A-Posteriori Error Analysis of a Mixed Method for Linear Parabolic Problem

M.I. Asensio, J.M. Cascón, and L. Ferragut

Departamento de Matemática Aplicada, Universidad de Salamanca,
37008 Salamanca, Spain
{mas, casbar, ferragut}@usal.es

Abstract. In this paper we present a-posteriori error estimator for the mixed formulation of linear parabolic problem, and we use them in designing an efficient adaptive algorithm. Our space-time discretization consist of lowest order Raviart-Thomas finite element over graded meshes, and discontinuous Galerkin method with varying time-steps.

1 Introduction

A-posteriori error estimates are an essential component in the design of reliable and efficient adaptive algorithms for the numerical solutions of PDEs. At the same time, mixed formulations are suitable for certain problems, as they allow to directly approach certain derivatives of the solution.

In this paper we introduce an a-posteriori error estimation for mixed formulation of the linear parabolic problem obtaining,

$$\|u - U\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{p} - \mathbf{P}\|_{L^2(0,T;H^{-1}(\text{div},\Omega))} \leq \mathcal{E}(u_0, f, T, \Omega; U, \mathbf{P}, h, k),$$

where u is the scalar variable, and $\mathbf{p}$ its gradient. Capital letters represent the numerical approximations obtained with the lowest order Raviart-Thomas finite element in space and a variable step discontinuous Galerkin method in time. The estimator $\mathcal{E}$ is computable in terms of: given data u_0, f, Ω, T , computed solutions U and $\mathbf{P}$, meshsize h and time step k .

As in [4], we obtain the error representation formulas of $u - U$ and $\mathbf{p} - \mathbf{P}$ from the residual equations. The evaluation of the residual in the corresponding norms and the stability properties of the associated dual problem allow the estimations to be concluded. The scalar error bound is an extension to the mixed formulation of the results developed in [3]. For the error estimation of $\mathbf{p} - \mathbf{P}$ we use the Helmholtz decomposition in $L^2(\Omega; \mathbb{R}^2)$, as in [1] and [2] for the stationary case.

2 Continuous Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polyhedral domain, let $T > 0$ be the final time, and set $Q_T := \Omega \times (0, T)$ and $\Gamma_T := \partial\Omega \times (0, T)$. Given an initial condition

u_0 and a source term $f(x, t)$, let $(u, \mathbf{p})$ be the solution of the problem,

$$u_t + \operatorname{div} \mathbf{p} = f \quad \text{in} \quad Q_T, \quad (1)$$

$$\mathbf{p} + \nabla u = 0 \quad \text{in} \quad Q_T, \quad (2)$$

$$\nabla u \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_T, \quad (3)$$

$$u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega, \quad (4)$$

where $\boldsymbol{\nu}$ is the unit normal vector to $\partial\Omega$.

In the following, $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega; \mathbb{R}^2)$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the corresponding spaces. We define the space $H_0(\operatorname{div}, \Omega)$ as,

$$H_0(\operatorname{div}, \Omega) := \{\mathbf{q} \in L^2(\Omega; \mathbb{R}^2) : \operatorname{div} \mathbf{q} \in L^2(\Omega), \langle \mathbf{q} \cdot \boldsymbol{\nu}, v \rangle = 0 \quad \forall v \in H^1(\Omega)\}$$

and denote by $H^{-1}(\operatorname{div}, \Omega)$ its dual space. For the corresponding variational formulation of this problem we use the bilinear operator B , defined on $L^2(\Omega) \times H_0(\operatorname{div}, \Omega)$ as,

$$B(w, \mathbf{q}; r, \mathbf{s}) = (w, r) + (\operatorname{div} \mathbf{q}, r) + (\operatorname{div} \mathbf{s}, w) - (\mathbf{q}, \mathbf{s}).$$

Adding the equations (1) and (2) we obtain the following variational problem,

Continuous problem. Find u and $\mathbf{p}$ such that

$$u \in H^1(0, T; L^2(\Omega)), \quad \mathbf{p} \in L^2(0, T; H_0(\operatorname{div}, \Omega)), \quad u(\cdot, 0) = u_0, \quad (5)$$

$$B(u_t, \mathbf{p}; v, \mathbf{q}) = (f, v) \quad \forall (v, \mathbf{q}) \in L^2(\Omega) \times H_0(\operatorname{div}, \Omega) \text{ a.e. } t \in (0, T). \quad (6)$$

3 Discretization

The fully discrete problem combines finite elements in space with the Galerkin discontinuous method in time.

