$\mathcal{NP}$ -Hardness of Equilibria in Case of Risk-Averse Players

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Abstract. We study the complexity of deciding the existence of *mixed equilibria* for minimization games where players use *valuations* other than expectation to evaluate their *costs*. We consider *risk-averse* players seeking to minimize the sum $V = E + R$ of *expectation* E and a *risk valuation* R of their costs; R is non-negative and vanishes exactly when the cost incurred to a player is constant over all choices of strategies by the other players. In a V -*equilibrium*, no player can unilaterally reduce her cost.

The results presented in this paper show that the question whether a 2-player game with such risk-modeling valuations V has a V -equilibrium is strongly $\mathcal{NP}$ -hard under very mild assumptions. We only have to ask that the valuation functions are strictly quasiconcave or that they fulfill the Weak-Equilibrium-for-Expectation property and that additionally some 2-strategy game has no V -equilibrium. These conditions are fulfilled for the functions from Markowitz's Mean-Variance approach like $E + \text{Var}$, $E + \text{SD}$ and the Sharpe Ratio, and also for Conditional Value-at-Risk, recently very popular for modeling volatile economic circumstances.

1 Introduction

1.1 Motivation

In a *game*, each *player* is using a *mixed strategy*, a probability distribution over her *strategies*; her *cost* depends on the choices of all players and is evaluated by a *valuation*, a function from probability distributions to reals. The most prominent valuation in *Non-Cooperative Game Theory* is *expectation*, where each player minimizes her expected cost. A drawback of expectation is that it may not accommodate *risk* and its impact on strategic decision (cf. [3]). Indeed, *risk-averse* players [1] are willing to accept a larger amount of payment rather than to gamble and take the risk of a larger cost. According to [15], “a risk-averse

Partially supported by the German Research Foundation (DFG) within the Collaborative Research Centre “On-the-Fly-Computing” (SFB 901), and by funds for the promotion of research at University of Cyprus.

player is willing to pay something for certainty”. In many experiments (see [19]) it has been demonstrated that expectation is not well-suited to model human behaviour under risk. This was observed by Daniel Bernoulli already in 1739. A short description of corresponding results is given below in Subsect. 1.2.

So, valuations other than expectation have been sought (cf. [1, 10, 22, 35, 37]). *Concave* valuations, such as *variance* and *standard deviation*, are well-suited to model risk-averse minimizing players. Already in 1906, Fisher [14] proposed attaching standard deviation to expectation as an additive measure of risk. In his seminal paper [22], Markowitz introduced the *Mean-Variance* approach to portfolio maximization, advocating the minimization of variance constrained on some lower bound on the expected return. Popular valuations are (i) $E - \gamma \cdot \text{Var}$, where E and Var stand for *expectation* and *variance*, respectively, and $\gamma > 0$ describes the risk tolerance (see [10]), and (ii) the *Sharpe Ratio* $SR = E/SD$ [35], where SD stands for *standard deviation*. The *Mean-Variance* paradigm [22] created Modern Portfolio Theory [10] as a new field and initiated a tremendous amount of research — see the surveys [20, 36] for an overview. However, in the *Mean-Variance* paradigm [22], only expectation and variance were used for evaluating the return; this choice is justified only if the return is normally distributed [17].

Therefore this approach is not-well suited to model the losses that might be incurred in finance or in the insurance industry. Real financial data are characterized by fat tails. For these applications Conditional Value at Risk became the most prominent measure during the last years. CVaR is recognized as a suitable model of risk in volatile economic circumstance [20, 31, 32, 34]; it is widely used for modeling and optimizing hedge and credit risks. There were recently two quite notable developments in the insurance domain and the financial domain, namely the issue of the *Solvency II Directive* in the insurance domain and of *Basel III Rules* in the financial domain, which enforce a tremendous increase to the role and applicability of CVaR.

A significant advantage of expectation is that it guarantees the existence of a *Nash equilibrium* [26, 27], where each player is playing a *best-response* mixed strategy and could not unilaterally reduce her expected cost. Existence of equilibria (for minimization games) extends to *convex* valuations [7, 11], but may fail for non-convex and even for *concave* ones. *Crawford’s game* [6, Sect. 4] was the *first* counterexample game with no equilibrium for a certain valuation; for more counterexamples, see [9, 23].

1.2 Human Behaviour Under Risk

In Experimental Economics, experimental methods are applied to study economic questions, e.g., in order to test the validity of economic theories.

The paper of Kahnemann and Tversky [19] presents a critique of expected utility theory as a descriptive model of decision-making under risk.

- Possibility of winning, certainty effect
 - A: 80% chance of winning \$4,000
B: \$3,000 with certainty
20% choose A, 80% choose B
 - A: 50% chance of winning a three-week tour of England, France, and Italy
B: One-week tour of England with certainty
22% choose A, 78% choose B
- Possibility of losing
 - A: 80% chance of losing \$4,000
B: Losing \$3,000 with certainty
92% choose A, 8% choose B

Already in 1739, in his paper “Exposition of a New Theory on the Measure of Risk” Daniel Bernoulli [3] studies an example where expectation is not suited to model human behaviour.

A person owns a lottery ticket:

- It yields with equal probability either nothing or 20,000 ducats.

A poor person may want to sell this ticket for 9,000 ducats. While a rich man may want to buy it for this price.

