The New Graphic Description of the Haar Wavelet Transform

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Abstract. The image processing and analysis based on the continuous or discrete image transforms are the classic processing technique. The image transforms are widely used in image filtering, data description, etc. The image transform theory is a well known area, but in many cases some transforms have particular properties which are not still investigated. This paper for the first time presents graphic dependences between parts of Haar and wavelets images. The extraction of image features immediately from spectral coefficients distribution has been shown. In this paper it has been presented that two-dimensional both, the Haar and wavelets functions products, can be treated as extractors of particular image features.

1 Introduction

The computer and video-media applications have developed rapidly the field of multimedia, which requires the high performance, speedy digital video and audio capabilities. The digital signal processing is widely used in many areas of electronics, communication and information techniques [1,2,3,6,12]. In the signals compression, filtration, systems identification, the commonly used transforms are based on sinusoidal basic functions such as: Discrete Fourier, Sine or Cosine Transform or rectangular basic functions: Discrete Walsh and Wavelet Transform, (Haar, Daubechies, etc.) [2,3,7]. All these functions are orthogonal, and their transforms require only additions and subtractions. It makes that it is easy to implement them on the computer. It not only simplifies computations but also permits to use different (linear and nonlinear) filters [3,4,9] to get the spectrum. One should remember that researches in this topic are still in progress and new improvements have been found [5,8,9].

Fourier methods are not always good tools to recapture the non-smooth signal [2]; too much information is needed to reconstruct the signal locally. In these cases the wavelet analysis is often very effective because it provides a simple approach for dealing with the local aspects of signal, therefore particular properties of the Haar or wavelet transforms allow analyzing original image on spectral domain effectively.

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2 The Discrete Haar and Wavelet Transforms

Alfred Haar in [7] has defined a complete orthogonal system of functions in $L^p([0,1])$, $p \in [1,\infty]$. Nowadays, in the literature there are some other definitions of the Haar functions [3,12]. Discrete Haar functions can be defined as functions determined by sampling the Haar functions at 2^n points. These functions can be conveniently represented by means of matrix form. Each row of the matrix $\mathbf{H}(n)$ includes the discrete Haar sequence $haar(w,t)$ (or otherwise the discrete Haar function). In this notation, index w identifies the number of the Haar function and index t discrete point of the function determination interval. In this case, the Haar matrix of any dimension can be obtained by the following recurrence relation:

$$\mathbf{H}(n) = \begin{bmatrix} \mathbf{H}(n-1) & \otimes[1 \quad 1] \\ 2^{(n-1)/2}\mathbf{I}(n-1) & \otimes[1 \quad -1] \end{bmatrix}, \qquad \mathbf{H}(0) = 1 \tag{1}$$

and: $\quad \mathbf{H}(n) \neq \mathbf{H}(n)^T$ for $n > 1$ and $[\mathbf{H}(n)]^{-1} = 2^{-n} \cdot \mathbf{H}(n)^T$,
where: $\mathbf{H}(n)-$ matrix of the discrete Haar functions of degree 2^n, $\mathbf{I}(n)-$ identity matrix of degree 2^n, $\otimes -$ the Kronecker (tensor) product.

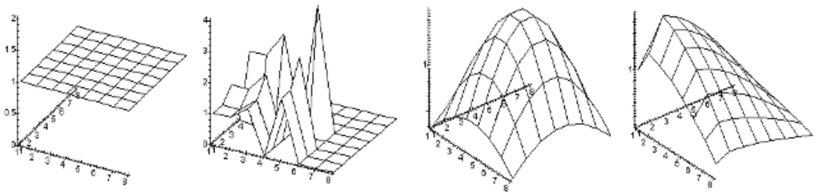
Definition 1. Two-dimensional $N \times N = 2^n \times 2^n$ forward and inverse Discrete Haar Transform can be defined in matrix notation as:

$$\mathbf{S} = a \cdot \mathbf{H}(n) \cdot \mathbf{F} \cdot a \cdot \mathbf{H}(n)^T, \qquad \mathbf{F} = b \cdot \mathbf{H}(n)^T \cdot \mathbf{S} \cdot b \cdot \mathbf{H}(n), \tag{2}$$

where: **F** – the image in matrix form. The matrix has dimension $N \times N$ pixels. **S** – the spectrum matrix, and $a \cdot b = 1/N$. Hence a or b parameters can be defined as values: 1/N, $1/\sqrt{N}$ or 1, $n = \log_2 N$.