Let k_n denotes the time step at the n -th step and set $t_n = \sum_{i=1}^n k_i$. Let N be the total number of time steps, that is, $t_N = T$. To each time step, $(t_{n-1}, t_n]$ we associate $(\mathcal{T}_n, \mathbb{M}_n, \mathbb{X}_n)$, where $\mathcal{T}_n = \{\mathcal{S}_n\}$ is a uniformly regular partition of Ω in triangles. We denote h_n a piecewise constant function with $h_n|_{\mathcal{S}} = \operatorname{diam}(\mathcal{S})$ for all $\mathcal{S} \in \mathcal{T}_n$, $\mathcal{B}_n$ is the set of all interior edges e of $\mathcal{T}_n$; h_e stands for the size of e . The mesh $\mathcal{T}_n$ is obtained from $\mathcal{T}_{n-1}$ by refining/coarsening. $\mathbb{M}_n \in L^2(\Omega)$ is the usual space of piecewise constant finite elements over $\mathcal{T}_n$. Finally, $\mathbb{X}_n \in H(\operatorname{div}, \Omega)$ is the lowest order Raviart-Thomas finite element. Then, we have the problem, **Discrete problem.** Given $U^{n-1} \in \mathbb{M}_{n-1}$, $\mathbf{P}^{n-1} \in \mathbb{X}_{n-1}$, then k_{n-1} and $\mathcal{T}_{n-1}$ are modified as described in section §5 to get k_n and $\mathcal{T}_n$ and thereafter $U^n \in \mathbb{M}_n$, $\mathbf{P}^n \in \mathbb{X}_n$ computed according to,

$$\frac{1}{k_n} (U^n - \pi_{\Omega} U^{n-1}, V) + (\operatorname{div} \mathbf{P}^n, V) = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} (f, V) dt, \quad \forall V \in \mathbb{M}_n, \quad (7)$$

$$(\mathbf{P}^n, \mathbf{Q}) - (\operatorname{div} \mathbf{Q}, U^n) = 0, \quad \forall \mathbf{Q} \in \mathbb{X}_n, \quad (8)$$

where π_{Ω}^n denotes the usual $L^2(\Omega)$ -projection operator over $\mathbb{M}_n$.

4 A-Posteriori Estimator

In the following Theorems we present the a-posteriori error estimates for the schemes developed bellow.

Theorem 1. *Let $(u, \mathbf{p})$ be the solution of (5-6) and $(U, \mathbf{P})$ the solution of (7-8). If Ω is convex, for $N \geq 1$,*

$$\|e_u^{t_N}\|_{L^2(\Omega)} \leq \|e_u^0\|_{L^2(\Omega)} + \sum_{i=1}^5 C_i \max_{1 \leq n \leq N} \mathcal{E}_i,$$

with,

$$\begin{aligned} \mathcal{E}_1 &:= \|U^n - \pi_\Omega^n U^{n-1}\|_{L^2(\Omega)} & \mathcal{E}_2 &:= \|U^{n-1} - \pi_\Omega^n U^{n-1}\|_{L^2(\Omega)}, \\ \mathcal{E}_3 &:= \|h_n \mathbf{P}\|_{L^2(\Omega)}, & \mathcal{E}_4 &:= \max_{t \in [t_{n-1}, t_n)} \|h_n(f - \operatorname{div} \mathbf{P})\|_{L^2(\Omega)}, \\ \mathcal{E}_5 &:= \int_{t_{n-1}}^{t_n} \|f - \operatorname{div} \mathbf{P}\|_{L^2(\Omega)} dt. \end{aligned}$$

