma^2}{2} T^{2\mathbb{H}}\} \right) e^{-\frac{y^2}{2}} dy . \quad (2)$$

In the present paper we focus on path dependent options, i.e. options whose payoffs depend on the whole path of S . For example, the payoff of a *lookback option* depends on the minimum or maximum price of S , while the payoff of an *Asian option* depends on the average price of S during the life of option. Evaluation of path dependent options in fractional models is difficult and analytic formulas like (2) are not known. In the paper we will estimate $C(f_T)$ using the Monte Carlo method.

2 Approximation of fBm

In the paper we introduce a new method of approximating of $B^{\mathbb{H}}$ based on the integral representation of fBm as provided in [2]. Define the so-called fundamental martingale (M_t) by

$$M_t = \frac{c_{\mathbb{H}}}{2\mathbb{H}} \int_0^t s^{1/2-\mathbb{H}} dW_s , \quad (3)$$

where $c_{\mathbb{H}}$ is a constant depending on $\mathbb{H}$ and W is a standard Brownian motion. M is a Gaussian martingale with independent increments. Now, if we put

$$Y_t = 2\mathbb{H} \int_0^t (t-s)^{\mathbb{H}-1/2} dM_s$$

then the process

$$B_t^{\mathbb{H}} = \int_0^t s^{\mathbb{H}-1/2} dY_s$$

is a fractional Brownian motion (for the proof we refer to [2]).

To implement this method we proceed as follows.

1. Let M^n be a discretization of M , i.e. $M_t^n = M_{\frac{k}{n}}^n$ for $t \in [t_k, t_{k+1})$, where $t_k = \frac{k}{n}$, $k = 0, 1, \dots, [nT]$. We can simulate the increments of M_t^n using the fact that $M_{t_k}^n - M_{t_{k-1}}^n \sim \mathcal{N}(0, C(t_k^{2-2\mathbb{H}} - t_{k-1}^{2-2\mathbb{H}}))$.

2. Next, we define the approximation Y^n of Y by

$$Y_t^n = 2\mathbb{H} \sum_{t_k \leq t^*} (t^* - t_{k-1})^{\mathbb{H}-1/2} (M_{t_k}^n - M_{t_{k-1}}^n) ,$$

where $t^* = \max\{t_k; t_k \leq t\}$. It is worth noting that $(t^* - t_k)^{\mathbb{H}-1/2}$ can be written in the form $(\frac{m}{n})^{\mathbb{H}-1/2}$, where $m, n \in \mathbb{N}$, $m \leq [nT]$, so to compute Y^n it is sufficient to compute $m^{\mathbb{H}-1/2}$ for $m = 1, \dots, [nT]$.

3. Finally, we approximate $B^{\mathbb{H}}$ by sums of the form

$$B_t^n = \sum_{t_k \leq t^*} t_{k-1}^{\mathbb{H}-\frac{1}{2}} (Y_{t_k}^n - Y_{t_{k-1}}^n) . \quad (4)$$

Let $\|\cdot\|_p$ denotes the usual $\mathbb{L}^p$ -norm.

Theorem 1. *Let (B_t^n) be a process defined by (4). Then for any $p \geq 1$, $\epsilon > 0$ and $T \in \mathbb{R}^+$*

$$\|\sup_{t \leq T} |B_t^{\mathbb{H}} - B_t^n| \|_p = \mathcal{O} \left(\frac{1}{n^{\mathbb{H}^* - \epsilon}} \right) ,$$

where $\mathbb{H}^* = \min(1 - \mathbb{H}, \mathbb{H} - 1/2)$.

