Chapter 14

Examples of Klein's Practice *Elementary Mathematics from a Higher Standpoint: Volume I*

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Abstract In the first volume of *Elementary Mathematics from a Higher Standpoint: Arithmetic, Algebra and Analysis*, Klein closely adheres to several principles which contribute a great deal to the understanding of Klein's higher standpoint—such as the principle of mathematical interconnectedness, the principle of intuition, the principle of application-orientation, and the genetic method of teaching. In addition, Klein conveys not only a mathematical but also a historical and a didactical perspective, all of which broaden this standpoint. This versatile approach to the mathematical content will be illustrated in this article by taking a closer look at the chapter on logarithmic and exponential functions.

Keywords Felix Klein • Elementary mathematics • Higher standpoint
Logarithms • Perspectives

14.1 Introduction

The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fall in with the time-honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching. (Klein 1908/2016, p. 1)

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Klein called this situation a *double discontinuity*¹ and met this problem by giving a series of lectures entitled *Elementarmathematik vom höheren Standpunkte aus* (*Elementary Mathematics from a Higher Standpoint*).² In total, three manuscripts were published: one on arithmetic, algebra, and analysis; a second one on geometry; and a third one on precise and approximative mathematics.³ The third volume, however, aims to show the connection between approximative mathematics and pure mathematics. Klein does not cover questions on mathematics education in the last volume.

In the first two volumes, Klein provides an overview of school mathematics to connect the different mathematical branches and to point out the connection to school mathematics (see Klein 1908/2016, p. 2). As described in Schubring (this book), Klein expected his students to have a basic knowledge of higher level mathematics, such as functions theory, number theory, and differential equations:

I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the chief fields of mathematics. I shall often talk of problems of algebra, of number theory, of function theory, etc., without being able to go into details. You must, therefore, be moderately familiar with these fields, in order to follow me. (Klein 1908/2016, p. 1)

In teachers' education in Germany, there has been some intention to re-establish Felix Klein's ideas: In the COACTIV study, Krauss et al. (2008) stated that a large number of students lack profound knowledge in elementary mathematics and school mathematics when leaving university. In TEDS-M (Tatto et al. 2012) this was confirmed with representative samples. This led to the conclusion that

clearly, teachers' knowledge of the mathematical content covered in the school curriculum should be much deeper than that of their students. We conceptualised CK [content knowledge] as a deep understanding of the contents of the secondary school mathematics curriculum. It resembles the idea of "elementary mathematics from a higher viewpoint" (Krauss et al. 2008, p. 876)

In 2008, IMU and ICMI commissioned a project to revisit the intent of Felix Klein when he wrote *Elementary Mathematics from a Higher Standpoint*. The authors' aim was to write a book for secondary teachers that showed the connection between ongoing mathematical research and the senior secondary school curriculum (www.kleinproject.org).

However, in all discussions, the term *higher standpoint* was used intuitively and, without making it explicit or naming concrete characteristics, Klein's lectures were

¹In Kilpatrick (this book) Klein's notion of the double discontinuity is described precisely.

²As Kilpatrick (2014) noted, the original English translation of the title using the word *advanced* as translation for *höher* is misleading, as the term *advanced* could be interpreted as "more developed," which Klein, who was aiming for a panoramic view, did not have in mind. Taking Kilpatrick's concerns into account, the new edition of Klein's lectures was released under that newly translated title.

³The latter has recently been translated into English by Marta Menghini and Gert Schubring (Klein 1902/2016). It is based on a lecture Klein held in 1901. In Klein's last years, he decided to republish it as a third part of the *Elementary Mathematics from a Higher Standpoint* series. For more information on that volume, see Meghini (this book).

assumed to have functioned as a role model. In my Ph.D. thesis (Allmendinger 2014), I attempted to help close this academic gap by analysing the lectures of Klein in an attempt to answer the guiding question: What is Klein's understanding of the term *higher standpoint*?