Fig. 1 presents some known transforms of a test image. The test image contains a simple test impulse represented as 8×8 matrix, which has 0 values everywhere, except the upper left element, which has the value of 8. From Fig. 1 we can observe that all N^2 elements of these transforms are nonzero except the Haar transform, which has only 2N nonzero entries. These features are very important in image processing and convenient from image compression point of view. The energy distribution informs us where there are situated the important features of image [2,10,12].

It is easy to observe from Fig.1 that the Walsh transform gives the worst results here: distribution of spectral energy is uniformable. In c) and d) cases distribution of spectral energy has sharply outlined maximum, outside of which, one can observe the decrease of energy. The distribution of the Haar spectrum is not proper too, but we can treat this transform differently. Presented discrete transforms, enable us to observe where energy concentrations occur but from this representation, it is not possible to find more precisely information about real image. For example, it is difficult to point places, which describe horizontal, vertical, etc. details of real image. These troubles can be overcome by well known multiresolution analysis [3,5].



a) b) Note: $a = 1/\sqrt{N}$ c) d)

Fig. 1. The **S** transform of image containing the test impulse: a) Walsh-Hadamard; b) Haar; c) DST (Discrete Sine Transform); d) DCT (Discrete Cosine Transform)

The motivation for usage of the wavelet transform is to obtain information that is more discriminating by providing a different resolution at different parts of the time-frequency plane. The wavelet transforms allow partitioning of the time-frequency domain into non-uniform tiles in connection with the time-spectral contents of the signal. The wavelet methods are connected with classical basis of the Haar functions – scaling and dilation of a basic wavelet can generate the basis Haar functions.

Any Haar function basis (1) can be generated as: $\psi_i^j(t) = \sqrt{2^j}\,\psi(2^j t - i)$, $i = 0,1,...,2^j - 1$, $j = 0,1,...,\log_2 N - 1$, or generally $\psi_i^j(t) = haar(2^j + i, t)$. From this example follows that functions $\psi_i^j(t)$ are orthogonal to one another. Hence, we obtain linear span of vector space $W^j = spn\{\psi_i^j\}_{i=0,...,2^j-1}$. A collection of linearly independent functions $\{\psi_i^j(t)\}_{i=0,...,2^j-1}$ spanning W^j we called wavelets. The Haar scaling function is defined by the formula: $\phi_i^j(t) = \sqrt{2^j}\,\phi(2^j t - i)$, $i = 0,1,...,2^j - 1$, $j = 0,1,...,\log_2 N - 1$. The index j refers to dilation and index i refers to translation [3,11]. Hence, we obtain linear span of vector space $V^j = spn\{\phi_i^j\}_{i=0,...,2^j-1}$. The basic functions from the space V^j are called scaling functions. In multiresolution analysis the Haar basis has important property of orthogonality: $V^j = V^{j-1} \oplus W^{j-1}$. The space W^j can be treated as the orthogonal complement of V^j in V^{j+1}. So, the basis functions of W^j together with the basis functions of V^j form a basis for V^{j+1}.

3 The Haar and Wavelet Basic Images

Due to its low computing requirements, the Haar transform has been mainly used for image processing and pattern recognition. From this reason two dimensional signal processing is an area of efficient applications of Haar transforms due to their wavelet-like structure. Because $\mathbf{H}(n)$ and $\mathbf{H}(n)^T$ are the square matrices, their product is commutative, therefore equations (2) can be rewritten and expressed as:

$$s(k,m)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\times haar(k,x)\times haar(m,y) \quad (3)$$

where: $\mathbf{S}=[s_{km}]$, $\mathbf{F}=[f_{xy}]$, $x,y,k,m\in\{0,1,...,N-1\}$.

Basing on equation of analysis (2) we can conclude that in 2D spectral domain the values of coefficients s_{ij} depend on appropriate product of the two Haar functions. Fig. 2 presents an example of product of the arbitrary selected Haar functions.

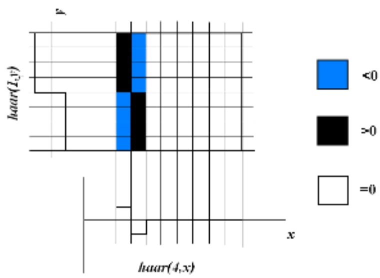


Fig. 2. The example of product of two discrete Haar functions

Because this product is multiplied by image matrix, the result of such multiplication can be treated as a particular extractor – it can be used to locate the specific edges hidden in image. By looking for all coefficients in the spectral space, one can find all important edge directions in the image. In this case, we must find decomposition matrices of matrix $\mathbf{H}(n)$. For last decomposition level, it can be noticed that $\mathbf{M}_n=\mathbf{H}(n)$. If each orthogonal matrix $\mathbf{M}_i$, $i=1,2,3$ one multiplies by $1/\sqrt{2}$ factor, then procedure of calculations will be according to the classical Mallat algorithm [11]. The product of the decomposition levels for all 2D Haar functions (for case $N=8$) is shown in Fig. 3 – the pictures have been composed on the basis of $\mathbf{M}_i$ matrices and the method shown in Fig. 2. From Fig. 3 we can conclude that the classical Haar transform gives different spectral coefficients on different decomposition levels. The construction of decomposition matrices can be as follows:

Step 1. According to the formula $V^n=V^{n-1}\oplus W^{n-1}$, the matrix $\mathbf{M}_1$ has a form $\mathbf{M}_1=[\phi^{n-1}_{j=0,...,2^{n-1}-1}\subset V^{n-1},\psi^{n-1}_{j=0,...,2^{n-1}-1}\subset W^{n-1}]^T$.

Step 2. Because $V^{n-1}=V^{n-2}\oplus W^{n-2}\oplus W^{n-1}$, the matrix $\mathbf{M}_2$ can be constructed as follows $\mathbf{M}_2=[\phi^{n-2}_{j=0,...,2^{n-2}-1}\subset V^{n-2},\psi^{n-2}_{j=0,...,2^{n-2}-1}\subset W^{n-2},\psi^{n-1}_{j=0...,2^{n-1}-1}\subset W^{n-1}]^T$.

Step n. Finally, after n steps of calculations, we can construct the formula $V^1=V^0\oplus W^0\oplus W^1\oplus W^2\oplus...\oplus W^{n-1}$, hence the matrix $\mathbf{M}_n$ has a structure $\mathbf{M}_n=[\phi^0_0\subset V^0,\psi^0_0\subset W^0,\psi^1_{j=0,1}\subset W^1,\psi^2_{j=0,...,3}\subset W^2,...,\psi^{n-1}_{j=0,...,2^{n-1}-1}\subset W^{n-1}]^T$.

Example 1. Let $n = 3$ then:

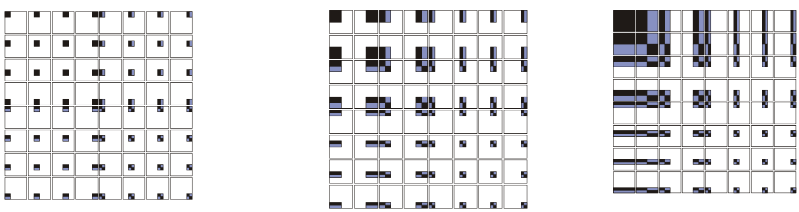
$$V^3 = V^2 \oplus W^2, \qquad \mathbf{M}_1 = [\phi_0^2, \phi_1^2, \phi_2^2, \phi_3^2, \psi_0^2, \psi_1^2, \psi_2^2, \psi_3^2]^T,$$

$$V^2 = V^1 \oplus W^1 \oplus W^2, \qquad \mathbf{M}_2 = [\phi_0^1, \phi_1^1, \psi_0^1, \psi_1^1, \psi_{j=0\ldots,3}^2 \subset W^2]^T,$$

$$V^1 = V^0 \oplus W^0 \oplus W^1 \oplus W^2, \qquad \mathbf{M}_3 = [\phi_0^0, \psi_0^0, \psi_{j=0,1}^1 \subset W^1, \psi_{j=0,\ldots3}^2 \subset W^2]^T,$$

$$\mathbf{M}_1 = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}, \mathbf{M}_2 = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix},$$

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}.$$



a) b) c)

Fig. 3. The 2D Haar functions product treated as extractors. Decomposition levels: a) first, b) second, c) third

One advantage of the method presented above is that often a large number of the detail coefficients turn out to be very small in magnitude, as in the example of Fig. 1. Truncating, or removing, these small coefficients introduce only small errors in the reconstructed image. Additionally, we can control which coefficients will be removed, because its distribution is known (Fig. 3).

Basing on the facts that $W^j = spn\{\varphi_i^j\}_{i=0,\ldots,2^j-1}$, $V^j = spn\{\phi_i^j\}_{i=0,\ldots,2^j-1}$ we can express functions ϕ and ψ as a linear combination of the basis functions from V and W spaces. Let us denote **F** as an image in matrix form and define the operators:

$$\mathbf{A}(i) = 1/\sqrt{2} \cdot [\mathbf{F}(2i) + \mathbf{F}(2i+1)], \qquad \mathbf{D}(i) = 1/\sqrt{2} \cdot [\mathbf{F}(2i) - \mathbf{F}(2i+1)], \qquad (4)$$

where: $\mathbf{F}(i)$ – vector of size *N*, containing row or column of matrix **F**, $i \in \{0,1,\ldots,N/2-1\}$, $\mathbf{A}(i)$ – vector of size *N*/2, containing approximation coefficients, $\mathbf{D}(i)$ – vector of size *N*/2, containing detail coefficients.