1.3 Framework

We embarked on a research direction making different-from-classical assumptions. We model the valuation of each player as the sum $V = E + R$, where E and R are the *expectation* and *risk valuation*, respectively. R has the *Risk-Positivity* property: the value of R is non-negative – it is 0, yielding no risk, exactly when the cost incurred to a player is constant over all choices of strategies by the other players. We focus on the associated decision problem, denoted as $\exists V$ -EQUILIBRIUM, asking, given a game G , whether G has a V -equilibrium, where no player could unilaterally reduce her cost (as evaluated by V).

We shall focus on strictly quasiconcave valuations. A key property of a strictly quasiconcave valuation, called *Optimal-Value*, we prove and exploit is that it maintains the same optimal value over all convex combinations of strategies supported in a given best-response mixed strategy (Theorem 1).

The *Weak-Equilibrium-for-Expectation* property [23, Sect. 2.6] requires that all strategies supported in a player’s best-response mixed strategy induce the same conditional expectation, taken over all random choices of the other players, for her cost. Thus the player is holding a *unique* expectation for her cost no matter which of her supported strategies she ends up choosing. This property holds vacuously for Nash equilibria [26, 27] and it holds for several of the valuation functions we consider, but not for all. It is an open problem to classify the subclass of strictly quasiconcave valuations with this property.

1.4 Contribution and Related Work

Fiat and Papadimitriou [13] were the first in Computer Science to consider algorithmic problems about equilibria for games where players are not expectation-optimizers. Specifically, they introduced the equilibrium computation problem in games where risk-averse players use valuations more general than expectation, and addressed the complexity of deciding the existence of such equilibria. Subsequently, Mavronicolas and Monien [23] focused on the concave valuation *expectation plus variance*, and established structural and complexity results [23, Sect. 3].

Mavronicolas and Monien [24, 25] developed a framework for games with players minimizing an $(E + R)$ -valuation V . Our framework enabled proving the strong $\mathcal{NP}$ -hardness of $\exists V$ -EQUILIBRIUM in the simple case of games with two players, and for many significant choices of $(E + R)$ -valuations V . These were the *first* complexity results for deciding the existence of equilibria in the context of risk-modeling valuations.

The results from these 3 papers are presented in this article in a unified approach. The results show that the question whether a 2-player game with such risk-modeling valuations V has a V -equilibrium is strongly $\mathcal{NP}$ -hard under very mild assumptions. We only have to ask that the valuation functions are strictly quasiconcave or that they fulfill the Weak-Equilibrium-for-Expectation property and that additionally some 2-strategy game has no V -equilibrium. These conditions are fulfilled for the functions from Markowitz's Mean-Variance approach like $E + \text{Var}$, $E + \text{SD}$ and the Sharpe Ratio, and also for Conditional Value-at-Risk, recently very popular for modeling volatile economic circumstances.

Complexity results for the related optimization problems do also exist: Maximizing the expectation of portfolio-return of an investment for a given threshold on variance and for a given threshold on Var_α , respectively, were shown to be $\mathcal{NP}$ -hard in [21] and in [2], respectively.

A different aspect is studied by Brautbar et al. [5]. They introduce a class of function valuations coming from Markowitz's Mean-Variance paradigm [22] and show that for valuations from this class non-pure equilibria are very rare.

In [5] it is also shown, following a proof idea described already by Crawford [6], that for concave valuations a correlated equilibrium always exists.

Finally we want to mention that [24] also contains the result that the problem $\exists V$ -EQUILIBRIUM is strongly $\mathcal{NP}$ -hard also for 2-strategy games provided that $V = E + \text{Var}$ or $V = E + \text{SD}$.

2 (Strict) Convexity and (Strict) Concavity

Definition 1. A function $f: D \rightarrow \mathbb{R}$ on a convex set $D \subset \mathbb{R}^n$ is *convex* (resp., strictly convex) if for every pair of points $x, y \in D$, for all $\lambda \in [0, 1]$ (resp., $\lambda \in (0, 1)$), $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ (resp., $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$). A function $f: D \rightarrow \mathbb{R}$ on a convex set D is *concave* (resp., strictly concave) if $-f$ is convex (resp., strictly convex).

We give a characterization of convexity (resp., concavity) for differentiable functions.

Lemma 1. *Assume that the function $f: D \rightarrow \mathbb{R}$ on the convex set $D \subset \mathbb{R}^n$ is differentiable. Then, f is convex (resp., concave) if and only if for all pairs $x, y \in D$, $f(y) \geq f(x) + \nabla f(x)^T(y - x)$ (resp., $f(y) \leq f(x) + \nabla f(x)^T(y - x)$).*

2.1 Quasiconvexity and Quasiconcavity

Quasiconvexity (resp., quasiconcavity) is a generalization of convexity (resp., concavity).

Definition 2. *A function $f: D \rightarrow \mathbb{R}$ on a convex set $D \subset \mathbb{R}^n$ is quasiconvex (resp., quasiconcave) if for every pair of points $x, y \in D$, for all $\lambda \in (0, 1)$, $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$ (resp., $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$).*

An introduction into quasiconvexity/quasiconcavity and many examples can be found in [4].

As a nice historical survey in [16] shows, the notion of quasiconcavity emerged in Economics and Mathematics from the works of three authors, namely von Neumann [28], DeFinetti [8] and Fenchel [12], and started the research field of *generalized convexity*. Quasiconcave functions make a very rich class with many applications in Mathematical Economics; for example, in Microeconomics, the convexity of customers' preference relations is a very desirable feature, implying the quasiconcavity of their utilities [33].