I decided to focus on the first volume of Klein's lecture notes, as the different approaches in all three volumes—described by Kilpatrick (this book)—make it difficult to compare them directly. In the first volume on arithmetic, algebra, and analysis, Klein includes pedagogical remarks throughout the whole lecture, while in the second volume on geometry Klein focuses in the first chapters on the mathematical aspects and then discusses pedagogical questions in a final chapter. Kilpatrick even concludes, that “the organization of the first volume allows Klein to make specific suggestions for instruction and references to textbooks and historical treatments of topics, whereas the comments in the second volume tend to be more general” (Kilpatrick 2014, p. 34).

For the analysis of the manuscript of the first volume, I used a phenomenological approach, as found in Seiffert (1970, p. 42). This approach analyses a source in its historical sense, as it concentrates on the source itself and does not focus on the historical background in the first place.

Additionally, I integrated didactic concepts and vocabulary to describe and specify Klein's procedure. I was able to show that today's movement towards improved mathematical university studies for teacher trainees bears some resemblance to and coherence with Klein's ideas.

As Klein directly comments on his intentions in his lecture notes, analysing these seems to be a possible procedure to locate the characteristics. But especially with regard to how Klein's concept is being adapted today, it is important to understand the circumstances that led Klein to construct this lecture and the conditions he faced. In Klein's days, there was no distinction between teacher trainees and mathematics students pursuing a scientific career. Therefore, the students in Klein's lectures had relatively broader background knowledge than today's teacher students. That is why I embedded my analysis in its historical context.⁴

In order to holistically describe Klein's understanding of the term *higher standpoint*, one must also take into account its counterpart—*elementary mathematics* (see Schubring in this book). As this term, like *higher standpoint*, has always been used quite intuitively, it is not possible to give a concrete definition.⁵ For this article, I will use a preliminary definition: Everything is “elementary” that can be made accessible to an “averagely talented pupil” (Klein 1904, p. 9). Klein's lectures cover both subjects of the established school curriculum and subjects that Klein felt should be part of school curriculum, for example, calculus (see Meran Curriculum 1905).

The results of my analysis show that on the one hand Klein closely adheres to several principles, such as the principle of mathematical interconnectedness, the principle of intuition, the principle of application orientation, and the genetic method of

⁴A good overview of this historical context can be found in Schubring (2007).

⁵In the beginning of the 20th century some mathematicians aimed to give a definition of elementary mathematics (e.g., Weber 1903; Meyer and Mohrmann 1914), as discussed in Allmendinger (2014).

teaching. Those principles contributed greatly to the development of Klein's *Higher Standpoint*. In addition, Klein conveys a multitude of perspectives—mathematical, historical, and didactic—that widen this higher standpoint.

In the present chapter, I focus on presenting the different perspectives that characterise Klein's *Higher Standpoint*. I specify them generically by reference to the chapter in Klein's lecture notes on logarithmic and exponential functions (Klein 1908/2016, pp. 153–174) described by Kilpatrick (this book). This chapter is paradigmatic and outstanding at the same time, as all characteristics I found in Klein's lecture cumulate in this chapter. Therefore, it seems appropriate to outline Klein's intentions and his proceeding.

All three perspectives can clearly be noted in this chapter: Klein starts reviewing and reflecting the current teaching practice and making suggestions on how to improve the introduction of this theme in school. On the one hand, he regards the subject virtually from a didactic perspective. On the other hand, he gives an overview of its historical development and thus an insight in his historical perspective. Finally, Klein enhances the knowledge on logarithmic functions by adopting the standpoint of function theory.

14.2 Klein's Didactic Perspective

I start by taking a closer look at the didactic perspective: the standpoint of mathematical pedagogy. Klein had always shown great interest in questions of mathematical education [as noted, for example, in Schubring (2007) and Mattheis (2000)]. He was one of the main protagonists in the Meran reform, supporting and accelerating the integration of perception of space and the prominence to the notion of function, which culminates in the introduction of calculus. In my analysis, I was able to show that all the demands made in the Meran reform strongly influence Klein's lecture: Klein adheres closely to the "primacy of intuition" ("*Primat der Anschauung*"), and nearly all aspects of the notion of function that Krüger (2000) carved out in her Ph.D. thesis can be detected.

However, Klein's *Higher Standpoint* can be understood in the first place from the didactic perspective as a methodological one: Klein intends to prepare future teachers for their upcoming tasks and to provide them with the necessary overview and background. However, eventually he also criticises the common procedures in school and presents alternatives.

