Chapter 7
Aspects of "Anschauung" in the Work of Felix Klein

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Abstract Aimed at modernizing the teaching of mathematics at German secondary schools around 1900, The "Kleinian Reform Movement" was characterized by Felix Klein's two key demands: "strengthening spatial intuition" and "training the habit of functional reasoning". This paper presents a number of examples demonstrating the importance of the concept of intuition (*Anschauung*) for Klein and explains the role he assigned to intuition in mathematics instruction at school and university.

Keywords Anschauung · Intuition · Space intuition · Kleinian reform movement
Meran curriculum proposal

7.1 Core Demands for Modernizing the Teaching of Mathematics at Secondary Schools

For Felix Klein, the insistence that intuition (*Anschauung*)[1]—or more precisely "space intuition" (*Raumanschauung*)—should be given a greater role in mathematics and mathematics teaching was one of the core demands in the process of modernizing mathematical teaching not only at universities and engineering colleges but also at secondary schools. Felix Klein played a key role in drawing up the Meran Curriculum Proposal of the 1905 Breslau Teaching Commission of the Gesellschaft Deutscher Naturforscher und Ärzte (GDNÄ), which essentially formulated two key demands regarding mathematics teaching at secondary schools: "Strengthening the capacity to think in three dimensions and training the habit of functional reasoning."

[1] In translating Felix Klein's ideas into English, I largely use the term "intuition" to convey the concept of "Anschauung". The term "intuition" originates in the Latin for "consideration/looking at" and is employed here not in the sense of a sudden insight without any conscious reasoning but follows its use in the philosophical tradition as a discovering of truth through contemplation, i.e. rational intuition.

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> A further aspect will be to dispense with all one-sided and practically insignificant specialized knowledge while fully recognizing the formal educational value of mathematics and to facilitate an optimal development of the ability to view the world of phenomena surrounding us from a mathematical angle. This entails two special objectives: enhancing the capacity to think in three dimensions and training the habit of functional reasoning. – This does not affect the objective of training logics, which has always been assigned to mathematics teaching. In fact, it can even be said that this objective is enhanced by giving more attention to the above-mentioned orientation of mathematics teaching. (Breslau Teaching Commission 1905, p. 154, translated by Martin Mattheis)

The Curriculum Proposal of the Teaching Commission initiated at the GDNÄ Assembly in Breslau in 1904 followed a lengthy and fundamental dispute over the principal goals of mathematics instruction at secondary schools and the value of mathematics for achieving a general education. The general educational character of mathematics lay for many, beyond the circle of neo-humanist minded classical philologists, above all in a formal development of the intellect.

Particularly from the second half of the 19th century on, the development and consolidation of educational institutions aimed at preparing young people for life and work in the modern world, the "realistische Bildungsanstalten", was accompanied by a heated debate over whether the leaving qualifications conferred by the three different types of secondary school (*Gymnasium*, *Realgymnasium* and *Oberrealschule*) prepared their graduates equally for admission to higher education and to professional fields in the public services. In this context, especially in the period before 1900, debate also centered on the issue of what general knowledge mathematics teaching should look like. This is why efforts to modernize the mathematics syllabus in secondary schools, which many mathematicians thought necessary, were always potentially or actually confronted by the accusation of reducing the subject's general knowledge character to a narrow specialism. Since this would have led to the status of the respective institutions being lowered from providers of general education to technical or vocational institutions, the attempt was often made to argue that, in addition to the goals proper of the desired change, the modernization would retain or even enhance the formal education value of mathematics teaching. The different objectives of the ever-increasing number of reform advocates ultimately culminated in the two bold and simple demands that calculus and analytical geometry should be introduced in secondary school curricula.

After the *3rd Prussian School Conference on Secondary School Teaching*, in June 1900, had come out in favor of giving equal status to the qualifications conferred by all three types of secondary school, this controversy was resolved, as far as Prussia was concerned, on the 26th November 1900 by decree of the Prussian King, who took a keen interest in secondary school education. In his role as state ruler, Wilhelm II declared the equivalence of degrees conferred by *Gymnasium*, *Realgymnasium* and *Oberrealschule*. In the years after 1900, the other constituent states in the confederation followed the Prussian example with respect to the equivalence of leaving qualifications conferred by the three types of secondary school (Mattheis 2000a, pp. 18–20).

