Chapter 25

Relevance of Learning Logical Analysis of Mathematical Statements

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Abstract Our work focuses on logic and language at a university in Cameroon. The mathematical discourse, carried by the language, generates ambiguities. At the university level, symbolism is introduced to clarify it. Because it is not taught in secondary school, it becomes a source of difficulties for students. Our thesis is as follows: “The determination of the logical structure of mathematical statements is necessary in order to properly use them in mathematics.” We conducted our study in the predicate calculus theory. In the first part of the paper, a summary of the theory is presented, followed by a logical analysis of two complex mathematical statements. The second part is a report of two sequences of an experiment that was conducted with first-year students that shows that knowledge of the logical structure of a statement enables students to clarify the ambiguities raised by language.

Keywords Logic and language • Symbolism • Logical structure of statement
Didactics

25.1 Introduction

The mathematical discourse is carried by language. As such, linguistic ambiguities are unavoidable. We can quote as examples the phrases “two by two” and “all . . . are not,” which may have different¹ meanings according to the context. Besides, interpretation of statements whose quantification is implicit is problematic for a number of students.

The logico-mathematical symbolism introduced in mathematics courses in order to sort out these ambiguities is far from being shared by learners and even represents an obstacle in their understanding of statements. The handling of symbols is

¹See Durand-Guerrier (2013) and Fuchs (1996), respectively.

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learned neither in secondary school nor by university students. Switching from one language to another, namely, from a statement in a natural language to one written exclusively with mathematical variables or other relationship or operation symbols, constitutes for many students a difficult obstacle to overcome (Duval 1988).

Regarding the construction of a proof, Selden and Selden (1995) argue that when students cannot easily make the structure of a logical informal² statement explicit, they cannot easily determine the structure of the proof of these statements. Indeed, the logical structure of statements provides indications of how the proof can be undertaken.

The results of the studies that we listed above and a number of others that we are going to present further lead us to propose the following thesis:

The identification of the logical structure of mathematical statements is necessary for the good use of these statements in the learning of mathematics.

A research question that emerges is:

To what extent will conducting a logical analysis enable us to anticipate and analyze the difficulties students face in handling of mathematical statements?

We are carrying out our research in the framework of predicate calculus, which, according to Durand-Guerrier (2003), is the theory of reference for the analysis of mathematical discourse.

In the first part of this paper, we present some elements of predicate calculus that we used as tools to analyze statements. In the second part, we will illustrate with two examples the relevance of logical analysis of mathematical statements as a tool to anticipate students' difficulties. Indeed, the logical analysis of statements can help to anticipate the difficulties a priori in the determination of the structure of a sentence. In these analyses, we lay emphasis on *logical structure and proof* and *logical structure and language switching*.

In the third part, we present the result of an experiment involving first-year mathematics students. We conclude with the perspectives of the research.

25.2 Predicate Calculus as a Tool for Didactic Analyzing of Mathematic Statements

According to Cori and Lascar (2003), predicate calculus is somehow the first step into formalizing the mathematical activity.

²A statement that deviates from a version in the language of predicate calculus, i.e., it does not use such expressions as “for every,” “there exists,” “and,” “or,” “if . . . then,” or “if and only if” with their variants (Selden and Selden 1995).

25.2.1 *Some Elements of Predicate Calculus*

In predicate calculus, the formal language consists of letters for variables and predicates, symbols for logical connectors, and both existential and universal quantifiers. The fundamental elements are atomic formulas, which are built with predicate letters and variables together, and terms. From atomic formulas, logical connectors, and quantifiers, complex statements can be built. But determining the truth value of these statements no longer obeys in most cases the principle of verifunctionality, as was the case in proposition calculus because in predicate calculus, “the complex propositions are not the aggregates of simpler propositions” (Tarski 1936/1972). Indeed, many complex statements are made of intertwined statements. The notion of satisfaction of the propositional function by an element of the discourse universe initiated by Tarski (1944/1972) allows giving a semantic definition of truth as an extension of propositional calculus.

In mathematics, sentences such as:

Some integers are even. (1)

All integers are even numbers. (2)

are respectively true and false statements. They contain, in the first, the existential quantifier “some” and, in the second, the universal quantifier “all.” These quantifiers are not part of the alphabet of propositional calculus. These sentences are considered in this system as entities and formalized by a letter, which is a propositional variable.

Let consider the mathematical negation of sentence (2):

There is at least one integer that is not even number. (3)

This can be formalized as $\neg p$, where p is interpreted using statement (2).

This formalization does not allow us to notice the change of quantifier from statement (2) to statement (3) and, *a fortiori*, the structure of the two sentences. Therefore, we cannot analyze them.

25.2.2 *Quantification*

In the standard language of predicate calculus, there exist two quantifiers: the noted universal quantifier $\forall$, whose meaning in natural language is “all,” and the existential quantifier $\exists$, which means in spoken language “there exists at least one.”