As in many other chapters, Klein starts his chapter on logarithmic and exponential functions by giving a short overview of the curriculum and teaching practice: "Let me recall briefly the familiar curriculum of the school, and the continuation of it to the point at which the so-called algebraic analysis begins" (Klein 1908/2016, p. 155).

By starting with powers of the form $a = b^c$ with c a positive integer, Klein describes how one extends the notion for negative, fractional, and finally irrational values. The logarithm is then defined as that value c , which gives a solution to the named equation.

What matters is that he critically reflects on this procedure: To uniquely extend the values to fractional values, stipulations have to be made⁶ that—in Klein's opinion—"appear to be quite arbitrary...and can be made clear only with the profounder resources of function theory" (Klein 1908/2016, p. 156).

If we now admit all real, including irrational, values of y , it is certainly not immediately clear why the principal values which we have been marking on the right now constitute a continuous curve and whether or not the set of negative values which we have marked on the left do similarly permit such a completion. (Klein 1908/2016, p. 156)

As we will see in the latter sections, he analyses the mathematical content from a historical and mathematical point of view in order to develop an alternative that avoids the emphasised problems.

Today, Klein's suggested approach of introducing the logarithmic function as the integral of $1/x$ is often used as an example for a concept, which Freudenthal (1973) called *antididactical inversion*, which means that the smoothened end product of a historical learning process becomes the point of departure in education (e.g., Kirsch 1977). Nevertheless, Klein presents mathematical reorganisations of school mathematical contents in order to bypass obstacles that the students might face. He presents a mathematical analysis with an interest its impact on school practice.

Although Klein dedicates the implementation in the classroom to the "experienced school man" (Klein 1908/2016, p. 168), he has concrete ideas towards teaching methods, which he mentions in remarks throughout the whole lecture:

I am thinking, above all, of an impregnation with the genetic method of teaching, of a stronger emphasis upon space perception, as such, and, particularly, of giving prominence to the notion of function, under fusion of space perception and number perception! (Klein 1908/2016, p. 88)

14.3 Klein's Historical Perspective

Klein has always shown a strong interest in historical development (e.g., Klein 1926). He is said to be one of the first representatives of a historical genetic method of teaching, as shown in Schubring (1978). Klein vindicates his approach with the help of the biogenetic fundamental law "according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species" (Klein 1908/2016, p. 292).⁷ The lectures in *Elementary Mathematics from a Higher Standpoint* can be seen as an example of Klein's understanding of this historical genetic method itself.

⁶For example, there are two values of x that solve the equation $x = b^2$. Generally, in the equation for every $y = m/n$, completely reduced, with an even value for n there are n different solutions. One decides to stipulate that that should be the so-called principal root.

⁷Nowadays this law is highly criticised, as it suggests that every individual has to go through the same learning process (see Wittmann 1981, p. 133).

Fig. 14.1 Generalisation of
Bürge's approach (Klein
1908/2016, p. 167)

In Klein's opinion, following historical development is the "only scientific" way of teaching mathematics. That is why he aims to provide the future teachers with the necessary background to use this method in school. This requires profound knowledge of historical development, which Klein allocates by steadily integrating historic remarks and overviews:

An essential obstacle to the spreading of such a natural and truly scientific method of instruction is the lack of historical knowledge which so often makes itself felt. In order to combat this, I have made a point of introducing historical remarks into my presentation. (Klein 1908/2016, p. 292)

Klein therefore constantly adds historical remarks and digression, which are both rich in content and distinguished by a rather scarce depiction. They are sophisticated sections that demand intensive post-processing from the students. In the chapter on logarithmic and exponential functions, the reader will find one of the rarer parts of the lecture, where Klein extensively shows his understanding of a historical genetic approach: "In short, if we really wish to press forward to a full understanding of the theory of logarithms, it will be the best to follow the historical development in its broad outlines" (Klein 1908/2016, p. 157).

Klein shows a different approach to the definition of the logarithmic function by describing the historical development of the theory: The main idea Bürge had when he was calculating his logarithmic tables, was to avoid the stipulation that was shown above by choosing a basis b close to 1. In this way, the calculation with integer valued y 's lead to a table, where the distance between neighbouring values of x was rather small.