Felix Klein, who had initially dealt primarily with mathematics in higher education and had—on this question and through his efforts to turn the University of Göttingen into a center of mathematics—been in close contact with the Prussian Ministry of Cultural Affairs, was requested at relatively short notice prior to the 3rd Prussian School Conference to produce two expert assessments of issues raised by mathematics teaching in secondary schools (printed in Schubring 2000). Having compiled his assessments, Felix Klein was the only university mathematician, among the total of 34 participants, to attend the Schools Conference held in Berlin from the 6th to the 8th June 1900.

In his reports for the Schools Conference, Klein explained what he thought a reform of mathematics teaching in secondary schools should look like. His demands centered on graduates from all three types of secondary school being qualified to study at both a university and a college of engineering (*Technische Hochschule*). Having failed in his initial attempt, by way of negotiations with the Ministry of Cultural Affairs, to have the modifications he desired implemented directly into the new curriculum to take effect from 1901, he followed other channels: rallying support for his ideas among teachers working in schools, the experimental schools established by the Ministry, and associations such as the Gesellschaft Deutscher Naturforscher und Ärzte. These ideas centered above all on the concept of functional reasoning (*funktionales Denken*),[2] i.e. that the concept of function should run through school mathematics right from the start, and on the need to strengthen spatial intuition (*räumliche Anschauung*), i.e. the capacity to think in three dimensions. Both ideas were then prominently included as a core demand in the Meran Curriculum Proposal in 1905 (Schubring 2007, pp. 5–8).

7.2 Intuition in Mathematics Teaching in Higher Education

In his inaugural lecture on assuming his first professorship in Erlangen, the 1872 Antrittsrede, Felix Klein was already emphasizing the importance of applications and intuition for mathematics and the teaching of mathematics at universities. To him, applying mathematics went significantly beyond “the predictive calculations of the astronomer, […] the precision of geodetic measurements, […] [or] the accomplishments of the engineering art”. In the second half of the 19th century, the formal educational value (*formaler Bildungswert*) of mathematics had come to be seen as an essential to an advanced education in Germany, and Klein believed this value lay in the “application of mathematical conceptions” above all in the fields of physics and the natural sciences, but also in medicine (Rowe 1985, p. 137).

Klein characterized a mathematician’s work as such as “drawing further conclusions from precisely formulated foundations”. To the mathematician, it was irrelevant whether the foundations were derived from hypotheses or from observed facts, i.e. from intuition (Rowe 1985, p. 137). However, in contrast to the actual work math-

[2] On *functional reasoning*, see the corresponding chapter in this volume.

ematician, this issue was, he said, relevant in applications such as mathematical physics where "applying abstract mathematical thinking to a sensate (better said: intuitive) domain" could be done in the same manner as in geometry (Rowe 1985, p. 138). For both fields, Klein insisted that, having actually drawn mathematical conclusions, the results gained should be referred "back to the vivid realm of sensate intuition" unless intuition and mathematical investigation happened to go hand in hand (Rowe 1985, p. 138).

Viewing mathematics from the opposite perspective, Felix Klein highlighted the considerable role played by the "intuition-oriented disciplines" in the progress made by mathematics in recent centuries: The questions raised by astronomy, mathematical physics and geometry had led to considerable advances in mathematics through the 18th and 19th century (Rowe 1985, p. 138).

However, when assessing Klein's very broad definition of mathematics in 1872 and his remarks on its applications, one should always bear in mind that Klein was seeking with his Erlangen Antrittsrede, above all to raise funding for the changes he envisaged in mathematics instruction at Erlangen when assuming his professorship. This overriding goal is reflected in his chain of reasoning: "If we educate better teachers, then mathematics instruction will improve by itself". This underpinned Klein's demand for improved teacher training to include mathematics seminars and, importantly, "exercises in drawing and in building models" (Rowe 1985, p. 139).

The argument that mathematics had a formal educational value was also intended to support this project. To humanist-oriented academics, i.e. to many among the audience at his Erlanger Antrittsrede, the formal educational value of mathematics was crucial to the characterization of mathematics instruction as a general educational task not only at the *Gymnasium* but also at all other secondary schools and in higher education (Rowe 1985, p. 124pp; Mattheis 2000b, p. 42).