Given an interpreting domain, the universal quantifier changes an open statement into a true proposition when all the elements in the discourse universe satisfy the open statement;³ if not, the proposition is false. The formalization of a universally

³An open statement is a statement containing a free variable, i.e., a variable that is not in the scope of a quantifier. For instance, “ x is an even number.”

quantified statement is “ $\forall x, P(x)$,” where x is a variable and P is a propositional function.

The existential quantifier transforms an open sentence into a true proposition if at least one element of the discourse universe satisfies the open sentence. In a case where no object satisfies the open sentence, the proposition is false. A formalization of an existential statement is “ $\exists x, P(x)$,” where x and P are as previously defined.

It is worth noting that in common language, the existential quantifier is not always explicit. It is the case of the following statement:

The set A has an upper bound.

To convert a given common language statement into predicate calculus language, we have to clarify its meaning, as we will see later. Let us consider that the implicit quantification of statements can have a major influence on the construction of the negation of such statements.

25.2.3 Implication

A formula of the type $P(x) \Rightarrow Q(x)$, where P and Q are predicates, is interpreted with an open statement. For any element a in the discourse universe, $P(a) \Rightarrow Q(a)$ is a material implication. It is false only if $P(a)$ is true and if $Q(a)$ is false. In the other cases, it is true. We will say in these cases that a satisfies the formula $P(x) \Rightarrow Q(x)$. Therefore, the connector $\Rightarrow$ in predicate calculus is defined from the material implication and is called *open implication*. As in proposition calculus, the contrapositive of the open implication $P(x) \Rightarrow Q(x)$ is the formula $\neg Q(x) \Rightarrow \neg P(x)$. It is an open implication equivalent to the preceding formula.

The formula $P(x) \Rightarrow Q(x)$ is interpreted in a structure by an open statement; it can be closed with a universal or existential quantifier.

The universal enclosure of the previous statement is $\forall x, P(x) \Rightarrow Q(x)$, which is called *formal implication* (Russell 1910/1989) or *conditional cluster* (Quine 1950). This proposition is true when in every instance of x the derived material implication is true. Therefore, it is obvious that to define the formal implication $\forall x, P(x) \Rightarrow Q(x)$, one should introduce each material implication $P(a) \Rightarrow Q(a)$, defined for a given series of objects.

Formal implication will generate two fundamental rules of deduction:

1. If $\forall x(P(x) \Rightarrow Q(x))$ and $P(a)$, then $Q(a)$.
2. If $\forall x(P(x) \Rightarrow Q(x))$ and $\neg Q(a)$, then $\neg P(a)$.

A formal implication being true can be inferred only in two cases:

When for an instance a of x , $P(a)$ is true, or when $\neg Q(a)$ is true.

For the rest, it is not possible to decide without further information.

It is worth noting that mathematical theorems are generally given in the form of a formal implication, but very often the quantifier is omitted. This expert practice does not always enable students to draw the distinction between an open statement and its universal enclosure: This can generate errors in the use of those statements.

25.2.4 Conclusion

We have made a short presentation of some elements of predicate calculus that make it possible to specify the vocabulary that we will use thereafter. Furthermore, unlike proposition calculus, where the sentence is considered as an entity, predicate calculus takes into account quantification and the status of the letters. It provides tools for analyzing complex statements.

The concepts encountered in this framework present a certain complexity in their use (Ben Kilani 2005; Durand-Guerrier 2003; Epp 1999; Njomgang Ngansop 2013; Durand-Guerrier et al. 2014). One finds them in statements whose logical level of complexity is high, because of the structure of these statements and in the way in which concepts are interwoven with them. We intend to highlight the complexity of two mathematical statements based on their logical analysis. This research shall be based on the logical elements presented above; they shall equally enable us do a priori and a posteriori analysis.

25.3 Examples of Logical Analysis of Mathematical Statements

As the students progress in their curriculum, they face mathematical statements that have increasing complexity. This is the case at the university level with the definition of the continuity of a numerical function of a real variable at point x_0 . In secondary school, this definition is introduced with the notion of limit, while at university, it is the mixed or formal language that is used (Bloch and Ghedamsi 2005), but it is not always within the students' reach. This linguistic and mathematical complexity reinforces difficulties in the treatment of statements, but we are not going to linger on it.

In accordance with Quine (1950), we hold that the formalization of mathematical statements contributes to conceptual clarification. This is what guides the logical analyses of the two statements that we propose to examine.

The first statement that we suggest for our analyses is in elementary number theory, and the second one is in calculus.

25.3.1 *The First Conjecture of Goldbach*

(Pb1) An even integer greater or equal to 4 is the sum of two prime numbers.

We specify that the universe in consideration is the set of natural numbers.

We chose this statement for the following reasons:

- It is stated in common language and is apparently simple and understandable by the reader.
- It is a universally quantified conditional statement whose quantification is implicit. As pointed out in Sect. 25.1.2., this practice is a source of difficulties for students.
- Its initial form hides what must be done to prove this conjecture, while its logical structure shows it.

