Network of Networks^{*,**}

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Abstract. We consider the categorical concepts of a ‘network of networks’: (a) each node is a host network (1-network or 1-graph) and super-links are analogous to a graph-functor, *i.e.* this is (1, 1)-network; (b) 2-network where there are 2-links among 1-links. The general notion of network-morphism is proposed.

1 Network

The dominant technological structure is a network (a graph), e.g. World Wide Web graph, each web page is a node and each (hyper)link is a directed edge; internet, social networks, networks in the molecular biology, a biological cell as a network of genes, neural networks, metabolic network, scientific citation network, energetic networks, phone calls, linguistics networks, networks in natural languages, ecological eco-networks, computer circuits, www.nd.edu/networks or <http://www.internetmathematics.org>.

A graph underline a category [4,6,7, Burroni 1981, Lawvere 1989]. Graphs provide useful concept in a computational science (integrating science with a computation), braided logic [5, Chávez Rodríguez et al. 2001], etc. Lawvere consider the category of directed graphs and their morphisms [Lawvere 1989]. The categorical aspects of graphs is not familiar outside of category theory. The value of categories for computational science is that the category theory, invented by Eilenberg and MacLane in 1945, has developed *language* for the study of structures, language for the formulation of problems, language for development of methods of calculation and deduction, language for discovering and exploiting analogies between various inter-disciplinary fields of science. Attraction of category theory is that the same algebraic tool=language is applicable in a variety of multi-disciplinary science. Study of the structure (process) involves studying the pre-categories ($\equiv$ graphs) and pre-functors $\equiv$ graph-morphisms.

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A graph consists of collections of variously called objects/nodes/vertices/states/0-cells V/G_0 , and arrows/edges/links/transitions/1-cells E/G_1 , with two operations: domain/source $s : G_1 \rightarrow G_0$ and codomain/target $t : G_1 \rightarrow G_0$ [4,9, Burroni 1981, Mac Lane 1998]. Elements of G_0 are said also to be 0-morphisms/0-transitions, elements of G_1 are said to be 1-morphisms/1-transitions. For a cell $f \in G$, the value sf is said to be a domain or a source of f , and tf is a co-domain or target of f .

A (time-dependent) out-degree of a cell $x \in G$ is a cardinality of a fiber $s^{-1}x$. An in-degree of x is a cardinality of $t^{-1}x$. The power-law degree says that a distribution probability of the connecting links to the new node y (to a new 0-cell y) for ty is $\simeq (\text{degree of } y)^{-(1+4)}$, known as a 'scale-free'. A power-law in-degree distribution means that the percentage of the number of cells with a given in-degree is proportional to the power (in-degree)^{-constant}. A power-law degree was discovered by Lotka [8, 1926] in a scientific productivity.

The models of the time evolution of the networks, models of the graph-process and models of the interactions among nodes, attract most attention. The best known is the Barabási-Albert model, based on two axioms one new vertex $x \in V$ at a time with a fixed node-independent in-degree $t^{-1}x \in \mathbb{N}$, [1,2, Barabási & Albert 1999; Barabási 2002].

1.1 Aim of This Note

It was observed by Bharat et al. [3, 2001], that web-graph is a 2-level structure. We consider the categorical concepts of a 'network of networks' or an hierarchical network. For a natural number $m \in \mathbb{N}$ we are introducing a concept of a m -network, and moreover for $m, n \in \mathbb{N}$ a fine concept of (m, n) -network. In particular, our aim is to elucidate the differences and similarities among the following hierarchical structures.

- (a) Each node is a usual (connected) host network (1-network or 1-graph) and super-links are analogous to a graph-functor. Such structure we call $(1, 1)$ -network or $(1, 1)$ -graph, or 'graph of host-graphs', or a network of clusters
- (b) 2-network (2-graph) where there are links among specified links of the same type.

The study of categorical aspects of networks leads to new insights. A graph is the same as 1-graph. We developed conceptual notion of a graph of graphs, $(1, 1)$ -graph, multi-graph m -graph, multi-graph of multi-graphs (m, n) -graph, and iterations $(n_1, \dots, n_k)$ -graph for $\forall n_i \in \mathbb{N}$. Graph of graphs can be used for refinement of the hierarchical structures and/or 'network of networks'. In particular the conceptual hierarchical structure 'graph of graphs' can be useful for description of the networks where the links are attribute-dependent and therefore are natural 'higher' links between attributes leading to multi-level systems [10,11, Obtulowicz 2001], for example the attribute-dependent links among employers and workers, professors and students.

2 Graph-Morphism

A multi-network G is a sequence of collections of i -cells/ i -morphisms $\{G_i, i \in \mathbb{N}\}$, a sequence of surjective sources $\{s_i \equiv s|G_i\}$ and targets $\{t_i \equiv t|G_i\}$. A 0-cell in G_0 is variously also called an object, 0-morphism, node; 1-cell - a morphism, 1-morphism, an arrow, a functor. A 2-cell is called a 2-morphism, morphism of morphism, natural transformation, etc.