On assuming a professorship in geometry in Leipzig in 1880, Felix Klein once again delivered a programmatic *Antrittsrede*, although this inaugural lecture was not published until 1895. In the Leipzig Antrittsrede he presented a set of mathematical models, compiled in collaboration with Alexander von Brill at the Technische Hochschule in Munich, in order to reaffirm the importance of intuition in geometry. In particular, he criticized the way sensate objects such as fourth order curves or third order surfaces, although developed out of mathematical propositions, were not being brought into any relationship with the intuitive geometric objects on which they were originally based (Klein 1895, p. 538).

He contrasted this observation with the approach he had chosen, together with Brill, of using drawings and models—both for teaching purposes and for his own research. Taking on the possible counter-argument that more intuition would reduce mathematical abstraction, making mathematics more accessible and lowering standards, Klein stressed that the desired "visualization" should only be viewed as a "complementary intervention" and that such an argument failed to see that modeling can bring forth new ideas for abstract research (Klein 1895, pp. 539–540). Thus, Felix Klein did not regard the application of mathematical models in mathematics in higher education merely as a means of visualizing and clarifying familiar contents;

rather, models were, to him, also objects for stimulated ideas in the pursuit of new research findings (cf. Rowe 2013).

Here again, when assessing Klein's Leipzig Antrittsrede, one should not forget that, on assuming the professorship in Leipzig, he was not only presenting his notions of mathematics teaching in higher education but very directly making a case for the additional funding required for the changes he envisaged—such as compiling a collection of models.

On November 2, 1895, in the same year as the first publication of his Leipzig Antrittsrede from 1880, Felix Klein delivered a lecture at the public session of the Königliche Gesellschaft der Wissenschaften zu Göttingen under the title "On the Arithmetization of Mathematics" (*Über Arithmetisierung der Mathematik*). Here, Klein examines the role of intuition (*Anschauung*) in mathematics. He begins by pointing out that in the work of Gauß we still find the incautious use of spatial intuition (*Raumanschauung*) as proof of the universal validity of propositions that were not at all universally valid. Klein argued that this had led to demands for exclusively arithmetical reasoning in mathematics. But this was unfortunate, so Klein, for his part, now sought to demonstrate "that mathematics is certainly not exhausted in logical deduction but that, alongside the latter, intuition completely retains its specific importance" (Klein 1896, p. 144, translated by Martin Mattheis).

Especially in the case of geometry, Klein called for results gained through arithmetical approaches to be reconnected with spatial intuition. He argued that imprecise spatial intuition should first be idealized in the axioms in order to proceed with a mathematical approach. This, Klein emphasized, gave rise to new concepts and insights. Such an approach should, he said, also be pursued in mechanics and mathematical physics (Klein 1896, p. 146). Here, the crux of his argument is that he wants to see logical deduction and intuition given equal status alongside each other, demanding that the role of intuition as both a source of ideas for reaching logical conclusions and as a form of application through deduction, be understood as acquired mathematical knowledge (Klein 1896, p. 149).

However, in addition to this view of intuition as something closely bound up with logical deduction, Klein also stressed the importance of what he called naïve intuition (*naive Anschauung*):

> Incidentally, naive intuition, which is in large part an inherited talent, emerges unconsciously from the in-depth study of this or that field of science. The word 'Anschauung' has not perhaps been suitably chosen. I would like to include here the motoric sensation with which an engineer assesses the distribution of forces in something he is designing, and even that vague feeling possessed by the experienced number cruncher about the convergence of infinite processes with which he is confronted. I am saying that, in its fields of application, mathematical intuition understood in this way rushes ahead of logical thinking and in each moment has a wider scope than the latter. (Klein 1896, p. 147, translated by Martin Mattheis)

Thus, Felix Klein subsumes under the general term of "Anschauung" a certain degree of intuition gained through experience.

7.3 Intuition in Felix Klein's Lectures

Having looked at Felix Klein's programmatic statements in his Antrittsreden and papers, we will now consider some concrete examples taken from lecture courses during Klein's period at the University of Göttingen. Already in the 1898/90 winter semester lectures on non-Euclidian geometry, published in handwriting in 1892, Klein described his view of the interplay between axioms and spatial intuition. He explicitly criticized the notion that intuition merely played a role in setting up the axioms, insisting instead that, especially in geometric considerations, mathematicians should always draw on intuition. Above all he thought the role of axioms was to counter the inexactness of intuition and create exactness (Klein 1892, p. 354).