The formalization of this statement requires removing the implicit aspects inherent to natural language.

The proposed statement is in the form “Every A is B ,” where A stands for “even integer greater or equal to 4” and B stands for “the sum of two prime numbers.” According to Epp (1999), this form can be changed to “for all x , if $A(x)$, then $B(x)$,” which is formalized as:

$$\forall x, A(x) \Rightarrow B(x).$$

We are going to paraphrase statement (Pb1) in view of determining the logical structure.

Making explicit the conditional

Suppressing the bounded quantification and introducing a variable

(P1) “For every integer n , if n is even and greater or equal to 4 then, n is the sum of two prime numbers.”

The variable n takes its values from the set of integers.

We have a formal implication where the universal quantification depends on the variable n . The antecedent is “ n is even and greater than 4” and the consequent is “ n can be written as the sum of two prime numbers.”

Up to that point, the formulation of the consequent is not explicit; it concerns formalizing this property: “to be the sum of two prime numbers” by introducing two letters of variable.

Making explicit the existential quantifier

To say “the integer n is the sum of two prime numbers” implies that “one can find two prime numbers of which n is the sum,” or, still, that “there are two prime numbers p and q such that their sum is equal to n .” This clarification thus highlights the underlying existential quantifier.

This conjecture is stated thus:

For every integer n , if n is even and greater than 4, then there are two prime numbers p and q such as their sum is equal to n .

We note:

P : the property interpreted as “be even”

Q : the property interpreted as “be greater than or equal to 4”

S : the ternary relation interpreted as “be the sum of . . . and . . .”

We obtain the formal writing:

$$(P2) \forall n ((P(n) \wedge Q(n)) \Rightarrow (\exists p, \exists q, (P(p) \wedge P(q) \wedge S(n, p, q)))$$

This highlights the statement form, which is a universally quantified conditional. Its antecedent is the conjunction of two atomic formulas, and its consequent is an existential statement.

In the clarification of the conditional which is done above, the limited quantification can be maintained within the set of even integers. The formulation obtained is:

(P3) For every even integer n , if it is greater or equal to 4, then there are two prime numbers whose sum is n .

After the clarification of the existential quantifier in the consequent it becomes:

(P4) For every even integer n , if it is greater or equal to 4, then, there exists two prime numbers whose n is the sum.

We can still delete the limited quantification; this brings about the appearance of a new implication:

(P5) For every integer n , (if n is even, then (if n is greater or equal to 4, then there exist two prime numbers whose sum is n)).

Written formally:

$$(P6) \forall n, [P(n) \Rightarrow (Q(n) \Rightarrow (\exists p, \exists q, P(p) \wedge (q) \wedge S(n, p, q)))]$$

The (P6) written form is equivalent to (P2), for we have the logical equivalence:

$$[p \Rightarrow (q \Rightarrow r)] \equiv [(p \wedge q) \Rightarrow r]$$

The only variable that appears in the writing of (P1) is n , yet in (P2), we need three variables (n, p , and q) defined in $\mathbb{N} \times P \times P$. It is possible to have more variables by raising the formalization level of the statement: That is the case if we have to clarify that “ p and q are prime numbers” and “ n is even.”

The paraphrase and the formalization helped us highlight:

1. The logical structure of the statement,
2. The pertinent variables for its treatment, and
3. The hidden existential quantifier and implicit universal quantifier at the beginning of the statement.

The logical structure of this conjecture gives us guidelines on what to do to make sure a given integer satisfies the implication.

25.3.2 A Fixed-Point Theorem

(u_n) designates a series defined by recurrence with the form “ $u_{n+1} = f(u_n)$ ”, where f is a continuous function in $\mathbb{R}$. We therefore have the following result:

(Pb2) If the series (u_n) is convergent, then its limit is the solution to the equation $f(x) = x$.

The logical reasons for the choice of this statement stem from the fact that:

- we are dealing with a theorem that is stated in combined language and simple at first sight;
- it contains non-explicit quantifiers, which makes its formalization complex; and
- the construction of its contrapositive in common language raises a problem caused by the presence of the anaphora.

Let us start with the study of the logical structure of statement (Pb2).

The study relies on the analyses of Durand-Guerrier (1996, pp. 151–153).

The initial formulation is not the same, but the changes bring it to the same formulation as ours.

We specify that the stated general theorem is well known in Terminal class. It is found in the *Terminale C* mathematics book in the syllabus in Cameroon⁴ and also in the first-year university calculus course.

We are in the presence of a conditional statement whose structure is complex. It involves three distinct mathematical objects: the series (u_n) , an equation, and the numerical function f which links the first two objects. The limit, which is mentioned in the consequent, is implicit in the antecedent. Indeed, to say a series converges means it admits a limit.

(Pb2) can boil down to the minimal statement where the equation is no longer explicit:

If the series (u_n) converges, then its limit is a fixed point of the function.