Proof. From (3) it follows that $a_t = [M]_t = Ct^{2-2\mathbb{H}}$, where C is a constant depending on $\mathbb{H}$. Hence $M_t = \bar{W}_{a_t}$ where $\bar{W}$ is a standard Brownian motion. Using the known estimates on the modulus of continuity of $\bar{W}$ (see e.g. [5]) we get

$$\begin{aligned} \|\sup_{t \leq T} |M_t - M_t^n| \|_p &\leq \|\omega_M(1/n, T)\|_p = \|\omega_{\bar{W}_a}(1/n, T)\|_p \\ &\leq \|\omega_{\bar{W}}(a(1/n), T)\|_p \leq \text{Const} (a(1/n) \ln(a(1/n)^{-1}))^{1/2} \\ &\leq \text{Const} \frac{\sqrt{\ln n}}{n^{1-\mathbb{H}}} , \end{aligned} \quad (5)$$

where $\omega_x(\delta, T) = \sup\{|x_t - x_s|; s, t \in [0, T], |s - t| \leq \delta\}$, $\delta > 0$, $T \in \mathbb{R}^+$, $x \in C([0, T], \mathbb{R}^+)$.

We next turn to estimating $\|\sup_{t \leq T} |Y_t - Y_t^n| \|_p$. Using the integration by parts formula, we obtain

$$Y_t = 2\mathbb{H} \int_0^t (t-s)^{\mathbb{H}-1/2} dM_s = -2\mathbb{H} \int_0^t M_s d(t-s)^{\mathbb{H}-1/2} . \quad (6)$$

The process Y^n can be written as

$$\begin{aligned} Y_t^n &= 2\mathbb{H} \sum_{t_k \leq t^*} (t^* - t_{k-1})^{\mathbb{H}-1/2} (M_{t_k}^n - M_{t_{k-1}}^n) \\ &= -2\mathbb{H} \sum_{t_k \leq t^*} M_{t_k}^n [(t^* - t_k)^{\mathbb{H}-1/2} - (t^* - t_{k-1})^{\mathbb{H}-1/2}] \\ &= -2\mathbb{H} \int_0^{t^*} \hat{M}_s^n d(t^* - s)^{\mathbb{H}-1/2} , \end{aligned} \quad (7)$$

where $\hat{M}_t^n = M_{\frac{k+1}{n}}$ for $t \in [\frac{k}{n}, \frac{k+1}{n})$. Note that (5) still holds if we replace M^n by $\hat{M}^n$. Now,

$$\begin{aligned} \left\| \sup_{t \leq T} |Y_t - Y_t^n| \right\|_p &\leq \text{Const} \left(\left\| \sup_{t \leq T} |Y_t - Y_{t^*}| \right\|_p + \left\| \sup_{t \leq T} |Y_{t^*} - Y_t^n| \right\|_p \right) \\ &= I_1 + I_2 . \end{aligned}$$

Combining (5) with (6) and (7) gives

$$\begin{aligned} I_2 &\leq \text{Const} \left\| \sup_{t \leq T} \left| \int_0^{t^*} (M_s - \hat{M}_s^n) d(t^* - s)^{\mathbb{H}-1/2} \right| \right\|_p \\ &\leq \text{Const} \left\| \sup_{t \leq T} E |M_t - \hat{M}_t^n| \right\|_p \\ &\leq \text{Const} \frac{\sqrt{\ln n}}{n^{1-\mathbb{H}}} . \end{aligned} \tag{8}$$

To estimate I_1 , we first prove that for any $t_1, t_2 \in [0, T]$ and $q \geq 1$

$$E |Y_{t_2} - Y_{t_1}|^q \leq \text{Const} |t_2 - t_1|^{q(\mathbb{H}-1/2)} .$$

Indeed, by the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned} E |Y_{t_2} - Y_{t_1}|^q &= E \left| \int_0^{t_1} [(t_2 - s)^{\mathbb{H}-1/2} - (t_1 - s)^{\mathbb{H}-1/2}] dM_s \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\mathbb{H}-1/2} dM_s \right|^q \\ &\leq \text{Const} \left(E \left| \int_0^{t_1} [(t_2 - s)^{\mathbb{H}-1/2} - (t_1 - s)^{\mathbb{H}-1/2}]^2 d[M]_s \right|^{q/2} \right. \\ &\quad \left. + E \left| \int_{t_1}^{t_2} (t_2 - s)^{2\mathbb{H}-1} d[M]_s \right|^{q/2} \right) \\ &\leq \text{Const} \left((t_2 - t_1)^{q(\mathbb{H}-1/2)} \left| \int_0^{t_1} C s^{1-2\mathbb{H}} ds \right|^{q/2} \right. \\ &\quad \left. + \left| \int_{t_1}^{t_2} C s^{1-2\mathbb{H}} ds \right|^{q/2} \right) \\ &\leq \text{Const} (t_2 - t_1)^{q(\mathbb{H}-1/2)} \left| \int_0^T C s^{1-2\mathbb{H}} ds \right|^{q/2} \\ &\leq \text{Const} (t_2 - t_1)^{q(\mathbb{H}-1/2)} . \end{aligned}$$