> Rather, in true geometric thinking, spatial intuition accompanies us at every step we take. […] I assign the axioms the role that they represent postulations with the aid of which we transcend the inaccuracy of intuition or the limitations of intuition in order to achieve unlimited accuracy. (Klein 1892, p. 354, translated by Martin Mattheis)

A few pages on, taking the case of a common tangent of two curves, Klein then discussed the issue of the extent to which proofs can be obtained from intuition in pure mathematics. In the case of the two curves in Fig. 7.1, one may assume, based on intuition, that they share a common tangent. From Klein's viewpoint, however, the drawing merely represents a "sensualization" (*Versinnlichung*) of the true curves. He explicitly stated that without knowing which "mathematically precise law" the two curves followed, one could not make any statement as to whether a common tangent actually exists (Klein 1892, p. 359).

> With our notion of the essence of intuition, an intuitive treatment of figurative representations will tend to yield a certain general guide on which mathematical laws apply and how their general proof may be structured. However, true proof will only be obtained if the given figures are replaced with figures generated by laws based on the axioms and these are then taken to carry through the general train of thought in an explicit case. Dealing with sensate objects gives the mathematician an impetus and an idea of the problems to be tackled, but it does not pre-empt the mathematical process itself. (Klein 1892, pp. 359–360, translated by Martin Mattheis)

Fig. 7.1 In the case of a common tangent (Klein 1892, p. 359)

So Felix Klein regarded intuition as a useful heuristic aid for mathematicians seeking to reach mathematical conclusions but, in his view, it was by no means a substitute for correct proof.

7.3.1 Sensate, Idealizing and Abstract Intuition

The concept of intuition (*Anschauung*) is found in Felix Klein’s work in different contexts of meaning. First, there is “sensate intuition” (*sinnliche Anschauung*), for which, in his 1895 presentation, he also used the term “naive intuition” (*naive Anschauung*) (Klein 1896, p. 147). Sensate intuition comprises everything that surrounds us in real space and that we can touch and measure. In his lecture course on “Elementary Mathematics from a Higher Standpoint”, he additionally presented a further interpretation of spatial intuition in the form of “idealizing spatial intuition” (*idealisierende Raumanschauung*), which addresses the abstract notion of geometrical objects, i.e. the mathematical idea freed from the error-prone inexactness of the real objects (Klein 1908, p. 88).

> This is the proper place to say a word about the nature of space intuition. It is variously ascribed to two different sources of knowledge. One the sensibly immediate, the empirical intuition of space, which we can control by means of measurement. The other is quite different, and consists in a subjective idealizing intuition, one might say, perhaps, our inherent idea of space, which goes beyond the inexactness of sense observation. (Klein 1908, p. 88 or Klein 2016a, p. 37)

Such a distinction between the sensately immediate intuition and the idealizing inner intuition goes back to the respective concepts developed by Kant (Allmendinger 2014, pp. 52–53).

In relation to the development of infinitesimal calculus, Felix Klein introduced a further term to the circle of concepts that differentiate “intuition”. In the context, he places alongside “sensate intuition” the notion of “abstract intuition” to refer to what was in fact anything but an intuitive process of abstraction.

> It is precisely in the discovery and in the development of the infinitesimal calculus that this inductive process, built up without compelling logical steps, played such a great role; and the *most effective heuristic aid was very often sense intuition*. And I mean here the *immediate* sense intuition, with all its inexactness, for which a curve is a stroke of definite width, *not* the *abstract* intuition, which postulates a completed passage to the limit, yielding a one-dimensional line. (Klein 1908, pp. 455–456 or Klein 2016a, p. 226)

Klein exemplified this line of thought, using the integral being defined as the limit of a sum of rectangles. From the perspective of sensate intuition, he thought it was reasonable to define the surface area “as the sum of a large number of quite narrow rectangles”, since the width of the rectangles is obviously limited by the degree of drawing accuracy. He gave further examples of the significance of the respective use of sensate intuition in the emergence of infinitesimal calculus, including Kepler’s measuring of barrels and spheres, the “method of exhaustion” applied by

Archimedes, Cavalieri's principle or the differential quotient of a function (Klein 1908, pp. 456–460 or Klein 2016a, pp. 226–227). Ultimately however, when speaking of *abstract* intuition, he meant the same as what he had already assigned the concept of *idealizing* intuition to.