To get wavelet decomposition on the first level of an image **F** (the spectrum matrix called $\mathbf{S}_1$) we first apply the operators (4) to all columns of the matrix and then to all rows [3,8,11]. To get the second level of wavelet decomposition (matrix $\mathbf{S}_2$) one can apply similar analysis to upper left sub-matrix of size $\frac{N}{2}\times\frac{N}{2}$ of matrix $\mathbf{S}_1$. And generally, to get k-th level – matrix $\mathbf{S}_k$, one can apply this analysis to upper left sub-matrix of size $\frac{N}{2^{k-1}}\times\frac{N}{2^{k-1}}$ of matrix $\mathbf{S}_{k-1}$, where $k \in \{1,...,\log_2 N\}$.

Note, that applying filters (4) to an image, give the same results as multiplying matrices $\mathbf{S}_1 = \frac{1}{8}\mathbf{M}_1 \cdot \mathbf{F} \cdot \mathbf{M}_1^T$, where matrix $\mathbf{M}_1$ is taken from Example 1. Therefore, $\mathbf{S}_1$ may be treated as extractor of image features on the first level of wavelet decomposition, similar as above in the Haar decomposition case. Because on the second and next levels only the part of a matrix is transformed (opposite to Haar decomposition) these extractors on these levels are different. For example, for N=8 the products of the non-standard wavelet decomposition levels are shown in Fig. 4.

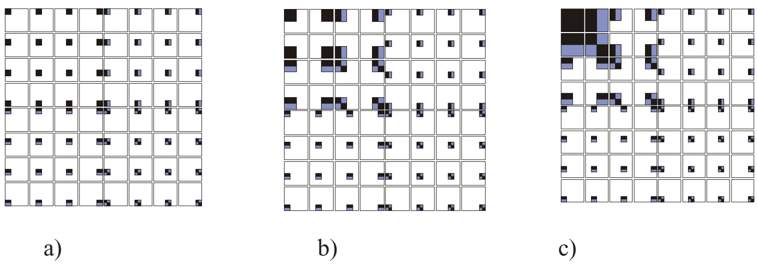


Fig. 4. The 2D wavelet functions product treated as extractors. Decomposition levels: a) first, b) second, c) third

All considerations, which have been presented until now for the classical of Haar functions, have applications in that case too, with the exception of extractors' distribution (Fig. 4). The Haar decomposition can be simply implemented as matrix multiplication. The wavelet algorithm is a little more efficient.

4 Experimental Results

To test our method the well known benchmarks have been used. Each of these images was of size $a \times a \times 8$ bits, where $a \in \{32, 64, 128, 256\}$. By analysing the Figs. 3-4 we can divide areas of a figure into 4 equal pieces. Each piece has dimension $(N/2)\times(N/2)$ and is called A, H, V and D. Location of these areas presents Fig. 5.

Each piece (A, H, V or D) for $N = 8$ includes sixteen appropriate sub-squares from Fig. 3-4. According to presented arguments, mentioned areas possess different features: A (Approximation), H (Horizontal), V (Vertical), D (Diagonal). Fig. 5 presents “Baboon” – one of the grey-level test images and its wavelet and Haar spectra. The spectra images are different what directly follows from Figs. 3-4. Taking into account mentioned features of areas some differences between spectres can be shown.

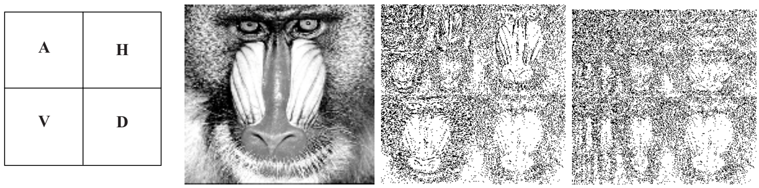


Fig. 5. Principle of spectra partitioning; original image and its wavelet and Haar spectra respectively

In Fig. 6 are shown differences between origin image and compressed one for wavelet and Haar method of analysis, respectively after removing some coefficients.