Before finishing his historic overview, Klein shows how to set up a differential equation by generalising Bürge's approach (see Fig. 14.1). His analysis and calculations lead to the definition of the natural logarithm as

$$\int \frac{1}{x} dx$$

This leads Klein to conclude:

I should like to outline briefly once more my plan for introducing the logarithm into the schools, in this simple and natural way. The first principle is that the proper source from which to bring in new functions is the quadrature of known curves. (Klein 1908/2016, p. 167)

In this way, the historic parts in Klein's lecture notes have a special meaning for mathematical education in general. They lead to new approaches to different subjects. But the historic remarks have a benefit for mathematics teachers' education as well. Nickel (2013) gave a classification on how and why the integration of the history of mathematics should be part of teachers' education. One can place Klein's historical perspective clearly in this suggested classification: Klein uses the history of mathematics as a tool of comfort and motivation by presenting fascinating anecdotes and as a tool to improve insightful contact with mathematics by reliving its historical development. It becomes obvious that Klein does not teach the history of mathematics as an autonomous learning subject.⁸

14.4 Klein's Mathematical Perspective

Last, Klein's understanding of a higher standpoint on elementary mathematics involves being capable of connecting school mathematics with the higher mathematics taught at university. This especially involves having background knowledge. Therefore higher mathematics becomes a tool to explain the contents of school mathematics.

In the chapter on logarithmic and exponential functions, the section on the standpoint of function theory is a typical example. Function theory is not part of school mathematics—neither in Klein's day nor today—but in Klein's opinion, the teacher has to have a basic knowledge of that subject to adequately understand the definition of the logarithm: "Let us, finally, see how the modern theory of functions disposes of the logarithm. We shall find that all the difficulties which we met in our earlier discussion will be fully cleared away" (Klein 1908/2016, p. 168).

Klein shows that the logarithm is defined by means of the integral

$$\omega = \int \frac{d\zeta}{\zeta}$$

"where the path of integration is any curve in the ζ -plane joining $\zeta = 1$ to $\zeta = z$ " (Klein 1908/2016, p. 168; see Fig. 14.2). He further states, that the integral has an infinite number of values. A definite value can be determined, if we "slit the plane along the negative real axis and agree that the path integration shall not cross this cut" (Klein 1908/2016, p. 168). Giving an insight into the means of function theory, Klein is able to show that "if n is even, the negative real values of $b^{m/n}$ will constitute a set which is everywhere dense, but they belong to an entirely different one of our infinitely many functions, and cannot possibly combine to form a continuous analytic curve" (Klein 1908/2016, p. 170).

By this, he provides an answer to question of why the principal roots were chosen to build the logarithm function.

⁸The complete classification can be found in Nickel (2013).

Fig. 14.2 Komplexe Logarithms (Klein 1908/2016, p. 170)

In this last part, one aspect of Klein's understanding of the higher standpoint becomes evident. Klein does not expect his students to teach this prospectively to their pupils:

I hardly believe, however, that the average pupils, even in the Prima, can be carried so far.... I am sure all the more desirous that the teacher shall be in full possession of all the function-theoretic connections that come up here: For the teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them. (Klein 1908/2016, p. 162)

Additionally, Klein uses higher mathematics and its vocabulary for a precise and significant representation of school mathematics. He occasionally has to discuss up-to-date research, as in his remarks on the *Logical Foundations of Operations with Integers* (Klein 1908/2016, pp. 10–16). School mathematics is also shown to be the origin of research: The search for algebraic solutions of equations is an interrogation that is easily accessible to pupils and is covered in school. However to understand that an equation of the fifth degree or higher is not algebraically soluble, one has to have profound knowledge of Galois' theory.

All these examples give evidence of a mathematical perspective on the contents of math classes. Klein shows how university studies are connected to mathematical school contents in order to oppose the double discontinuity: He connects elementary mathematics with "higher" mathematics—literally discussing elementary mathematics from a higher standpoint. It can be assumed, that this mathematical perspective shows Klein's higher standpoint in the narrow sense of the word.

