Chapter 11
The Role of Algebra in School Mathematics

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Abstract Algebra can be viewed as a language of mathematics; playing a major role for students' opportunities to pursue many different types of education in a modern society. It may therefore seem obvious that algebra should play a major role in school mathematics. However, analyses based on data from several international large-scale studies have shown that there are great differences between countries when it comes to algebra; in some countries algebra plays a major role, while this is not the case in other countries. These differences have been shown consistent over time and at different levels in school. This paper points out and discusses how these differences may interfere with individual students' rights and opportunities to pursue the education they want, and how this may interfere with the societies' need to recruit people to a number of professions.

Keywords Equal rights to education · Low emphasise on algebra
Daily life mathematics · Different profiles in mathematics education

11.1 Introduction

Algebra can be viewed as *a language of mathematics*. It is commonly accepted that competence in a countries language is essential for your opportunities in that country. The same may be said about algebra. Competence in algebra is essential for people across all types of education and professions where they use this language. To learn a country's language takes time, and it matures over time by intensive training through listening and by training to use it yourself, and it is

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G. Kaiser et al. (eds.), *Invited Lectures from the 13th International Congress on Mathematical Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-72170-5_11

usually easier for young children to start learning it than for adults. To some extent the same can be said about algebra; except that in mathematics you learn arithmetic first, as arithmetic is the basis for algebra, so it seems reasonable to start with algebra after some fluency in arithmetic.

In a modern society, everyone goes to school for a long time; a school preparing them for being responsible citizens taking care of their own daily life as well as having a job to support themselves and to contribute to society. We have to ask what type of competence is it reasonable that school emphasises in our societies. Is it enough to teach them some arithmetic and statistics in mathematics to prepare them for their daily life, or do we have to put more effort into learning them the mathematical language algebra? A modern society needs a lot of people well educated in different types of technology such as computer science and engineering. A modern society faces problems related to the environment and economy. In all these domains, competence in the mathematical language algebra is essential. Algebra is an important tool for pursuing a profession in so many domains in our society. It is also important for all types of education in natural sciences as physics, biology, chemistry, or in mathematics itself. If you want to study geometry at a university, you need to be fluent in the language of algebra.

The school is responsible for giving students competence in algebra, and it is, for good reasons, part of school curriculum all over the world. Nevertheless, a number of analyses have shown that the emphasis on students learning algebra varies quite a lot around the world. This paper presents the results of a number of such analyses completed over the two last decades, based on data from different studies and at different levels in school. Drawing on these results, some consequences for individual students and societies not emphasizing the learning of algebra in their schools will be pointed out and discussed.

This include discussions of students equal rights to pursue all types of education and by that have the opportunity for a number of different positions in the society, possible reasons for the low emphasis on algebra in some countries, the relation between pure and applied mathematics, and also some reflection about teaching and learning algebra from the perspective that algebra is a language.

11.2 Different Profiles in Mathematics Education

Since it is commonly accepted that competence in algebra is an important tool for pursuing a number of types of education and profession in a modern high technology society, it may seem obvious that algebra should play a major role in school mathematics. However, a number of analyses based on data from several international large-scale studies have shown that there are great differences between countries when it comes to emphasis on algebra in school mathematics; in some countries algebra plays a major role, while this is not the case in other countries.

International assessment surveys such as TIMSS, TIMSS Advanced, and PISA (IEA 2017a; OECD 2017) aim at establishing reliable and valid scores for students' achievement which can be compared across countries or across groups of pupils within countries, and to relate achievement to various background and context variables that may give ideas about possible indicators for characterization of high performance in mathematics. There has also been an international comparative study of teacher education in mathematics (IEA 2017b; Tatto et al. 2012) collecting the same types of data for students in this type of tertiary education. All these studies also offer opportunities for secondary analyses to answer a number of other research questions. An important research question that has been asked is if it is reasonable to distinguish between different profiles of mathematics education in various countries or groups of countries, and to what extent such profiles seems to be consistent over time and at different levels in school. This paper will especially pay attention to the role of algebra in different groups of countries.

A number of analyses have been conducted based on data from all the studies mentioned above, looking for patterns in what type of content different countries seem to emphasis in mathematics in school. A method commonly used in these analyses is a type of cluster analysis looking for "item-by-country interactions" to investigate similarities and differences between countries or groups of countries across cognitive items. For more about these types of cluster analyses, see Olsen (2006). It has to be recognized that in these analyses, one is talking about *relative performance*. Countries at different levels of performance can therefore show equal patterns for the *type of mathematical content that is emphasized*, since the cluster analysis displays groupings of countries according to similarities in *relative response patterns* across items. Countries in the same group tend to have *relative* strengths and weaknesses in the same items. These types of analyses have been conducted on data from the first TIMSS-study in 1995, and later on data from a number of international studies at different levels in school and with different framework according to the type of mathematical competence that is measured in the study. The analyses of data from TIMSS 1995 concluded that the following clusters of countries formed meaningful profiles from a geographical, cultural or political point of view: English-speaking, German-speaking, East European, Nordic, and East Asian countries (Grønmo et al. 2004). In the following, the paper will concentrate on the four profiles that have revealed consistent profiles in a number of later analyses. The German-speaking profile will not be included because this profile has not been that consistent in later studies. Grønmo et al. also used the residuals in the matrix which was the basis for the cluster analysis in the previous section in order to identify items for which a certain group of countries achieved particularly well or badly. They concluded that typical for the items where East European and East Asian countries seemed to perform relatively best, was that they all focused on classical, pure and abstract mathematics such as algebra and geometry. Contrary to this finding, the Nordic group as well as the English-speaking group performed relatively better on items *closer to daily life mathematics* like estimation and rounding of numbers. The Nordic and the English-speaking groups also scored relatively low on items dealing with more classical abstract mathematics like fractions and algebra.