Let $E \equiv (E_0 \rightrightarrows E_1 \rightrightarrows \cdots \rightrightarrows E_k)$ and $F \equiv (F_0 \rightrightarrows F_1 \rightrightarrows \cdots \rightrightarrows F_k)$ be two k -graphs. A k -graph morphism $f : E \rightarrow F$, $f \in \text{graph}(E, F)$, is a collection of functions $f_i : E_i \rightarrow F_i$, intertwining the source s and the target t maps (in this case a morphism is said to be totally covariant),

$$\begin{array}{ccccccc} s \circ f & = & f \circ s & \text{ and } & t \circ f & = & f \circ t, \\ E_0 & \rightrightarrows & E_1 & \rightrightarrows & E_2 & \cdots & \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ F_0 & \rightrightarrows & F_1 & \rightrightarrows & F_2 & \cdots & \end{array}$$

Set $E, F \in G_0$ to be 0-cells, and $\text{graph}(E, F) \subset G_1$, then $G \equiv (G_0 \rightrightarrows G_1)$ is a graph of k -graphs, that we denote by $(1, k)$ -graph. This means that every 0-cell of G is a k -graph and every 1-cell of $(1, k)$ -graph G is a k -graph morphisms, $f \in G_0(E, F) \equiv \text{graph}(E, F) \subset G_1$.

A 1-graph $G = (G_0 \rightrightarrows G_1)$ whose every 0-cell is a graph, and every 1-cell is a graph morphism, is said to be a graph of graphs. In particular, $(0, 1)$ -graph is a collection of 1-graphs, and a 1-graph whose every 0-cell is a 1-graph and whose every 1-cell is 1-graph morphism is said to be a $(1, 1)$ -graph. Therefore we interpret $(1, 0)$ -graph as a more particular concept that a general 1-graph.

2.1 2-Graph of Graphs

Let G be a 2-graph such that every 0-cell is a k -graph and every 1-cell is a k -graph morphism, *i.e.* G is a $(2, k)$ -graph. We need to define 2-cell as a k -graph natural transformation. The following definition generalize the notion of the natural transformation of (pre)functors from the case 2-graph of $k = 1$ -graphs invented by Eilenberg and MacLane [1945], to the case of arbitrary natural number $k \in \mathbb{N}$.

Let $f, g \in G_0(E, F) \subset G_1$. This means

$$G_0(E, F)(f, g) \subset G_1(f, g) \subset G_2.$$

A k -graph natural transformation $t \in \text{nat}(f, g) \equiv G_1(f, g) \subset G_2$, is a collection of $(i + 1)$ -cells-valued map intertwining f with g ,

$$\begin{array}{ccccccc} \cdots & \rightrightarrows & E_i & \rightrightarrows & E_{i+1} & \rightrightarrows & E_{i+2} & \cdots \\ & & \downarrow & \searrow t & \downarrow & \searrow t & \downarrow & \searrow t \\ \cdots & \rightrightarrows & F_i & \rightrightarrows & F_{i+1} & \rightrightarrows & F_{i+2} & \cdots \end{array}$$

$$E_i \ni x \xrightarrow{t} tx \in F_i(fx, gx) \subset F_{i+1},$$

$$\forall a \in E_i(x, y) \subset E_{i+1}, \quad (ga) \circ (tx) = (ty) \circ (fa) \in F_{i+1}. \quad (1)$$

The inter-twining condition (1) is referred as the naturality condition.

Let s_F denote F -source map. Then $s_F \circ t = f = s_G t$, therefore s_G is given be push-forward, $s_G = (s_F)_*$.

The k -graph natural transformation $t \in \text{nat}(f, g)$ for the case of a 'graph of 1-graphs' is analogous to natural transformation of functors. A natural transformation $t \in \text{nat}(f, g)$ can be illustrated also by means of a commutative diagram with pull-back and push-forward,

$$\begin{array}{ccc} E(x, y) & \xrightarrow{f} & F(fx, fy) \\ g \downarrow & & (ty)_* \downarrow \\ F(gx, gy) & \xrightarrow{(tx)^*} & F(fx, gy) \end{array}$$

Therefore $G_0(E, F) \subset G_1$ can be seen as an 1-graph (= pre-functor graph): vertices are the morphisms of graphs, $f, g \in \text{graph}(E, F)$, and edges are natural transformations $\text{nat}(f, g) \subset G_2$.

A $(0, 2)$ -graph is a collection G_0 whose every element is a 2-graph. A $(2, 0)$ -graph is not the same as a 2-graph, *i.e.* a 2-graph $G_0 \Leftarrow G_1 \Leftarrow G_2$ whose every 0-cell in G_0 is a 0-graph $\simeq \text{set}$, 1-cells are mappings and 2-cells are operators.

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