⁴In the collection CIAM manual *Terminal S*, it is in Chap. 13 (numerical series), paragraph 3, (complements on series), p. 286.

According to Durand-Guerrier (1996),

La simplicité apparente de cet énoncé cache en fait une structure complexe qui apparaît lorsqu'on cherche à le formaliser, même partiellement. L'énoncé donné est d'ailleurs un intermédiaire nécessaire; en effet, pour formaliser l'énoncé, la présence d'un pronom nous oblige à introduire l'objet "limite." (p. 151)⁵

In fact, saying that the function converges, means admitting the existence of a real number l such that $\lim_{n \rightarrow +\infty} u_n = l$. In order to formalize this, the author uses as a discourse universe the reunion of the following sets: the set $\mathbb{R}$ of real numbers, the set of defined and continuous functions in $\mathbb{R}$ and with values in $\mathbb{R}$, and the set of numerical series.

She also chooses:

- a symbol for a two-place relation, R , that states that a series converges towards a given real; $R(u, l)$ is interpreted by “the series $u = (u_n)$ converges towards the real l ;
- a predicate with two places denoted as S that expresses the relation between a series and the associated function: $S(u, f)$ is interpreted as “ $u_{n+1} = f(u_n)$ ”; and
- a two-place predicate T that expressing the relation between a function and a fixed point: $T(f, l)$, which is interpreted as “ l is a fixed point in function f .”

When we are in the discourse universe, the theorem is formalized as:

$$\forall u, \forall f, \forall l, S(u, f) \wedge R(u, l) \Rightarrow T(f, l) \quad (a)$$

Given that $S(u, f)$ is true, the statement (Pb2) is going to be written:

$$R(u, l) \Rightarrow T(f, l) \quad (b)$$

Which is interpreted as “If the series (u_n) converges towards l , then l is a fixed point of f .” This statement is actually quantified. It is written as:

$$\forall l, R(u, l) \Rightarrow T(f, l) \quad (c)$$

And it is interpreted as:

$$\forall l, \lim_{n \rightarrow +\infty} u_n = l \Rightarrow f(l) = l \quad (d)$$

The real l is an intermediary object, necessary in the treatment of this situation, where the objects at stake are the series (u_n) and the equation $f(x) = x$.

⁵The apparent simplicity of this statement hides in fact a complex structure that appears in the formalization process, even partially. The given statement is moreover a necessary intermediary; indeed, to formalize the statement, the use of a pronoun makes us present the object “limit.” [our translation].

Contrary to what one might think, the letter l is connected to the universal quantifier, which the whole statement is on the scope. If that letter is introduced with the existential quantifier, which refers to the antecedent to convey the convergence of the series, we obtain the following open statement in l :

$$\forall f, \forall (u_n), (\exists l, \lim_{n \rightarrow +\infty} u_n = l) \Rightarrow f(l) = l \quad (\text{e})$$

This is in contradiction with the fact that a theorem is a closed statement. Besides, this formulation produces a contrapositive which no longer bears its original meaning, namely, “if the equation $f(x) = x$ does not have a solution, then the series (u_n) does not converge.”

25.3.3 Conclusion

We have analyzed two mathematical statements and highlighted their logical structures.

In the first statement, we move from a sentence with a linear structure (subject/copula/attribute) to a universally quantified conditional whose consequent is an existential statement. The analysis reveals complex logical structure. We will see in Sect. 25.4.2. that many students do not succeed recognizing the logical structure of this statement.

The second statement, given in mixed language, contains non-explicit quantifiers. To make them appear and to determine their real scope is fundamental for the use of the statement, mainly for the construction of its contrapositive, as we will see later.

25.4 An Experiment with Mathematics Undergraduate Students

In December 2010, we administered a questionnaire to 68 mathematics undergraduate students from the Higher Teachers Training College of Yaounde. After administering it, in January 2011 we organized a follow-up module with eight voluntary students who had previously answered the questionnaire. The aim consisted of identifying the representations that these students had when using logical concepts on the one hand and in teaching situations to clarify such concepts on the other hand. In this paper, we are interested in the justifications that the students gave.

The findings that we give stem from:

- For Problem 1, answers to the aforementioned questionnaire by 68 students, referred to as S1 to S68.
- For Problem 2, a task with eight voluntary students who had answered the questionnaire. We divided them into two groups of four people each. The work

was first carried out by each group, and then we put together the results of the two groups. The following results concern the students from the second group, who will be referred to as E5 to E8.

There was a week gap between the two experiments. Results from Problem 1 are from the questionnaire, while those from Problem 2 come from the workshop that followed the questionnaire administration.

25.4.1 Problem 1

Students were asked to write in formal language the following statement:

(Pb1) Every even integer n greater or equal to 4 is the sum of two prime numbers.

While administering the questionnaire, we specified to students that the scope was the set of integers.