Let $q > \max(p, (\mathbb{H} - 1/2)^{-1})$. By the Kolmogorov continuity theorem,

$$E \left(\sup_{\substack{t_1, t_2 \leq T \\ t_1 \neq t_2}} \frac{|Y_{t_2} - Y_{t_1}|}{|t_2 - t_1|^\alpha} \right)^q < +\infty$$

for $0 < \alpha < (\mathbb{H} - 1/2) - 1/q$. Since $p < q$, from the above it follows that

$$I_1 \leq (E \sup_{t \leq T} |Y_t - Y_{t^*}|^q)^{1/q} \leq \text{Const} \frac{1}{n^{\mathbb{H}-1/2-\epsilon}} \quad (9)$$

for every $\epsilon > 0$. By (8) and (9),

$$\left\| \sup_{t \leq T} |Y_t - Y_t^n| \right\|_p \leq \text{Const} \frac{1}{n^{\mathbb{H}^*-\epsilon}} . \quad (10)$$

It remains to show that $\left\| \sup_{t \leq T} |B_t^{\mathbb{H}} - B_t^n| \right\|_p \leq \text{Const} \left\| \sup_{t \leq T} |Y_t - Y_t^n| \right\|_p$. Using once again the integration by parts formula we obtain

$$B_t^{\mathbb{H}} = \int_0^t s^{\mathbb{H}-1/2} dY_s = t^{\mathbb{H}-1/2} Y_t - \int_0^t Y_s ds^{\mathbb{H}-1/2} ,$$

and

$$\begin{aligned} B_t^n &= \int_0^t s^{\mathbb{H}-1/2} dY_s^n = t^{\mathbb{H}-1/2} Y_t^n - \int_0^t Y_s^n ds^{\mathbb{H}-1/2} \\ &\quad - \sum_{t_k \leq t} (Y_{t_k}^n - Y_{t_{k-1}}^n) (t_k^{\mathbb{H}-1/2} - t_{k-1}^{\mathbb{H}-1/2}) . \end{aligned} \quad (11)$$

Hence

$$\begin{aligned} \left\| \sup_{t \leq T} |B_t^{\mathbb{H}} - B_t^n| \right\|_p &\leq \text{Const} \left(\left\| \sup_{t \leq T} |t^{\mathbb{H}-1/2} Y_t - t^{\mathbb{H}-1/2} Y_t^n| \right\|_p \right. \\ &\quad + \left\| \sup_{t \leq T} \left| \int_0^t (Y_s - Y_s^n) ds^{\mathbb{H}-1/2} \right| \right\|_p \\ &\quad + \left\| \sup_{t \leq T} \left| \sum_{t_k \leq t} (Y_{t_k}^n - Y_{t_{k-1}}^n) (t_k^{\mathbb{H}-1/2} - t_{k-1}^{\mathbb{H}-1/2}) \right| \right\|_p \Bigg) \\ &= I_4 + I_5 + I_6 . \end{aligned}$$

It is easily seen that I_4 and I_5 are bounded by $\text{Const} \left\| \sup_{t \leq T} |Y_t - Y_t^n| \right\|_p$. To estimate I_6 we observe first that

$$\begin{aligned} \sup_{t_k \leq T} |Y_{t_k}^n - Y_{t_{k-1}}^n| &= \sup_{t \leq T} |Y_t^n - Y_{t-}^n| = \sup_{t \leq T} |(Y_t^n - Y_{t-}^n) - (Y_t - Y_{t-})| \\ &\leq 2 \sup_{t \leq T} |Y_t - Y_t^n| . \end{aligned}$$

Hence

$$I_6 \leq T^{\mathbb{H}-1/2} \left\| \sup_{t_k \leq T} |Y_{t_k}^n - Y_{t_{k-1}}^n| \right\|_p \leq \text{Const} \left\| \sup_{t \leq T} |Y_t - Y_t^n| \right\|_p ,$$

which completes the proof. $\square$

3 Approximation of Prices for Path Dependent Options

The method described in the previous section can be used to approximate the price evolution process S . Let S^n be the approximation of S defined by

$$S_t^n = S_0 e^{\mu t + \sigma B_t^n}$$

for all $t \in [0, T]$. The following result is a consequence of Theorem 1.