2.2 Strict Quasiconvexity and Strict Quasiconcavity

Definition 3. *A quasiconvex (resp., quasiconcave) function $f: D \rightarrow \mathbb{R}$ on a convex set $D \subset \mathbb{R}^n$ is strictly quasiconvex (resp., strictly quasiconcave) [30] if for every pair of points $x, y \in D$, $f(x) \neq f(y)$ implies that for all $\lambda \in (0, 1)$, $f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\}$ (resp., $f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\}$).*

A strictly quasiconcave valuation incurs to a player mixing two particular mixed strategies a cost larger than the minimum of the individual costs incurred by the two mixed strategies when the two costs are not equal. A significant property of strictly quasiconcave valuations, making them interesting to consider in the context of equilibrium computation, is that they do not exclude the existence of mixed equilibria, whereas strictly concave valuations do.

The notion of strictness for strictly quasiconcave functions is weaker than for strictly concave functions: for the former, strictness allows that $f(\lambda x + (1 - \lambda)y) = \min\{f(x), f(y)\}$, with $\lambda \in (0, 1)$, in the case $f(x) = f(y)$, whereas the equality is excluded for the latter. A notable property of a strictly quasiconcave function is a property of concave functions: a local maximum is also a global maximum [30, Theorem 2].

We prove:

Proposition 1. *Consider a pair of a convex function $g: D \rightarrow \mathbb{R}$ and a concave function $h: D \rightarrow \mathbb{R}$. Then, the function $f := g/h$ is strictly quasiconvex.*

3 Definitions and Background

3.1 Games

For an integer $n \geq 2$, an n -players game G , or *game*, consists of (i) n finite sets $\{S_i\}_{i \in [n]}$ of *strategies*, and (ii) n *cost functions* $\{\mu_i\}_{i \in [n]}$, each mapping $\mathcal{S} = \times_{i \in [n]} S_i$ to the reals. So, $\mu_i(\mathbf{s})$ is the *cost* of player $i \in [n]$ on the *profile* $\mathbf{s} = \langle s_1, \dots, s_n \rangle$ of strategies, one per player.

A *mixed strategy* for player $i \in [n]$ is a probability distribution p_i on S_i ; the *support* of player i in p_i is the set $\sigma(p_i) = \{\ell \in S_i \mid p_i(\ell) > 0\}$. Denote as $\Delta_i = \Delta(S_i)$ the set of mixed strategies for player i . Player i is *pure* if for each strategy $s_i \in S_i$, $p_i(s_i) \in \{0, 1\}$; else she is *non-pure*. Denote as p_i^ℓ the *pure strategy* of player i choosing the strategy ℓ with probability 1.

A *mixed profile* is a tuple $\mathbf{p} = \langle p_1, \dots, p_n \rangle$ of n mixed strategies, one per player; denote as $\Delta = \Delta(\mathcal{S}) = \times_{i \in [n]} \Delta_i$ the set of mixed profiles. The mixed profile $\mathbf{p}$ induces probabilities $\mathbf{p}(\mathbf{s})$ for each profile $\mathbf{s} \in \mathcal{S}$ with $\mathbf{p}(\mathbf{s}) = \prod_{i' \in [n]} p_{i'}(s_{i'})$. For a player $i \in [n]$, the *partial profile* $\mathbf{s}_{-i}$ (resp., *partial mixed profile* $\mathbf{p}_{-i}$) results by eliminating the strategy s_i (resp., the mixed strategy p_i) from $\mathbf{s}$ (resp., $\mathbf{p}$). The partial mixed profile $\mathbf{p}_{-i}$ induces probabilities $\mathbf{p}(\mathbf{s}_{-i})$ for each partial profile $\mathbf{s}_{-i} \in \mathcal{S}_{-i} := \times_{i' \in [n] \setminus \{i\}} S_{i'}$ with $\mathbf{p}(\mathbf{s}_{-i}) = \prod_{i' \in [n] \setminus \{i\}} p_{i'}(s_{i'})$.

3.2 (E + R)-Valuations

For a player $i \in [n]$, a *valuation function* V_i or *valuation* for short, is a real-valued function on $\Delta(\mathcal{S})$, yielding a value $V_i(\mathbf{p})$ to each mixed profile $\mathbf{p}$, so that in the special case where $\mathbf{p}$ is a profile $\mathbf{s}$, $V_i(\mathbf{s}) = \mu_i(\mathbf{s})$. A *valuation* $\mathbf{V} = \langle V_1, \dots, V_n \rangle$ is a tuple of valuations, one per player; $\mathbf{G}^V$ denotes G together with $\mathbf{V}$.

An $(E + R)$ -*valuation* [24, Definition 2.1] is a valuation of the form $\mathbf{V} = \mathbf{E} + \mathbf{R}$, where $\mathbf{E}$ is the *expectation valuation* with $E_i(\mathbf{p}) = \sum_{\mathbf{s} \in \mathcal{S}} \mathbf{p}(\mathbf{s}) \mu_i(\mathbf{s})$ for $i \in [n]$, and $\mathbf{R}$ is the *risk valuation*, a continuous valuation with the *Risk-Positivity* property: For each player $i \in [n]$ and mixed profile $\mathbf{p}$, (C.1) $R_i(\mathbf{p}) \geq 0$ and (C.2) $R_i(\mathbf{p}) = 0$ if and only if for each profile $\mathbf{s} \in \mathcal{S}$ with $\mathbf{p}(\mathbf{s}) > 0$, $\mu_i(\mathbf{s})$ remains constant over all choices of strategies by the other players. In such a case, $V_i(\mathbf{p}) = E_i(\mathbf{p}) = \mu_i(\mathbf{s})$ for any profile $\mathbf{s} \in \mathcal{S}$ with $\mathbf{p}(\mathbf{s}) > 0$.