Let us recall the formal writing of that statement

$$(P6)\forall n, [\mathcal{P}(n) \Rightarrow (Q(n) \Rightarrow (\exists p, \exists q, P(p) \wedge P(q) \wedge S(n, p, q)))]$$

where:

- Property P is interpreted as “be prime,”
- Property $\mathcal{P}$ is interpreted as “be even,”
- Property Q is interpreted as “be greater or equal to 4,” and
- The ternary relation S is interpreted as “be the sum of . . . and . . . “

25.4.1.1 Results Analysis Grid

The clarification of both the antecedent and consequent underlines several levels of possible formalizations of this statement and brings us to consider possible answers according to two axes:

1. There is global structure of the sentence and explicit domain of quantification at the beginning of the sentence or not. We distinguish:
 - a. There are universally quantified conditional statements where the antecedent and the consequent are respectively the correct expression or not in the formalized language in “ n is an even integer greater than 4,” and “ n can be written as the sum of two prime numbers”
 - b. An equivalence
 - c. Formulations that are not conditional statements and that we have called “linear.” It is a series of conjunctions or statements separated with a comma.

2. Translation of properties and introduction of variables (with or without quantifier).

We will adopt the coding below.

Following the first axis:

UQS: universally quantified conditional statement
 NoQCS: non-quantified conditional statement
 EquQ: universally quantified equivalence
 EqunonQ: non-quantified equivalence
 LQS: linear universally quantified statement
 LnonQS: linear non-quantified statement

Following the second axis:

FrV: free variable

We do not signal bound variables because all should be bound given that we deal with a closed statement.

Examples of classification:

B designates the set of prime numbers

$$\forall n \in \mathbb{N}, (((\exists k \in \mathbb{N}, n = 2k) \text{ et } n \geq 4) \Rightarrow (\exists (p, q) \in \mathbb{N} \times \mathbb{N}, n = p + q)), \quad (25.1)$$

with p and q prime

$$\forall n \in \mathbb{N}, n = 2k, \text{ et } n \geq 4 \Rightarrow n = p + q, p \text{ and } q \text{ prime} \quad (25.2)$$

$$\forall n \in A, (\exists (p, q) \in B \times B, n = p + q) \text{ where } A = \{n \in \mathbb{N} / n \geq 4 \text{ and } n \text{ even}\} \quad (25.3)$$

$$\forall n \in 2\mathbb{N}, \text{ and } n \geq 4, \exists (p, q) \in B^2, n = p + q \quad (25.4)$$

Statement	Structure	Be even and greater than 4	Be the sum of two prime numbers
(1)	Universally quantified conditional in $\mathbb{N}$ UQS	Correctly stated	Prime numbers are introduced by the universal quantifier (UQ) but the property “to be prime” is stated at the end
(2)	A priori universally quantified conditional in $\mathbb{N}$ UQS	Is stated with a free variable to express that n is even k: FrV	The prime numbers are designated with letters of free variables and the property is stated at the end of the consequent FrV: p, q

(continued)

(continued)

Statement	Structure	Be even and greater than 4	Be the sum of two prime numbers
(3)	Linear universally quantified on the set of even numbers greater than 4 LQS	It characterizes the set A, and is correctly stated	Prime numbers are introduced at the beginning of the consequent by the UQ. The formulation is correct
(4)	Linear universally quantified statement in $2\mathbb{N}$	The domain is made up of even numbers, and it is stated that the property be greater than 4	The prime numbers are introduced by the UQ. The formulation is correct

In consideration of what precedes, beyond a small number of possible global structures, we can expect to come across a wide range of formulations for this statement; this is all the more so as the mathematical uses are not homogenous from this viewpoint.

25.4.1.2 A Posteriori Analysis of the Results

Among the 68 students who took the test, only 25 proposed a formally written version of the item.

As we might have expected, productions are different from one to another in general, but we all the same find similar structures. We will present the answers of the students according to the first axis of our a priori analysis. In this table we delete the EqunonQ and LnonQS because all the students who answered the questionnaire proposed the universally quantified statements (Table 25.1).

Analysis according to the first axis

There is global structure of the sentence and explicit domain of quantification at the beginning of the sentence or not.

We have come across four types of formulations:

- Universally quantified conditional statements
- Non-quantified conditional statement
- Equivalence
- Linear quantified statements.

About the quantity of the sentence, except for the answers of two students, the scope of the universal quantifier binding the integer n was not specified, thus making the status of the letter n ambiguous: One cannot say with certainty whether the variable is free or is a generic element. But in the formal point of view, the variable is considered free.

We can attribute relative imprecisions about the quantifiers to school habits where the use of parentheses to mark the scope of the quantifier at the beginning of the sentence is not very common. It is when the quantifier is “internal” to the

Table 25.1 Distribution of answers according to the structure

UQS	NoQCS	EquQ	LQS	No answer
10	1	1	13	43

statement that we specify the scope. Therefore, we can devise a hypothesis for those answers: that the quantifier covers the whole sentence.