Theorem 2. *Let $(u, \mathbf{p})$ be the solution of (5-6) and $(U, \mathbf{P})$ the solution of (7-8). Let J_τ be the tangential jump of $\mathbf{P} \in \mathbb{X}$ along e . If Ω is convex, for $N \geq 1$,*

$$\|\mathbf{E}_\mathbf{P}\|_{L^2(0,T;H^{-1}(\operatorname{div},\Omega))} \leq \|e_u^0\|_{L^2(\Omega)} + \sum_{i=6}^{11} C_i \mathcal{E}_i,$$

where,

$$\begin{aligned} \mathcal{E}_6 &:= \left(\sum_{n=1}^N k_n \sum_{e \in \mathcal{B}_n} h_e \|J_\tau\|_{L^2(e)}^2 \right)^{\frac{1}{2}}, \mathcal{E}_7 := \left(\sum_{n=1}^N k_n \|U^n - \pi_\Omega U^{n-1}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \\ \mathcal{E}_8 &:= \left(\sum_{n=1}^N k_n \|U^{n-1} - \pi_\Omega U^{n-1}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \mathcal{E}_9 := \left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|h_n(f - \operatorname{div} \mathbf{P})\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \\ \mathcal{E}_{10} &:= \left[\sum_{n=1}^N \left(\int_{t_{n-1}}^{t_n} k_n^{1/2} \|f - \operatorname{div} \mathbf{P}\|_{L^2(\Omega)} \right)^2 \right]^{\frac{1}{2}}, \mathcal{E}_{11} := \left(\sum_{n=1}^N k_n \|h_n \mathbf{P}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

5 Adaptive Algorithm and Numerical Examples

We propose two adaptive strategies based on the former estimates. The goal is to determine the time steps and the meshes in order to obtain a uniform error distribution. As usually in parabolic problem we distinguish four kinds of terms in the a-posteriori estimate: initial error, space discretization error, coarsening error and time discretization error,

$$\mathcal{E}_n \leq \mathcal{E}_0 + \mathcal{E}_{h,n} + \mathcal{E}_{c,n} + \mathcal{E}_{k,n}.$$

We also select the parameters Γ_0 , Γ_h , Γ_k , verifying, $\Gamma_0 + \Gamma_h + \Gamma_k \leq 1$, in a way that given a total error tolerance tol , the adaptive algorithm tries to select time steps and meshes verifying for all n ,

$$\mathcal{E}_n \sim tol, \quad \mathcal{E}_0 \sim \Gamma_0 tol, \quad \mathcal{E}_{h,n} + \mathcal{E}_{c,n} \sim \Gamma_h tol, \quad \mathcal{E}_{k,n} \sim \Gamma_k tol.$$

The algorithm begins with an initial time step k_0 . Given $\theta_1 \in (0, 1)$ and $\theta_2 \in (0, \theta_1)$, if $\mathcal{E}_{k,n} > \Gamma_k tol$, then the time step diminishes in a factor $\delta_1 \in (0, 1)$; on the contrary, if $\mathcal{E}_k < \Gamma_k tol$, the time step increases in a factor $\delta_2 > 1$. For the space refinement we use an error equidistribution strategy (see [3]).

We use an implicit adaptive strategy, that is for each time step we start from the previous step mesh and repeat the process solving-estimating-adapting until the estimated error being below the tolerance.

In the poster session we will present several examples showing that the proposed method is efficient and reliable. We compare the estimators with the exact errors, and we obtain an excelent correlation coefficient number (≈ 1). We also study the adaptation in time and in space.

The numerical experiments have been designed with the finite element toolbox ALBERT [5], extended with new designed tools for the lowest order Raviart-Thomas finite element and the a-posteriori error estimator developed in this paper.

Conclusions

We obtain a-posteriori error estimators for the mixed formulation of linear parabolic problem and we use them in designing an efficient adaptive algorithm.

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