We shall deal with cases where for a player $i \in [n]$ and a mixed profile $\mathbf{p}$, $\{\mu_i(\mathbf{s}) \mid \mathbf{p}(\mathbf{s}) > 0\} = \{a, b\}$ with $a < b$, so that $R_i(\mathbf{p})$ depends on the three parameters a , b and q , where $q := \sum_{\mathbf{s} \in \mathcal{S} \mid \mu_i(\mathbf{s}) = b} \mathbf{p}(\mathbf{s})$. Then, denote

$$\widehat{V}_i(a, b, q) := a + q(b - a) + \widehat{R}_i(a, b, q).$$

3.3 V-Equilibrium

Fix a player $i \in [n]$. The pure strategy p_i^ℓ is a V_i -*best pure response* to a partial mixed profile $\mathbf{p}_{-i}$ if

$$V_i(p_i^\ell, \mathbf{p}_{-i}) = \min \left\{ V_i(p_i^{\ell'}, \mathbf{p}_{-i}) \mid \ell' \in S_i \right\}.$$

So, the pure strategy p_i^ℓ minimizes the valuation $V_i(\cdot, \mathbf{p}_{-i})$ of player i over her pure strategies. The mixed strategy p_i is a V_i -best response to $\mathbf{p}_{-i}$ if

$$V_i(p_i, \mathbf{p}_{-i}) = \min \{V_i(p'_i, \mathbf{p}_{-i}) \mid p'_i \in \Delta(S_i)\}.$$

So, the mixed strategy p_i minimizes the valuation $V_i(\cdot, \mathbf{p}_{-i})$ of player i over her mixed strategies. The mixed profile $\mathbf{p}$ is a V -equilibrium if for each player i , the mixed strategy p_i is a V_i -best response to $\mathbf{p}_{-i}$. So, no player could unilaterally deviate to another mixed strategy to reduce her cost. Denote as $\exists V$ -EQUILIBRIUM the algorithmic problem of deciding, given a game G , the existence of a V -equilibrium for G^V .

3.4 The Weak-Equilibrium-for-Expectation Property

The mixed profile $\mathbf{p}$ has the *Weak Equilibrium* property [23, Sect.2.6] for player $i \in [n]$ in the game G^V if for each pair of strategies $\ell, \ell' \in \sigma(p_i)$, $V_i(p_i^\ell, \mathbf{p}_{-i}) = V_i(p_i^{\ell'}, \mathbf{p}_{-i})$. The mixed profile $\mathbf{p}$ has the *Weak Equilibrium* property [23, Sect.2.6] in G^V if it has the *Weak Equilibrium* property for each player $i \in [n]$ in G^V . The valuation V has the *Weak-Equilibrium-for-Expectation* property if the following condition holds for every game G . For each player $i \in [n]$, if p_i is a V_i -best-response to $\mathbf{p}_{-i}$, then $\mathbf{p}$ has the *Weak Equilibrium* property for player i in the game G^E : for each pair of strategies $\ell, \ell' \in \sigma(p_i)$, $E_i(p_i^\ell, \mathbf{p}_{-i}) = E_i(p_i^{\ell'}, \mathbf{p}_{-i})$.

3.5 The Optimal-Value Property

We show in [25, Theorem 9]:

Theorem 1 (The Optimal-Value Property). *Fix a game G , a player $i \in [n]$ and a partial mixed profile $\mathbf{p}_{-i}$. Assume that (A.1) the valuation $V_i(p_i, \mathbf{p}_{-i})$ is strictly quasiconcave in p_i , and (A.2) $\hat{p}_i$ is a V_i -best response to $\mathbf{p}_{-i}$. Then, for any mixed strategy q_i with $\sigma(q_i) \subseteq \sigma(\hat{p}_i)$, $V_i(q_i, \mathbf{p}_{-i}) = V_i(\hat{p}_i, \mathbf{p}_{-i})$.*

4 Valuations

4.1 Definition

We introduce E-strict concavity from [24, Definition 2.2]:

Definition 4 (E-Strict Convexity, E-Strict Concavity). *Fix a player i . The $(E + R)$ -valuation V_i is E-strictly convex (resp., E-strictly concave) if for every game G , the following conditions hold for a fixed partial mixed profile $\mathbf{p}_{-i}$:*

- (1) V_i is convex (resp., concave) in the mixed strategy p_i .
- (2) For a pair of mixed strategies $p'_i, p''_i \in \Delta(S_i)$, if $E_i(p'_i, \mathbf{p}_{-i}) \neq E_i(p''_i, \mathbf{p}_{-i})$, then for any $\lambda \in (0, 1)$,

$$V_i(\lambda p'_i + (1 - \lambda)p''_i, \mathbf{p}_{-i}) < \max\{V_i(p'_i, \mathbf{p}_{-i}), V_i(p''_i, \mathbf{p}_{-i})\}$$

(resp., $V_i(\lambda p'_i + (1 - \lambda)p''_i, \mathbf{p}_{-i}) > \min\{V_i(p'_i, \mathbf{p}_{-i}), V_i(p''_i, \mathbf{p}_{-i})\}$).