Summarising, from a didactic perspective, Klein promotes a reflective attitude on the school curriculum and provides possible alternatives to the current teaching practice. Additionally, a historical perspective helps to place the object of investigation in an overall context and provide knowledge on the mathematical history of its development. Finally, from a mathematical perspective, the characteristics of Klein's higher standpoint on elementary mathematics are a high degree of abstraction, formal technical language, and a foundation of school mathematics' contents.

14.5 Higher Mathematics from an Elementary Standpoint?

In the chapter “Concerning the modern development and the general structure of mathematics” (Klein 1908/2016, pp. 81–88), described by Kilpatrick (this book), Klein introduces two different processes of growth in the history of mathematical development (calling them Plan A and Plan B), “which now change places, now run side by side independent of one another, now finally mingle” (Klein 1908/2016, p. 81). While in Plan A, each mathematical branch is developed separately using its own methods, Plan B aims at a “fusion of the perception of number and space” (Klein 1908/2016, p. 81)—mathematics is to be seen as a whole.

According to Klein, the education of mathematics in school and at university, should clearly be guided by Plan B: “Every movement for the reform of mathematical education must therefore for a stronger emphasis on the direction of B.... It is my aim that this lecture course shall serve this tendency” (Klein 1908/2016, p. 88).

In this chapter, Klein expresses his attitude towards mathematics education in general, as shown in the Meran reform, and legitimates the procedure in his *Elementary Mathematics from a Higher Standpoint* lectures (see Allmendinger and Spies 2013): The main principles, which are characteristic for the favoured Plan B and which Klein wants future teachers to implement in their school classes, are principles Klein attempts to pursue himself: the principle of interconnectedness, the principle of intuition, the principle of application orientation, and the genetic method of teaching.

By applying these principles, all “will all seem elementary and easily comprehensible” (Klein 1908/2016, p. 223). As a result, two different orientations can be identified in the lectures: Klein regards both elementary mathematics from a higher standpoint and higher mathematics from an elementary standpoint. This hypothesis can be underlined by Kirsch's aspects of simplification (see Kirsch 1977), as those show a striking resemblance to Klein's procedure in his lectures and his principles.

14.6 A Higher Standpoint: First Conclusions

As shown, Klein's *Elementary Mathematics from a Higher Standpoint* can be characterised by its underlying principles on the one hand and by a constant variation of different perspectives on the other hand. Both the principles and the perspectives aim for a connection between school and university mathematics, in order to overcome the double discontinuity: The mathematical, the historical, and the didactic perspectives help to restructure the higher standpoint on elementary mathematics. In particular, the didactic perspective shows an orientation that distinguishes Klein's lectures from other contemporary lectures on elementary mathematics. Moreover, the underlying principles show an additional orientation: Klein also demonstrates higher mathematics from an elementary standpoint.

Toeplitz (1932), however, questioned whether the establishment of lectures on elementary mathematics, which, for example, Klein had in mind, would be the right

way to prepare students for their future tasks. He criticised the selected contents: In Toeplitz's opinion, a teacher does not necessarily need to know the proof for the transcendence of e , to give just one example. He also had the opinion that Klein chose topics that required too much background knowledge for a lecture attempting to give an overview of the complete content of school mathematics' (see Toeplitz 1932, pp. 2f.). Toeplitz argues that a desirable higher standpoint cannot be taught in one single lecture, but has to be accomplished in every lecture of mathematical studies.

Nevertheless, the skills that accompany a higher standpoint in Toeplitz's understanding clearly resemble the ones Klein conveys in his *Elementary Mathematics from a Higher Standpoint*. Altogether, Klein's lectures can be understood as an archetype for current university studies, although adaptations have to be made depending on the given circumstances.

"It's not the task anymore to create new thoughts but to bring to light the right thoughts in the right way regarding the given circumstances" (Klein 1905, translated by the author).