After considering the various examples of intuition in the development of infinitesimal calculus, Klein then stressed that there were undoubtedly mathematical personalities who either found such a way of looking at things useful and or did not, and that the respective approach continued to play an important role in the period after 1900 in the development of new mathematical ideas in mathematical physics, mechanics and differential geometry (Klein 1908, p. 460 or Klein 2016a, p. 229).

> The *force of conviction* inherent in such naïve guiding reflections is, of course, different for different individuals. Some – and I include myself here – find them very satisfying. Others, again, who are gifted only on the purely logical side, find them thoroughly meaningless and are unable to see how anyone can consider them as a basis for mathematical thought. (Klein 1908, p. 460 or Klein 2016a, p. 229)

Referring to David Hilbert's paper "On the Foundations of Logics and Arithmetic" delivered at the International Congress of Mathematicians in Heidelberg in 1904, Klein argued that even at the highest level of abstraction when one attempts to break loose from any form of intuition, e.g. in the theory of numbers considered in purely formal terms, a certain minimum amount of intuition still has to remain, even if it is only to recognize the symbols with which one is operating merely in accordance with axiomatic rules (Klein 1908, pp. 32–35 or Klein 2016a, p. 16).

7.3.2 Intuition and the Function Concept

Felix Klein's lectures "On the teaching of mathematics at secondary schools", delivered through the 1904/05 winter semester, were the first course in which Felix Klein dealt not only with mathematics but also, explicitly, with questions of post-primary education. He discussed not only higher secondary schooling for boys but also the role of mathematics in compulsory public schools (Volksschulen), girl's schools (*Mädchenschulen*), intermediate-level vocational schools (*mittlere Fachschulen*), universities and engineering colleges (*technische Hochschulen*). Indeed, he also outlined the historical development of mathematics teaching and examined the reforms that were proposed for higher-level math teaching. The course was divided into two parts: eight weeks of lectures on school education and, for the rest of the semester, an actual mathematics part dealing with "elementary mathematics from a higher standpoint" (Schimmack 1911, p. 40f.).

The first part, covering mathematics teaching, was later edited by Rudolf Schimmack for publication, appearing in 1907 under the title "Lectures on mathematical teaching at secondary schools" (*Vorträge über den mathematischen Unterricht an den höheren Schulen*), published by Teubner-Verlag. Schimmack was a close collaborator of Felix Klein's and the first person to gain a post-doctoral award in the

Didactics of Mathematics, receiving the *Habilitation* in 1911. Following the structure of the lecture course itself, the published work (Klein 1907) was subtitled Part 1: On the Organization of Mathematics Teaching (*Teil 1 Von der Organisation des mathematischen Unterrichts*). However, the content of Klein's lectures on elementary mathematics from the 1904/05 winter semester did not come to publication.

In his introduction to the printed lectures, Felix Klein affirmed his full support for what he saw as the two primary demands of the Meran Curriculum Proposal (*Meraner Lehrplanvorschlag*): "strengthening the capacity to think in three dimensions (*räumliches Anschauungsver mögen*) and training the habit of functional reasoning", since these aspects of mathematics "played the most important role in modern life" (Klein 1907, p. 6, translated by Martin Mattheis).

In the second chapter, Klein explored *inter alia* the question of the function concept and the relationship between functional reasoning and intuition. His emphasis here lay on the need to ensure that the function concept always be introduced in lessons as a "function concept in geometric form", i.e. in today's terms as a "function graph" (Fig. 7.2) (Klein 1907, p. 21, translated by Martin Mattheis).