The conditional statements

Of the 10 students who produced universally quantified conditional statements, for eight of whom the quantification domain is $\mathbb{N}$. Among these eight answers, the clarification of the conjunction of “ n is even” and “ ≥ 4 ” is present only once. In the antecedents of other answers, these two statements are separated with a comma:

$$S15 : \forall n \in \mathbb{N}, (n \geq 4), \exists k \in \mathbb{N}^*, n = 2k \Rightarrow \exists p_1, p_2 \in P / n = p_1 + p_2$$

$$S30 : \forall n \in \mathbb{N}, n \text{ even}, n \geq 4 \Rightarrow \exists p, q \in P, n = p + q$$

$$S16 : \forall n \in \mathbb{N}, n = 2k, k \in \mathbb{N} \ n \geq 4 \Rightarrow \exists p_1 \text{ and } p_2 \in \mathbb{N}, \text{prime} / n = p_1 + p_2$$

We make the hypothesis that it is the literal version of “every even integer n , greater or equal to 4.”

Five consequents of the conditional statements are existential statements as in the three examples above; the others are not.

Linear statements

As is the case with conditional statements, we get the literal version “Every even integer n greater than or equal to 4” in some linear statements:

$$S27 : \forall n, n = 2k (k \in \mathbb{Z}_+^2) \ n > 4, \exists n_1 \text{ and } n_2 \text{ prime as } n = n_1 + n_2$$

$$S31 : \forall n \in 2\mathbb{N}, \exists (p, q) \in P^2 \text{ such as } \geq 4 \wedge n = p + q, \text{ with } P \text{ a set of prime numbers}$$

S31’s statement has incorrect syntax, and this leads to a modification in the meaning of the initial sentence. We paraphrase that answer thus:

For every even integer n , there is a couple of prime numbers (p, q) such that n is greater than 4 and is the sum of these two prime numbers.

The formulation is unsuitable because 2 is a counter-example to the associated open statement. Besides, we shall note the disappearance of the implication.

Analysis according to the second axis

Translation of properties and introduction of variables

Subsequently, we analyze the answers according to the second axis, that is to say, according to the manner in which the properties are expressed and the variables introduced. According to the clarification of the structure of the statement, except n , which is given in the initial sentence, subsidiary variables are introduced to define the two prime numbers and eventually the parity of an integer. The difficulty at this level could come from introduction of these hidden variables.

Table 25.2 Structure of statements and status of variables

	k FrV	p, q FrV	k, p, q FrV	n FrV	Closed statements	Total
The conditional statements	1	3	2	1	4	11
The linear statements	3	2	3	0	5	13
The equivalence	1					1

In Table 25.2, we have classified the students' answers according to the structure and the different free variables that they contain.

Let us give the name k to the variable which permits us to define the parity and p and q the variables that designate the two prime numbers.

According to the variable n

In S29's answer, if because n is a free variable in the consequent and is bound to the antecedent, then we have an open statement:

$$S29 : (\forall n \in \mathbb{N}, \exists p \in \mathbb{N} - \{0, 1\}, n = 2p) \Rightarrow (\exists p, q \in \mathbb{N}, p \text{ and } q \text{ prime} / n = p + q)$$

This is due to an error in writing the parentheses; the universal quantifier binding n only marks the antecedent.

According to the variable k

We recall that this variable is used to algebraically define the parity of the integer n . Fourteen students chose to explicate the parity as shown below.

Twelve students produced statements where k is a free variable (type (k FrV and k, p, q FrV)). Among their answers, incorrect syntaxes are found:

$$S16 : \forall n \in \mathbb{N}, n = 2k, k \in \mathbb{N} n \geq 4 \Rightarrow \exists p_1 \text{ and } p_2 \in \mathbb{N} \text{ and prime} / n = p_1 + p_2$$

$$S40 : \forall n \in \{2k\}, k \in \mathbb{N}, n \geq 4, n = p_1 + p_2 \text{ with } p_1, p_2 \text{ prime}$$

In S16's answer, one may think the two variables p_1 et p_2 have been bound, but the syntax is incorrect.

In S40's answer, the property "be prime" is at the end, whereas it ought to appear before the writing of n , and the prime numbers ought to have been bound with the existential quantifier.

Three students (S15, S29, and S42) used the existential quantifier to introduce it; k is a bound variable in their answer.

The other students did not make use of it as they have used sets in which the parity of the elements is a characteristic.