Theorem 2. *For any $p \geq 1$, $\epsilon > 0$ and $T \in \mathbb{R}^+$*

$$\left\| \sup_{t \leq T} |S_t - S_t^n| \right\|_p = \mathcal{O} \left(\frac{1}{n^{\mathbb{H}^* - \epsilon}} \right) .$$

Proof. Without loss of generality we may and will assume that $S_0 = 1, \mu = 0, \sigma = 1$. Observe first that by the Schwarz inequality,

$$\begin{aligned} \left\| \sup_{t \leq T} |S_t - S_t^n| \right\|_p &= \left\| \sup_{t \leq T} |e^{B_t^{\mathbb{H}}} [e^{(B_t^{\mathbb{H}} - B_t^n)} - 1]| \right\|_p \\ &\leq \left\| \sup_{t \leq T} |e^{B_t^{\mathbb{H}}}| \right\|_{2p} \left\| \sup_{t \leq T} |e^{(B_t^{\mathbb{H}} - B_t^n)} - 1| \right\|_{2p} . \end{aligned}$$

Since $|e^x - 1| \leq |x|e^{|x|}$, we have

$$\begin{aligned} \left\| \sup_{t \leq T} |e^{(B_t^{\mathbb{H}} - B_t^n)} - 1| \right\|_{2p} &\leq \left\| \sup_{t \leq T} |B_t^{\mathbb{H}} - B_t^n| \cdot e^{|B_t^{\mathbb{H}} - B_t^n|} \right\|_{2p} \\ &\leq \left\| \sup_{t \leq T} |B_t^{\mathbb{H}} - B_t^n| \right\|_{4p} \left\| \sup_{t \leq T} e^{|B_t^{\mathbb{H}} - B_t^n|} \right\|_{4p} . \end{aligned}$$

By Theorem 1,

$$\left\| \sup_{t \leq T} |B_t^{\mathbb{H}} - B_t^n| \right\|_{4p} = \mathcal{O} \left(\frac{1}{n^{\mathbb{H}^* - \epsilon}} \right) ,$$

so it remains to show that $\left\| \sup_{t \leq T} e^{B_t^{\mathbb{H}}} \right\|_{8p} < +\infty$ and $\sup_n \left\| \sup_{t \leq T} e^{B_t^n} \right\|_{8p} < +\infty$. Novikov and Valkeila [3] proved that $\sup_{t \leq T} |B_t^{\mathbb{H}}| \leq C \sup_{t \leq T} |M_t|$ for some constant $C > 0$. Therefore, by the Doob submartingale inequality,

$$\begin{aligned} \left\| \sup_{t \leq T} e^{B_t^{\mathbb{H}}} \right\|_{8p} &= (E e^{\sup_{t \leq T} (8p |B_t^{\mathbb{H}}|)})^{1/(8p)} \\ &\leq (E \sup_{t \leq T} e^{8p C |M_t|})^{1/(8p)} \\ &\leq (4 E e^{8p C |M_T|})^{1/(8p)} < +\infty , \end{aligned} \tag{12}$$

because M is a Gaussian martingale. Finally, by (11) and (7),

$$\begin{aligned} \sup_{t \leq T} |B_t^n| &\leq T^{\mathbb{H}-1/2} \sup_{t \leq T} |Y_t^n| + T^{\mathbb{H}-1/2} \sup_{t \leq T} |Y_t^n| \\ &\quad + T^{\mathbb{H}-1/2} 2 \sup_{t \leq T} |Y_t^n| \leq 4 T^{\mathbb{H}-1/2} \sup_{t \leq T} |Y_t^n| \\ &\leq 8 \mathbb{H} T^{2\mathbb{H}-1} \sup_{t \leq T} |M_t^n| . \end{aligned}$$