In contrast to current practice, where many school students seem to have the idea that the function concept can be reduced to the representation of a graph, Felix Klein stressed the representational form of the function graph and the importance of this form in mathematics and beyond: "After all, Gentlemen, graphic representations are found not only throughout the seminal modern literature of the exact subjects but, one may say, in all areas of present-day life!". (Klein 1907, p. 21, translated by Martin Mattheis) Klein was drawing attention here to a discrepancy in the 1901 Prussian curriculum for secondary schools, which did not even mention functions in course content requirements, yet demanded that they be grasped by students in the highest grade. The methodology guidelines stated that teachers should equip "the students with an in-depth understanding of the concept of function, with which they have

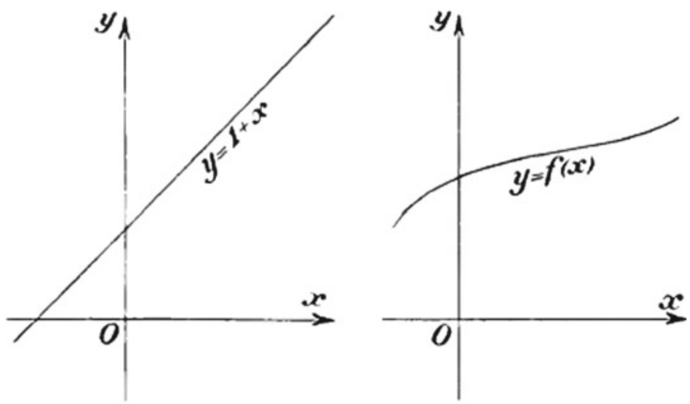

Fig. 7.2 The function concept in geometric form (Klein 1907, p. 21, reproduced with permission of Springer Nature Customer Service Center)

already been made familiar at earlier levels" (Klein 1907, pp. 21–22, translated by Martin Mattheis).

Klein again made the same connection between function concept and intuition (*Anschauung*) in the first part of his 1907/08 winter semester lecture course on "Elementary mathematics from a higher standpoint". This material first appeared in a handwritten edition, published by Teubner Verlag. It was published in print in 1924 by Springer Verlag. The first volume enjoyed a second printing, with the whole series becoming something of a bestseller that is still in demand today. Indeed, a complete retranslation appeared in English in 2016.

With regard to the graphic representation of functions, Klein again stressed in Elementary Mathematics that such representation was important in any practical application of mathematics. Moreover, he also called for school students to be acquainted as early as possible with the function concept (Klein 1908, p. 10 or Klein 2016a, p. 4).

> We, who used to be called the 'reformers', would put the function concept at the very center of teaching, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into teaching as early as possible with constant use of the graphical method, the representation of functional relations in the x-y system, which is used today as a matter of course in every practical application of mathematics. (Klein 1908, p. 10 or Klein 2016a, p. 4)

With regard to removing some of the traditional subject matter from the curriculum to make way for this approach, Klein believed that it was important that " *Intense formation of space intuition*, above all, will always be a prime task" (Klein 1908, p. 11 or Klein 2016a, p. 4).

7.3.3 *Proof Through Intuition*

We noted above that Klein rejected in principle the idea of accepting proofs from intuition, as argued in his lectures on non-Euclidian geometry that were delivered in the 1889–90 winter semester (Klein 1892, p. 359). However, in his "Elementarmathematik vom höheren Standpunkte" (Elementary Mathematics from a Higher Standpoint) he derives just such proofs, showing that an algebraic question can be resolved intuitively purely by graphic geometric presentation (Fig. 7.3) (Allmendinger 2014, pp. 47–50).

> 1. Given a>b and c>a, where a, b, c are positive. Then a − b is a positive number and is smaller than c, that is, c − (a − b) must exist as a positive number. Let us represent the

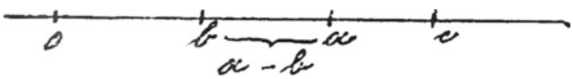

Fig. 7.3 Representation on the axis of abscissas (Klein 1908, p. 64, reproduced with permission of Springer Nature Customer Service Center)

> numbers on the axis of abscissas and note that the segment between the points b and a has the length a − b.
>
> A glance at the figure shows that, if we take away from c the segment a − b the result is the same as though we first took away the entire segment a and then restored the part b, i.e., (1) c − (a − b) = c − a + b. (Klein 1908, pp. 64–65 or Klein 2016a, p. 28)

Here, Klein proves the parenthesis rule c − (a − b) = c − a + b exclusively through an intuitive approach by considering the number line, without any need for algebraic transformations.

7.4 Intuition and the Genetic Method

On the question of the didactical method to be followed by teachers of mathematics at secondary schools, Felix Klein argued—alongside his demand for the “cultivation of spatial intuition”—for the course material to be designed in line with the respective age group of students. In the Volume II of “Elementary Mathematics from a Higher Standpoint”, he writes that instruction in geometry at secondary level should follow the basic principle of moving from the concrete to the abstract.