According to the variables p and q

The variables p and q should appear in the formalization of the statement in the writing of n as the sum of the two prime integers. They are introduced by the existential quantifier. A student (S02) has introduced them with the universal quantifier, which changes the significance of the statement, becoming: "every even

integer greater or equal to 4 is the sum of two prime numbers whatsoever.” The variables p and q appear as free variables in 10 statements produced by the students (eventually with another name). For example:

S26 : $\forall n \in 2\mathbb{Z}, n \geq 4, n = a + b$ with a, b of whole prime numbers

S67 : $\forall n = 2p, p \geq 2, p \in \mathbb{N} \Rightarrow n = T_1 + T_2$ and $D(T_1) = \{1, T_1\}; D(T_2) = \{1, T_2\}$

Among the 25 students who responded to this item, 16 produced open statements. Among the latter, four students (S40, S41, S44, and S67) neither introduced the letters referring to prime numbers nor the variable k ; three students (S26, S39, and S45) did not introduce the letters which refer to prime numbers; three students (S16, S42, S68) did not introduce k . Formally, the letters in their answer are free variables.

In the 16 responses aforementioned, the students specified the variables' domain after writing them:

$$= 2k, k \in \mathbb{N}$$

$$n = p + q, p, q \text{ prime}$$

$$n = p + q, \text{ where } p \text{ and } q \text{ are prime integers.} \dots$$

We can hypothesize that there are generic elements for those students that they introduced in some way.

25.4.1.3 Conclusion

This exercise has enabled us to account for the difficulties faced by a number of students to identify the implicits in the formulations in common language on the one hand and the management the students made of the variables on the other hand. We can draw the following conclusions:

- The transformation of a statement in the form “all A is B ”: into a statement in the form “ $\forall x, (A(x) \Rightarrow B(x))$ ” is not obvious: The students' answers are close the congruent statements of the initial statement, mostly regarding the antecedent. When the domain is $\mathbb{N}$, the formalization of the expression of “ n even and greater than 4” is not made in the form of a conjunction.
- None of the conditional statements which have been suggested is correct.
- The syntax in the use of symbols is approximate and the phenomenon of imitation that has been seen with students (Gueudet 2008) is clearly there. Before, some denotations that the teachers had used are found in their work.
- The status of variables is not always taken into consideration. Some students gave as a symbolic formulation open statements where several free or generic

variables often appear. The nature of these variables is specified, but in an incorrect syntax from a logic point of view.

- The absence of the existential quantifier produces statements congruent to the initial statement that do not express this statement.
- We find again in the students' productions the same phenomena spotted by Selden and Selden (1995), namely, the poor capacity of students in making explicit the logical structure of informal statements.

In testing our results, we questioned 25 second-year students studying mathematics at the Higher Teachers Training College of Yaounde in 2015 who have given responses similar to the those that undergraduate students gave: None were correct.

In general, the results of the test a priori show that the academic standard does not have any major influence on the students' ability to satisfactorily perform the language shift. We come across practically the same formulations as those of first-year students, whereas the practice of formalism for at least an academic year let us assume that they would be more capable of handling this issue.

25.4.2 Problem 2

This problem is about inference rules: We are interested in the issue of identifying situations that permit or do not permit making deductions. Let us recall the statement:

In what follows, (u_n) is a series defined by recurrence as " $u_{n+1} = f(u_n)$," where f is a continuous function on $\mathbb{R}$. We then have the following result:

(Pb2) If the series (u_n) is convergent, then its limit is the solution to the equation $f(x) = x$.

The question: What can we say about the convergence of the series (u_n) if equation " $f(x) = x$ " does not have a solution?

The table below presents the repartition of responses to this item in the questionnaire.

25.4.2.1 A Problematic Construction of the Contrapositive of (Pb2) by Students

We present and analyze in this part a sequence that happened between students of a group.

Answering the question asked, the students unanimously said that the series does not converge.

1 E6: If the equation $f(x) = x$ does not have a solution, we immediately deduce that f is not convergent. Because if we try to have a look at the . . . the reciprocal . . . the contrapositive of if f is convergent, its limit is the solution of $f(x) = x$, heuuu, no, wait a moment.

5 E5: $f(x) = x$. It is false. You see a little moment, so that seems a bit clear in my head if, that the position . . .

6.1,⁶ so, if $f(x) = x$ does not have a solution, then u_n is not convergent. To me, it looks crystal clear.

In their attempts to justify their answer, finally, the students decide to use the contrapositive of the statement that they formulate in common language as follows:

25 E5: And the contrapositive is very clear! The contrapositive says “if the limit of the series u_n ” . . .

26 E7: . . . is not a solution to the equation $f(x) = x$

The two interventions can be summarized thus:

$$\text{If } \lim_{n \rightarrow +\infty} u_n = l \text{ and } f(l) \neq l \text{ then, } \lim_{n \rightarrow +\infty} u_n \neq l \quad (f)$$

The construction of the contrapositive helps to discover the difficulties related to the implicit quantification. In common language, the literal expression of the contrapositive underlines a contradiction between the negation of the consequent “if the limit of the series does not satisfy the equation” and the negation of the antecedent “the series is not convergent,” which means that the series does not have a finite limit. This is due to the phenomenon of anaphora.

The formal writing permits clarification of the implicit quantification on the object *limit* in expressing the convergence of the series (u_n); this object might have been introduced by the universal quantifier. This writing permits building the contrapositive because it dispels the ambiguity on the status of limit. The difficulties due to passing to the contraposition are dealt with in the debates below.