4.2 The Weak-Equilibrium-for-Expectation Property

We show in [24, Proposition 3.2]:

Theorem 2. *Take a player $i \in [n]$ where V_i is E -strictly concave. Then, V has the Weak-Equilibrium-for-Expectation property for player i .*

4.3 Valuation Functions

We will consider the following valuation functions, where Var , SD and ESR denote variance, standard derivation and the Extended Sharpe Ratio, respectively. CVar denotes Conditional Value at Risk and will be defined and studied in Sect. 5. ESR is an adjustment of the Sharpe Ratio to minimization games.

- (1) $V = E + \gamma \cdot \text{Var}, \gamma > 0$
- (2) $V = E + \gamma \cdot \text{SD}, \gamma > 0$
- (3) $V^\nu = \nu^{-1}(\sum_{s \in S} p(s) \cdot \nu(\mu_i(s)))$
- (4) $\text{ESR} = \frac{M \cdot E}{M - \text{Var}}, M \text{ constant}$
- (5) CVar

Here $E(p) = \sum_{s \in S} p(s) \mu_i(s)$, $\text{Var}(p) = \sum_{s \in S} p(s) (\mu_i(s) - E_i(p))^2$, and $\text{SD}(p) = \sqrt{\text{Var}(p)}$. The function ν is increasing and a strictly convex function.

The valuation V_i^ν is popular in Acturial Risk Theory (see [18]). If $\nu(x) = x^2$, then $V^\nu = \sqrt{\sum_{s \in S} p(s) \mu_i^2(s)}$.

Proposition 2. *The valuation functions Var , SD , $\text{ESR} - E$ and $V^\nu - E$ fulfill the Risk Positivity Property.*

Proposition 3. *The valuation functions $E + \gamma \cdot \text{Var}$, $E + \gamma \cdot \text{SD}$, V^ν are concave. The valuation function ESR is strictly quasiconcave.*

Theorem 3 (Mavronicolas and Monien [24, Lemma 2.3, Corollary 3.3]).

- (i) *The valuation functions $E + \gamma \cdot \text{Var}$ and $E + \gamma \cdot \text{SD}$ are E -strictly concave.*
- (ii) *The valuation functions $E + \gamma \cdot \text{Var}$, $E + \gamma \cdot \text{SD}$ and ESR fulfill the Weak-Equilibrium-for-Expectation property.*

Consider a concave valuation V^ν , for an increasing and strictly convex function ν . Since ν^{-1} is also increasing, a mixed profile $\mathbf{p}$ is a V^ν -equilibrium for a game $\mathbf{G}$ if and only if $\mathbf{p}$ is an E -equilibrium for the game $\mathbf{G}^\nu$ constructed from $\mathbf{G}$ by setting for each player $i \in [n]$ and profile $\mathbf{s} \in S$, $\mu_i^\nu(\mathbf{s}) := \nu(\mu_i(\mathbf{s}))$. Since every game has an E -equilibrium, this implies that there is a V^ν -equilibrium for $\mathbf{G}$, and the associated search problem for a V^ν -equilibrium is total; it is in $\mathcal{PPAD}$ [29] for 2-player games.

The situation is different for the other valuation functions.

For $0 < \delta < 1$, the Crawford game $\mathbf{G}_C(\delta)$ is the 2-player game with bimatrix

$$\begin{pmatrix} \langle 1 + \delta, 1 + \delta \rangle & \langle 1, 1 + 2\delta \rangle \\ \langle 1, 1 + 2\delta \rangle & \langle 1 + 2\delta, 1 \rangle \end{pmatrix}$$

from [24]; it is a generalization of a game from [6, Sect. 4]. $G_C(\delta)$ has no pure equilibrium [24, Lemma 5.11]. We use the Weak-Equilibrium-for-Expectation property to prove:

Theorem 4 (Mavronicolas and Monien [24, Lemma 5.11] and [25, Theorem 12]). *For δ with $0 < \delta < 1$ and $V = E + \gamma \cdot \text{Var}$ or $V = E + \gamma \cdot \text{SD}$ or $V = \text{ESR}$ the game $G_C(\delta)$ has no V -equilibrium.*

4.4 Mean-Variance Preference Functions

Brautbar et al. [5, Sect. 3.1] study a class of valuations, coming from the Mean-Variance paradigm of Markowitz [22] and termed as Mean-Variance Preference Functions and show that for valuations from this class non-pure equilibria are very rare. We rephrase their definition [5, Definition 1] to fit into the adopted setting of minimization games:

Definition 5 (see Brautbar et al. [5]). *Fix a player $i \in [n]$. A Mean-Variance Preference Function is a valuation $V_i(p_i, \mathbf{p}_{-i}) := G_i(E_i(p_i, \mathbf{p}_{-i}), \text{Var}_i(p_i, \mathbf{p}_{-i}))$ which satisfies:*

- (1) $V_i(p_i, \mathbf{p}_{-i})$ is concave in p_i .
- (2) G_i is non-decreasing in its first argument ($E_i(p_i, \mathbf{p}_{-i})$).
- (3) Fix a partial mixed profile $\mathbf{p}_{-i}$ and a nonempty convex subset $\Delta \subseteq \Delta_i = \Delta(S_i)$ such that $V_i(p_i, \mathbf{p}_{-i})$ is constant on Δ . Then, both $E_i(p_i, \mathbf{p}_{-i})$ and $\text{Var}_i(p_i, \mathbf{p}_{-i})$ are constant on Δ .