References

- Allmendinger, H. (2014). *Felix Kleins Elementarmathematik vom höheren Standpunkte aus. Eine Analyse aus historischer und mathematikdidaktischer Sicht* (SieB Vol. 4). Siegen: universi.
- Allmendinger, H., & Spies, S. (2013). Über die moderne Entwicklung und den Aufbau der Mathematik überhaupt. In M. Rathgeb, M. Helmerich, R. Krömer, K. Legnink, & G. Nickel (Eds.), *Mathematik im Prozess. Philosophische, Historische und Didaktische Perspektiven* (pp. 177–194). Wiesbaden: Springer Spektrum.
- Freudenthal, H. (1973). *Mathematik als pädagogische Aufgabe*. Stuttgart: Ernst Klett.
- Kilpatrick, J. (2014). *A higher standpoint. Materials from ICME 11*. www.mathunion.org/icmi/publications/icme-proceedings/. Regular lectures, 27–43.
- Kirsch, A. (1977). Aspekte des Vereinfachens im Mathematikunterricht. In *Didaktik der Mathematik*, 2, 87–101.
- Klein, F. (1904). Über eine zeitgemäße Umgestaltung des mathematischen Unterrichts an höheren Schulen. In F. Klein & E. Riecke (Eds.), *Neue Beiträge zur Frage des Mathematischen und Physikalischen Unterrichts an den höheren Schulen*. Leipzig und Berlin: B. G. Teubner.
- Klein, F. (1905). Bericht an die Breslauer Naturforscherversammlung über den Stand des mathematischen und physikalischen Unterrichts an den höheren Schulen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 14, 33–47.
- Klein, F. (1926). *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (Vol. 1). Berlin: Julius Springer.
- Klein, F. (1908/2016). *Elementary mathematics from a higher standpoint, volume I: Arithmetics, algebra and analysis*. (G. Schubring, Trans.). Wiesbaden: Springer.
- Klein, F. (1902/2016). *Elementary mathematics from a higher standpoint, volume III: Precision Mathematics and Approximation Mathematics*. Translated by Marta Menghini and Gert Schubring. Wiesbaden: Springer.
- Krauss, S., Baumert, J., Brunner, M., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM Mathematics Education*, 40, 873–892.

- Krüger, K. (2000). *Erziehung zum funktionalen Denken. Zur Begriffsgeschichte eines didaktischen Prinzips*. Berlin: Logos-Verlag.
- Mattheis, M. (2000). Felix Kleins Gedanken zur Reform des mathematischen Unterrichtswesen vor 1900. *Der Mathematikunterricht*, 46(3), 41–61.
- Meran Curriculum (1905). Bericht betreffend den Unterricht in der Mathematik an den neunklassigen höheren Lehranstalten. In A. Gutzmer (Ed.), *Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte. Gesamtbericht* (pp. 104–114). Leipzig und Berlin: B. G. Teubner.
- Meyer, W. F., & Mohrmann, H. (1914). *Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*. Leipzig und Berlin: B. G. Teubner.
- Nickel, G. (2013). Vom Nutzen und Nachteil der Mathematikgeschichte für das Lehramtsstudium. In H. Allmendinger, Henrike, K. Lengnink, A. Vohns, & G. Wickel (Eds.), *Mathematik verständlich unterrichten—Perspektiven für Schule und Hochschule* (pp. 253–266). Wiesbaden: Springer Spektrum.
- Schubring, G. (1978). *Das genetische Prinzip in der Mathematikdidaktik*. Stuttgart: Klett-Cotta.
- Schubring, G. (2007). Der Aufbruch zum „funktionalen Denken“: Geschichte des Mathematikunterrichts im Kaiserreich. *N.T.M.*, 15, 1–17.
- Seiffert, H. (1970). *Einführung in die Wissenschaftstheorie. Band 2. Phänomenologie, Hermeneutik und historische Methode, Dialektik*. München: Beck.
- Tatto, M. T., Schwille, J., Senk, S. L., Ingvarson, L., Rowley, G., Peck, R., et al. (2012). *Policy, practice, and readiness to teach primary and secondary mathematics in 17 countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-M)*. Amsterdam: IEA.
- Toeplitz, O. (1932). Das Problem der “Elementarmathematik vom höheren Standpunkt aus”. *Semesterberichte zur Pflege des Zusammenhangs von Universität und Schule aus den mathematischen Seminaren*, 1, 1–15.
- Weber, H. (1903). Über die Stellung der Elementarmathematik in der mathematischen Wissenschaft. In *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 12, 398–401.
- Wittmann, E. (1981). *Grundfragen des Mathematikunterrichts*. Wiesbaden: Vieweg + Teubner.

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