Regarding the contradiction stated in 25 and 26:

34 E5: I don't think the word limit can be in . . .

35 E7: what will be the contrapositive?

36 E6: Because the limit must first of all exist. Because if they say limit . . ., if you say now that . . .

37 E5: Hum, if the limit . . ., if the limit of the series (u_n) is the solution of . . ., that is to say, . . .

38 E6: if the limit . . ., it already exists, you see, don't you? It already exists . . .

39 E5: . . . and to say after that the series (u_n) is not convergent, this doesn't have meaning. You are following me, so for me I now say that as contrapositive we must say that if the equation $f(x) = x$ does not admit a solution in $\mathbb{R}$, then the series is not convergent. To me, I think that it is the contrapositive. Because as soon as they put

⁶“6.1” is the number of the question in the questionnaire given to the students.

the word *limit*, it creates a sort of misunderstanding; we no longer understand anything.

Having identified the contradiction in the first proposition of the contrapositive, the students use pragmatic arguments to construct it (37, 38, and 39).

These exchanges underline the difficulties related to the relations between the different objects introduced. E5 must substitute (Line 39) the word *limit* for the transformation of the sentence in order to be able to state the contrapositive. He obtains the correct contrapositive that corresponds to the one we have proposed further to the formalization of the initial statement in universally quantifying the letter of the variable that designates the limit. Indeed, the contrapositive of (c) is:

$$\forall l, \text{non } T(f, l) \Rightarrow \text{non } R(u, l) \quad (g)$$

Which is interpreted as:

$$\forall l, f(l) \neq l \Rightarrow \lim_{n \rightarrow +\infty} u_n \neq l \quad (h)$$

That is to say, if the equation $f(x) = x$ does not have solution, then, the series (u_n) does not converge. This is E5's formulation. The latter highlights the difficulty created by the presence of *limit* (39).

25.4.2.2 Conclusion

Table 25.3 shows that, of the 47 students who answered to this question in the questionnaire that has been proposed, 83% answered correctly. This result can be explained by the fact that this theorem is well known to students.

Exchanges above highlight difficulties students feel in justifying their answers through the construction of contrapositive of (Pb2) because of:

- The presence of anaphora and
- The non-explicit quantifiers.

The strategy of the students will be to eliminate the word *limit* (Line 34) in the antecedent of the contrapositive.

Moreover, the status of the series and the variable x remain ambiguous in their proposition of the contrapositive; they seem to be generic elements.

- (1) From these exchanges, we can infer that the statement (Pb2) is appropriate to work, on one hand, on the choice of quantifiers in the formalization activities, and, on the other, on the importance of making quantifiers explicit in order to build the contrapositive of a statement. This problem could also permit making explicit some inference rules that will contribute to lighten certain reasoning. This problem is quite appropriate to work on the choice of quantifiers, firstly in formalization activities and secondly on the importance of clarification of quantifiers in constructing the contrapositive of a statement. This exercise could

Table 25.3 Students’ responses

	The series does not converge	Nothing can be said	Other answer	No answer
Size	39	3	5	21

also clarify certain rules of inference that would contribute to lighten certain reasoning.

Conclusion and perspectives

We have shown the logical analysis of two mathematical statements:

- The first statement is the first of Goldbach’s conjecturesThe hose clarification of the it logical structure that displays in the consequent, the existential quantifier that the formulation in common language was hiding;
- The second is an analysis theorem on $\mathbb{R}$ whose knowledge of the contrapositive is necessary in solving a problem.

These analyses highlight the complexity of those statements. This therefore urges us to question the capability of the students to effectively determine their own structure in their usage of proof-making activities.

Problem 1 shows that the transformation of a statement in the form “All A is B ” into a statement in the form “ $\forall x, (A(x) \Rightarrow B(x))$ ” is not obvious: The students’ answers are close to congruent statements to the initial statement. The syntax in the usage of symbols is approximate and the phenomenon of imitation seen in students by Gueudet (2008) is quite visible.

About Problem 2, the sequences of exchanges show that the knowledge of logical structure on one hand helps dispel the language ambiguities: An alternative among the possible interpretations of a given statement in common language has to be made. On the other, savings in the cognitive point of view can be achieved by students when they know the form of the statements they are working out.

Making the logical structure of mathematical statements explicit is an activity that, given its importance, should be regularly practiced, with an emphasis on the semantic aspect. As a matter of fact, the teacher lecturer has to led the students to give a meaning to the symbols that they use. A perspective of this work is to elaborate situations that will enable students to become familiar with this type of exercise based on statements used in mathematics classes. Another prospect would be to develop the reverse activity, which consists of moving from formal language to common language. Indeed, the switch from formal to common language permits a good understanding of a statement. We believe that such an activity can help develop linguistic and language competences of the subject being learned.

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