So, a Mean-Variance Preference Function simultaneously generalizes and restricts the $(E + R)$ -valuations; it generalizes “+” to G but restricts R to Var . Note that condition (3) in the above definition may be seen as a generalization of the Weak-Equilibrium-for-Expectation property conditioned on the assumption that $V_i(p_i, \mathbf{p}_{-i})$ is constant on a nonempty convex set $\Delta \subseteq \Delta_i$. It is proved in [5, Claim 1] that $E + \text{Var}$ is a Mean-Variance Preference Function. Since $E + \text{Var}$ is E -strictly concave (Theorem 3(i)) and has the Optimal-Value property (Theorem 1), the established condition (3) for $E + \text{Var}$ is a special case of our general result that every E -strictly concave valuation has the Weak-Equilibrium-for-Expectation property (Theorem 2).

Theorem 5 (Boyd and Vandenberghe [4, Theorem 1]). *Consider a two-player game with Mean-Variance Preference Functions V_1 and V_2 where, for each player $i \in \{1, 2\}$ and strategy profile $s \in S$, the cost $\mu_i(s) \in \mathbb{R}$ is drawn i.i.d. from an absolutely continuous distribution (with respect to Lebesgue). Then with probability 1 the game does not have a non-pure V -equilibrium.*

5 Conditional Value-at-Risk (CVaR_α)

Our presentation draws from [31, 32].

Fix throughout a confidence level $\alpha \in [0, 1)$. Some of the tools to study the properties of CVaR_α come from *Value-at-Risk*, denoted as VaR_α . They are both examples of a *valuation*, a function from probability distributions to reals. We consider a random variable P , representing a distribution of loss; the cumulative distribution function of P may be continuous or discontinuous. A *discrete* random variable P has a discrete distribution function, making a special case of a discontinuous cumulative distribution function: it takes on values $0 \leq a_1 < a_2 < \dots < a_\ell$ with probabilities $p_j := \mathbb{P}(P = a_j)$, $1 \leq j \leq \ell$; the values $a_1, a_2, \dots, a_\ell$ are called *scenaria* in the risk literature. In this case, we shall identify P with the probability vector $\mathbf{p}$. Note that in defining $\text{VaR}_\alpha(\mathbf{p})$ and $\text{CVaR}_\alpha(\mathbf{p})$, the values $a_1, \dots, a_\ell$ have to be fixed. Denote the cumulative distribution function of $\mathbf{p}$ as $\Pi(\mathbf{p}, z) = \sum_{k \in [\ell] | a_k \leq z} p_k$.

5.1 Value-at-Risk (VaR_α)

The *Value-at-Risk* of P at *confidence level* $\alpha \in [0, 1)$, denoted as $\text{VaR}_\alpha(P)$, is defined as $\text{VaR}_\alpha(P) = \inf \{ \zeta \mid \mathbb{P}[P \leq \zeta] \geq \alpha \}$; this definition applies to both continuous and discontinuous distributions. So, $\mathbb{P}[P \leq \text{VaR}_\alpha(P)] \geq \alpha$, with equality holding when the cumulative distribution function is continuous at $\text{VaR}_\alpha(P)$ — put in other words, when there is no *probability atom* at $\text{VaR}_\alpha(P)$. When P is a discrete random variable, $\text{VaR}_\alpha(\mathbf{p}) = \min \{ a_k \mid \sum_{j=1}^k p_j \geq \alpha \}$. Note that VaR_α is a discontinuous function of α . In this case, denote as $\kappa_\alpha = \kappa_\alpha(\mathbf{p})$ the unique index such that $\sum_{k \in [\kappa_\alpha]} p_k \geq \alpha > \sum_{k \in [\kappa_\alpha - 1]} p_k$. So, $a_{\kappa_\alpha} = \text{VaR}_\alpha(\mathbf{p})$. In the full version of the paper, we give an example establishing that VaR_α is not an $(E + R)$ -valuation. We shall later use an interesting property of VaR_α (cf. [4, Exercise 3.24 (f)]), for which we provide a new proof.

Lemma 2. *With $\alpha \in (0, 1)$, $\text{VaR}_\alpha(\mathbf{p})$ is quasiconcave and quasiconvex in the probabilities, but neither strictly quasiconcave nor strictly quasiconvex.*

5.2 Conditional Value-at-Risk (CVaR_α)

CVaR_α accounts for losses no smaller than VaR_α . For random variables with a continuous cumulative distribution function, the *Conditional Value-at-Risk* of P , denoted as $\text{CVaR}_\alpha(P)$, is the conditional expectation of P subject to $P \geq \text{VaR}_\alpha(P)$ [34, Definition 2]. So, $\text{CVaR}_\alpha(P) = \mathbb{E}(P \mid P \geq \text{VaR}_\alpha(P))$. Note that $\alpha = 0$ is a degenerate case with $\text{CVaR}_0(P) = \mathbb{E}(P)$. The general definition of CVaR_α for random variables with a possibly discontinuous cumulative distribution function is obtained as follows (cf. [32, Proposition 6]). Denote as $\text{CVaR}_\alpha^+(P)$ the conditional expectation of P subject to $P > \text{VaR}_\alpha(P)$; $\text{CVaR}_\alpha^+(P)$ is also called the *Conditional Tail Expectation* or *Upper CVaR* [34]. Set $\lambda_\alpha(P) := \frac{\mathbb{P}[P \leq \text{VaR}_\alpha(P)] - \alpha}{1 - \alpha}$. Note that $\lambda_\alpha(P)$ provides for the possibility that there is a probability atom (that is, the cumulative distribution function is discontinuous) at $\text{VaR}_\alpha(P)$. The numerator of $\lambda_\alpha(P)$ is the “excess” probability mass at $\text{VaR}_\alpha(P)$, i.e., the probability mass beyond α . The denominator $1 - \alpha$ is