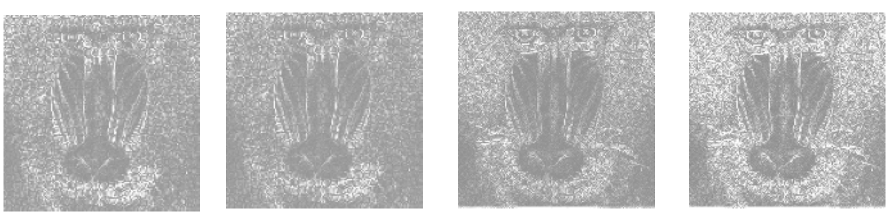
The exact information about distribution of spectral coefficients allows us to match easily up the compression ratio with the type of image. Obtained results for Haar matrix-based method and wavelet method were compared by means of PSNR coefficients. These results of investigations are collected in Tab. 1. From this table one can see that Haar reconstructed images have slightly better quality. From Tab. 1 (last column) follows, that after removing all horizontal and vertical details on the first level of decomposition we get exactly the same PSNR of both methods reconstructed images because of the proportionality of the diagonal detail coefficients.

The different cases of removing the spectral coefficients can be applied as well. These entire processes are based on the fact, that appropriate selection and modification of the spectral coefficients may preserve the contents of the image.

Between Haar matrix-based method and the wavelet one can be observed quantitative and graphic relationship. Let d_H and d_W stand for diagonal coefficients from Haar and wavelet spectrum matrix respectively, both of degree 2^n. Then $d_H = 2^n d_W$.

Table 1. The PSNR of reconstructed images after appropriate details elimination

Details / Method	Horizontal (H)	Vertical (V)	Diagonal (D)	Horizotal +Vertical (H+V)
Wavelet decomposition	29,7254	27,3697	31,4822	25,3813
Haar decomposition	29,7269	27,3702	31,4827	25,3813



a) b)

Fig. 6. Horizontal – (a) and horizontal with vertical – (b) details elimination and loosed information after applied wavelet and Haar matrix-based method, respectively

5 Concluding Remarks

In the paper it has been shown the new graphic way of presentation of decomposition levels for both the Haar matrix-based method and wavelets. As it has been shown both methods can be modelled on the basis of the wavelets theorem.

The 2D Haar matrix method of calculations like the 2D Fast Fourier Transform has complexity $O(4N^2 \log_2 N)$ [4], classical 2×1 D fast wavelet method of calculations has complexity $O(16/3N^2)$ only [3,11,12]. This complexity can be decreased to $O(14/3N^2)$ by suitable organization of calculations [10]. Described complexity factors are determined as number of additions and multiplications in computation process. The graphic distribution of the Haar-wavelet spectral coefficients also has been presented. Additionally, knowledge about spectra distribution allows us to point appropriate selection or modification (reduction) of the Haar-wavelet coefficients.

References

1. Addison P. S., Watson J. N., Feng T.: Low-Oscillation Complex Wavelets, Journal of Sound and Vibration, Vol. 254 (4), 733-762. (2002)
2. Ahmed N., Rao K. R.: Orthogonal Transforms for Digital Signals Processing. Springer-Verlag. Berlin, Heidelberg. (1975)
3. Castleman K.R.: Digital Image Processing. Prentice-Hall. New Jersey. (1996)
4. Daubechies I.: Recent results in wavelet applications, Journal of Electronic Imaging, Vol. 7 No. 4, pp. 719-724. (1998)
5. Drori I., Lischinski D.: Fast Multiresolution Image Operations in the Wavelet Domain, IEEE Transactions on Visualization and Computer Graphics, Vol. 9, No. 3, July-September, pp. 395-411. (2003)
6. Harmuth H.F.: Sequence Theory. Foundations and applications. Academic Press. New York. (1977)
7. Haar A.: Zur Theorie der orthogonalen Functionsysteme. Math Annal. No 69, pp. 331-371. (1910)
8. Jorgensen P.: Matrix Factorizations, Algorithms, Wavelets, Notices of the American Mathematical Society, Vol. 50, No 8, September, pp. 880-894. (2003)
9. Lisowska A.: Nonlinear Weighted Median Filters in Dyadic Decomposition of Images, Annales UMCS Informatica AI, Vol. 1, pp.157-164. (2003)
10. Lisowska A., Porwik P.: New Extended Wavelet Method of 2D Signal Decomposition Based on Haar Transform. Mathematics and Computers in Simulation. Elsevier Journal. (to appear)
11. Mallat S. A.: Theory for Multiresolution Signal Decomposition: The Wavelet Representation, IEEE Trans. Pattern Analysis and Machine Intelligence, Vol. 11,
12. Walker J. S.: Fourier Analysis and Wavelet Analysis. Notices of the American Mathematical Society. Vol. 44, No 6, pp. 658-670. (1997)