Therefore, similarly to (12) we get

$$\begin{aligned} \sup_n \left\| \sup_{t \leq T} e^{B_t^n} \right\|_{8p} &\leq \sup_n (4 E e^{64p \mathbb{H} T^{2\mathbb{H}-1} |M_T^n|})^{1/(8p)} \\ &\leq (4 E e^{64p \mathbb{H} T^{2\mathbb{H}-1} |M_T|})^{1/(8p)} < +\infty \end{aligned}$$

and the proof is complete. $\square$

Definition 3. We say that an operator $H : C([0, T], \mathbb{R}^+) \rightarrow \mathbb{R}$ is Lipschitz if there exists $L > 0$ such that for any $x, y \in C([0, T], \mathbb{R}^+)$

$$|H(x) - H(y)| \leq L \cdot \sup_{t \leq T} |x_t - y_t| .$$

From Theorem 1 and Definition 3 we obtain immediately the following corollary.

Corollary 1. If $H : C([0, T], \mathbb{R}^+) \rightarrow \mathbb{R}$ is Lipschitz then for any $p \geq 1$, $\epsilon > 0$

$$\|H(S) - H(S^n)\|_p = \mathcal{O} \left(\frac{1}{n^{\mathbb{H}^* - \epsilon}} \right) .$$

By choosing appropriately a Lipschitz operator H we obtain interesting examples of path dependent options of the form $f_T = H(S)$.

Example 1 (Floating strike lookback call option).

In this case $H(S) = S_T - \inf_{t \leq T} S_t$. Note that for any $x, y \in C([0, T], \mathbb{R}^+)$

$$\begin{aligned} |H(x) - H(y)| &= |(x_T - y_T) + (\inf_{t \leq T} y_t - \inf_{t \leq T} x_t)| \\ &\leq 2 \sup_{t \leq T} |x_t - y_t| . \end{aligned}$$

Example 2 (Fixed strike lookback call option with the strike price K).

Here $H(S) = (\sup_{t \leq T} S_t - K)^+$ and for any $x, y \in C([0, T], \mathbb{R}^+)$

$$|H(x) - H(y)| \leq \left| \sup_{t \leq T} x_t - \sup_{t \leq T} y_t \right| \leq \sup_{t \leq T} |x_t - y_t| .$$

Example 3 (Asian call option with the strike price K and continuous arithmetic averaging).

For such options

$$H(S) = \left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ ,$$

and it is easily seen that

$$\begin{aligned} |H(x) - H(y)| &= \left| \left(\frac{1}{T} \int_0^T x_t dt - K \right)^+ - \left(\frac{1}{T} \int_0^T y_t dt - K \right)^+ \right| \\ &\leq \left| \frac{1}{T} \int_0^T (x_t - y_t) dt \right| \leq \sup_{t \leq T} |x_t - y_t|, \end{aligned}$$

for any $x, y \in C([0, T], \mathbb{R}^+)$.

4 Numerical Results

The method described in Sections 2 and 3 we have applied to evaluate prices of the lookback and Asian options. We have simulated 100000 trajectories of S with step-size $1/1000$.

Table 1 gives prices of the floating strike lookback options (LB1), fixed strike lookback options with the strike price $K_1 = 5$ (LB2) and Asian options with the strike price $K_2 = 5$ (AS) for different choices of the Hurst index $\mathbb{H}$. The initial asset price S_0 is set equal to 5, parameters r and σ are equal 0.1 and 0.3 respectively.

Table 1. The prices of LB1, LB2 and AS ($S_0 = 5$, $K_1 = 5$, $K_2 = 5$, $\sigma = 0.3$, $r = 0.1$, $T = 1$).

$\mathbb{H}$	0.6	0.7	0.8	0.9
LB1	1.1381	1.0288	0.9399	0.8861
LB2	1.2727	1.1177	1.0013	0.9138
AS	0.4464	0.4358	0.4234	0.4207

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