the support size of $\mathbb{P}_\alpha$, the α -tail distribution of P , denoted as $\mathbb{P}_\alpha$ and defined as follows: $\mathbb{P}_\alpha[P \leq \zeta] = 0$ if $\zeta < \text{VaR}_\alpha(P)$, and $\mathbb{P}_\alpha[P \leq \zeta] = \frac{\mathbb{P}[P \leq \zeta] - \alpha}{1 - \alpha}$ if $\zeta \geq \text{VaR}_\alpha(P)$. Thus, $\lambda_\alpha(P) = \mathbb{P}_\alpha[P \leq \text{VaR}_\alpha(P)]$. Hence, we obtain the *weighted average formula* for $\text{CVaR}_\alpha(P)$ [32, Proposition 6]:

$$\text{CVaR}_\alpha(P) := \lambda_\alpha(P) \cdot \text{VaR}_\alpha(P) + (1 - \lambda_\alpha(P)) \cdot \text{CVaR}_\alpha^+(P).$$

So, CVaR_α is the weighted average of VaR_α and the expectation of losses strictly exceeding VaR_α . Note that $\text{CVaR}_\alpha(P) = \text{VaR}_\alpha(P)$ when $\lambda_\alpha(P) = 1$; note also that for continuous cumulative distributions, $\lambda_\alpha(P) = 0$ and $\text{CVaR}_\alpha(P) = \text{CVaR}_\alpha^+(P)$, the expected loss strictly exceeding VaR_α . For a discrete random variable P , identified with the probability vector $\mathbf{p}$, the weighted average formula reduces to the *discrete weighted average formula* (cf. [32, Proposition 8]):

$$\text{CVaR}_\alpha(\mathbf{p}) = \frac{1}{1 - \alpha} \left(\left(\sum_{k \in [\kappa_\alpha]} p_k - \alpha \right) a_{\kappa_\alpha} + \sum_{k \geq \kappa_\alpha + 1} p_k a_k \right).$$

Recall the *minimization function* $F_\alpha(\mathbf{p}, z) := z + \frac{1}{1 - \alpha} \sum_{k \in [n] | a_k > z} (a_k - z)p_k$, for a probability vector $\mathbf{p}$ and a number $z \in \mathbb{R}$, from [31, Sect. 2]. Note that F_α is linear in its first argument. The continuity of F_α in z follows from its convexity (in z) [32, Theorem 10]. We provide a direct and simple proof, bypassing convexity, for the discrete setting.

Lemma 3. *The function $F_\alpha(\mathbf{p}, z)$ is continuous in z .*

We now consider some monotonicity properties of $F_\alpha(\mathbf{p}, z)$ in z , drawing on the continuity of F_α (Lemma 3); they offer refinements of properties from [32, Proposition 5], which will be needed later. In order to take into account the possibility that some probabilities may be 0, denote, for an index $j \in [\ell]$, $\text{next}(\mathbf{p}, j) := \min\{k > j \mid p_k \neq 0\}$; $\text{next}(\mathbf{p}, j) := \ell + 1$ when $p_k = 0$ for all $j < k \leq \ell$.

Lemma 4. *The following implications hold:*

- (1) *If $\Pi(\mathbf{p}, \text{VaR}_\alpha(\mathbf{p})) > \alpha$, then:*
 - (1/a) $F_\alpha(\mathbf{p}, z)$ *increases strictly monotone in z for $z \geq \text{VaR}_\alpha(\mathbf{p})$.*
 - (1/b) $F_\alpha(\mathbf{p}, z)$ *decreases strictly monotone in z for $z \leq \text{VaR}_\alpha(\mathbf{p})$.*
- (2) *If $\Pi(\mathbf{p}, \text{VaR}_\alpha(\mathbf{p})) = \alpha$, then:*
 - (2/a) $F_\alpha(\mathbf{p}, z)$ *increases strictly monotone for $z \geq a_{\text{next}(\mathbf{p}, \kappa_\alpha(\mathbf{p}))}$.*
 - (2/b) $F_\alpha(\mathbf{p}, z)$ *is constant for $\text{VaR}_\alpha(\mathbf{p}) \leq z \leq a_{\text{next}(\mathbf{p}, \kappa_\alpha(\mathbf{p}))}$.*
 - (2/c) $F_\alpha(\mathbf{p}, z)$ *decreases strictly monotone for $z \leq \text{VaR}_\alpha(\mathbf{p})$.*

By Lemma 3, Lemma 4 immediately implies a result from [32, Theorem 10]:

Corollary 1. $\text{CVaR}_\alpha(\mathbf{p}) = \min_z F_\alpha(\mathbf{p}, z) = F_\alpha(\mathbf{p}, \text{VaR}_\alpha(\mathbf{p}))$.