> Let us first ask, what requirements should be made today of a sound geometrical education. Everyone will surely admit for this that: 1. *The psychological aspects* must substantially prevail. Teaching cannot only depend on the subject matter, but it depends above all on the *subject* that you have to teach: one will present the same topic to a six-year-old boy differently than to a ten-year-old-boy – and this, in turn, differently to a mature man.
>
> Applied in particular to geometry, this means that in schools you will always have to connect teaching at first with vivid concrete intuition and then only gradually bring logic elements to the fore; in general, the genetic method alone will provide a legitimate means slowly to develop a full understanding of concepts. (Klein 1909, pp. 435–436 or Klein 2016b, p. 238)

Felix Klein set out in detail the importance of the genetic teaching method in the first volume of “Elementary Mathematics from a Higher Standpoint”.

> In order to give precise expression to my own view on this point, I should like to bring forward the *biogenetic fundamental law*, according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species. […] Now, I think that instruction in mathematics, as well as in everything else, should follow this law, at least in general. *Taking into account the native ability of youth, instruction should guide it slowly to higher things, and finally to abstract formulations; and in doing this it should follow the same road along which the human race has striven from its naïve original state to higher forms of knowledge*. (Klein 1908, pp. 588–589 or Klein 2016a, pp. 291–292)

The genetic method of teaching was regarded by Klein as essential not only for geometry, but also for every aspect of mathematics instruction. He illustrated this with inter alia the example of the notion of number. A child understands numbers as numbers of concrete objects, like nuts or apples, and not as axiomatically defined objects devoid of intuitive meaning with which one can operate according to formal rules. (Klein 1908, p. 9 or Klein 2016a, p. 4)

Corresponding demands to apply the principle of guiding school students learning mathematics from the intuitively concrete the abstract can be found in Felix Klein's writings back in 1895 in his lecture "On the Arithmetization of Mathematics" (Klein 1896, p. 148). They also appear in his expertise on the Prussian Schools Conference of 1900 (Klein 1900, p. 70) and in his paper delivered at the assembly of the Gesellschaft Deutscher Naturforscher und Ärzte in Breslau in 1904 (Klein 1904, p. 135), which constituted the teaching commission that was to present the Meran Curriculum Proposal in the following year. Without explicitly referring to it by name, Klein outlined as early as 1895 the principle of the genetic method in saying that "learners will naturally pass through, on a small scale, the same developmental path that scholarship has passed through on a grand scale" (Klein 1896, p. 148, translated by Martin Mattheis). In his formulating his thoughts on the genetic method of teaching, Felix Klein clearly distinguished between a form of mathematics instruction suitable for secondary schools, which was to follow the basic principle of moving from the concrete to the abstract, and the deductive structure of teaching material—commonly used in higher education—aligned to the systematics of the discipline.

> The *manner of teaching* as it is carried on in this field in Germany can perhaps best be designated by the words *intuitive* and *genetic*, i.e., the entire structure is gradually erected on the basis of familiar, concrete things, in marked contrast to the customary *logical* and *systematic* method in higher education. (Klein 1908, p. 14 or Klein 2016a, p. 9)

It does seem doubtful, however, whether Klein was describing here the teaching really being practiced at secondary schools in 1908. It is more likely that he was presenting the way school mathematics should, in his view, be taught.

7.5 Conclusion

The calls to "strengthening spatial intuition" and "training the habit of functional reasoning" were not only central to the Meran Curriculum Proposal, in which Felix Klein played such a leading role, but had already been fundamental to his overall idea of what mathematics teaching should look like in secondary schools and higher education. For Felix Klein, however, the concept of "intuition" (Anschauung) referred to more than the physically sensate intuition one needs to describe concrete three-dimensional objects. While "intuition" was, of course, an important means for learners to gain new mathematical insights, he also saw it as a tool in research. Moreover, Felix Klein extends the concept of "intuition" to mean a source of inspiration with which people contemplating mathematical questions arrive intuitively at ideas for their solution. Thus, in this wider sense, intuition plays an important role in all fields of mathematics and not only—as one might initially expect—in geometry.

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