We continue to look at two significant properties:

Proposition 4. *With $\alpha \in (0, 1)$, CVaR_α is an $(E + R)$ -valuation.*

Proposition 5. *With $\alpha \in (0, 1)$, CVaR_α is concave in the probabilities.*

We continue with a characterization of the cases where CVaR_α is constant over all convex combinations of probability vectors $\mathbf{p}$ and $\mathbf{q}$; interestingly, one of the two cases requires that $\text{VaR}_\alpha(\mathbf{p}) = \text{VaR}_\alpha(\mathbf{q})$. We use the quasiconcavity and quasiconvexity of VaR_α in the probabilities (Lemma 2) to prove:

Theorem 6 (Mavronicolas and Monien [25, Theorem 3]). *Fix a confidence level $\alpha \in (0, 1)$ and consider probability vectors $\mathbf{p}$ and $\mathbf{q}$ with $\text{VaR}_\alpha(\mathbf{p}) \leq \text{VaR}_\alpha(\mathbf{q})$ and $\text{CVaR}_\alpha(\mathbf{p}) = \text{CVaR}_\alpha(\mathbf{q})$. Then, $\text{CVaR}_\alpha(\lambda \cdot \mathbf{p} + (1 - \lambda) \cdot \mathbf{q})$ is constant for $\lambda \in [0, 1]$ if and only if:*

- (1) $\text{VaR}_\alpha(\mathbf{p}) = \text{VaR}_\alpha(\mathbf{q})$, or:
- (2) $\alpha = \Pi(\mathbf{p}, \text{VaR}_\alpha(\mathbf{p}))$ and $\kappa_\alpha(\mathbf{q}) \leq \text{next}(\mathbf{p}, \kappa_\alpha(\mathbf{p}))$.

Because of Proposition 5 and Theorem 1 the Optimal-Value property holds and is used to prove:

Theorem 7 (Weak-Equilibrium-for- VaR_α Property [25, Theorem 4]). *Fix a confidence level $\alpha \in (0, 1)$. Take a CVaR_α -best response p_i of player $i \in [n]$ to the partial mixed profile $\mathbf{p}_{-i}$, where $p^1, p^2 \in \sigma(p_i)$ with $\text{VaR}_\alpha(\langle p^1, \mathbf{p}_{-i} \rangle) \leq \text{VaR}_\alpha(\langle p^2, \mathbf{p}_{-i} \rangle)$. Then:*

- (1) $\text{VaR}_\alpha(\langle p^1, \mathbf{p}_{-i} \rangle) = \text{VaR}_\alpha(\langle p^2, \mathbf{p}_{-i} \rangle)$, or:
- (2) $\alpha = \Pi(\langle p^1, \mathbf{p}_{-i} \rangle, \text{VaR}_\alpha(\langle p^1, \mathbf{p}_{-i} \rangle))$
and $\kappa_\alpha(\langle p^2, \mathbf{p}_{-i} \rangle) \in [\kappa_\alpha(\langle p^1, \mathbf{p}_{-i} \rangle), \text{next}(\mathbf{p}, \kappa_\alpha(\langle p^1, \mathbf{p}_{-i} \rangle))]$.

Using this theorem we show in [25, Theorem 6]:

Theorem 8. *With $\alpha \in (\frac{1}{3}, 1)$ and $\delta > 0$, $G_C(\delta)$ has no CVaR_α -equilibrium.*

6 Complexity Results

We state the *Master's Theorem for $\mathcal{NP}$ -hardness* from [24, 25]:

Theorem 9 (Mavronicolas and Monien [24, Theorem 5.1] and [25, Theorem 7]). *Fix an $(E + R)$ -valuation $\mathbb{V}$ such that:*

- (1) $\mathbb{V}$ has the Weak-Equilibrium-for-Expectation property or $\mathbb{V}$ has the Optimal-Value property.
- (2) There is a rational number δ with $0 < \delta \leq \frac{1}{4}$ such that:
 - (2/a) $\widehat{\mathbf{R}}(1, 1 + \rho, q) < \frac{1}{2}$ for each probability $q \in [0, 1]$ with $0 < \rho \leq 2\delta$.
 - (2/b) $\widehat{\mathbf{V}}(1, 1 + \rho, r) < \widehat{\mathbf{V}}(1, 2, q)$ for all $0 \leq r \leq q \leq 1$ with $0 < \rho \leq 2\delta$.

(2/c) The Crawford game $G_C(\delta)$ with bimatrix

$$\begin{pmatrix} \langle 1 + \delta, 1 + \delta \rangle & \langle 1, 1 + 2\delta \rangle \\ \langle 1, 1 + 2\delta \rangle & \langle 1 + 2\delta, 1 \rangle \end{pmatrix}$$

has no V-equilibrium.

Then, $\exists V$ -EQUILIBRIUM is strongly $\mathcal{NP}$ -hard for 2-player games.

Property (2/α) holds if $\hat{R}(1, 1 + 2\delta), q$ is continuous in δ and q . And, for the functions $E + \gamma \cdot \text{Var}, E + \gamma \cdot \text{SD}, \text{ESR}$ and CVaR_α property (2/b) can be shown easily. Therefore Theorems 4 and 8 imply:

Theorem 10 (Mavronicolas and Monien [24,25]). *Consider the valuation functions $V = E + \gamma \cdot \text{Var}, V = E + \gamma \cdot \text{SD}, V = \text{ESR}$ or $V = \text{CVaR}_\alpha$ with $1/3 < \alpha < 1$, then $\exists V$ -Equilibrium is strongly $\mathcal{NP}$ -hard for 2-player games.*

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