This type of analyses has later been conducted on data from TIMSS 2003, PISA 2003, TIMSS Advanced 2008, and TEDS-M 2008 (IEA 2017a, b; OECD 2017). The analyses have been conducted over an extended period of time, and there have been different countries participating in the studies which also influence the result. Nevertheless, *all these analyses* have concluded that it seems to be consistent patterns of countries clustering together in a Nordic group, an English-speaking group, an East European group, and an East Asian group (Grønmo et al. 2004; Olsen and Grønmo 2006; Grønmo and Olsen 2006a, b; Grønmo and Pedersen 2017; Blömeke et al. 2013) for mathematics in school. The analyses have been conducted at different levels in school, from lower secondary through upper secondary and even at the teacher education level of mathematics. The studies contributing to data for such analyses have a quite different framework for their testing, and different types of items for testing students' competence. PISA tests students' ability to solve problems presented mostly in some type of daily life context—or some in a more professional context (Wu 2009; Olsen and Grønmo 2006). The context is described with text, tables, and requires quite some reading, it also requires ability to relate and understand different types of information, before students use some mathematics to answer one or more questions. PISA does not have any items testing students' competence in pure algebra (ibid.). TIMSS in lower secondary school has items testing students in pure algebra, and items where algebra is tested in context, but less demanding when it comes to reading than items in PISA. TIMSS Advanced test students in a number of items in pure algebra, and some in context, but complexity in this study is in the mathematical domain, not in reading, as it is in PISA. TEDS-M test students' to become teachers in their understanding of pure algebra, in addition to also testing them from the perspectives of how to teach algebra in school (IEA 2017b). However, all these analyses give consistent results pointing out that it is meaningful to conclude that we have four different profiles in mathematics education that seem to be stable over time, at different levels in education and in different studies independent of the study framework or way of formulating the items of the tests.

Although the analyses reveal four different types of profiles, it is also meaningful to talk about two very different *types of profile*, a conclusion especially interesting from the *perspective of algebra in school*. One type of profile consists of East Asia and East Europe, the other type consists of the English-speaking and the Nordic countries. To summarise, even if there are *distinct differences between each of the four profiles*, we also find clear similarities between the two groups of countries we have linked as belonging to the *same type of profile*. The East Asian and the East European profiles are quite similar in the sense that both groups perform relatively better in traditional mathematical content areas like algebra than in mathematics more closely related to daily life such as data representation and probability. In the same way do the English-speaking and the Nordic profiles reveal similarities, both these groups of countries perform relatively better on data representation and probability and relatively worse in algebra.

The consistent difference between the two types of profiles according to algebra, based on a number of cluster analysis of data from TIMSS, TIMSS Advanced, PISA and TEDS (IEA 2017a, b; OECD 2017) forms an important basis for discussions and conclusions in this paper.

Conclusions about differences according to *emphasis on algebra in different countries* are also supported by other types of analyses. Items in TIMSS, TIMSS Advanced and PISA have been re-categorized according to requirement of algebraic manipulation or not to be solved, then compared with students' success in different countries in solving the items (Hole et al. 2015, 2017).

11.3 Equal Rights to Education

It is interesting from several perspectives that some groups of countries like the Nordic and the English-speaking countries, emphasis teaching and learning of algebra less than other group of countries like East-Asian and East-European countries. It is reasonable to discuss possible consequences of such priorities in school, both for individual students as well as for society at large. If some countries do not give their students the opportunities to achieve the type of competence they need to be successful in todays modern societies, students from these countries will have disadvantages compared to students from other countries where this type of competence is achieved. It may also influence students' possibilities within each country, because if this type of competence is not achieved in school, students with highly educated parents, or economically well suited students may get some help outside school, and for that reason have a much better position to pursue a number of educations and important professions (Grønmo 2015).

Questions about what type of competence and knowledge students need in their life, daily life and in their professional working life; and questions about how the school can give them this type of competence seems therefore highly relevant. Algebra is not likely to be needed that much in daily life, but in a modern highly developed technological society it might be essential for students' possibilities for further education and for getting the job they want. It can also be argued that algebraic competency underpins higher level abstract reasoning, especially when it involves unknowns and generalised relationships. On this basis, algebraic competency is necessary or highly desirable for professional occupations including medicine, management and administrative occupations. In the PISA-study in 2015, 29% of the Norwegian students answered that they saw themselves at the age of 30 having a job categorized as based on some competence in technology, natural science or mathematics (Kjærnsli and Jensen 2016), while only 20% answered the same in 2006. According to the changes seen in our societies, it seems realistic that a high number of people will have these types of professions. OECD have pointed out an increasing need for more education related to science, technology and mathematics to give people a fair opportunity in our modern societies (OECD 2017). They also make strategy reports for a number of countries all over the world, about how to improve their educational system. In the Norwegian report, more competence in mathematics is pointed out as being of essential importance (OECD 2014).

The important issue discussed in this paper is that many professions require some type of competence in mathematics, and especially in the mathematical language algebra, since all these types of professions in one way or another use this mathematical language as a tool. This is true for educations and professions as engineering, economy, computer science, and natural sciences as physics, chemistry or biology. The responsibility for teaching students the algebra they need for further education and professions lies with the school. If this is not provided in school, it will influence student's possibilities to pursue a number of educations based on their home background (Grønmo 2015). This is not in accordance with the goal of equity for access to educations and later professional work that are a main goal in education in so many countries (Ibid.).

The school's responsibility for providing this type of knowledge to their students' is therefore closely related to students' equal rights to education in a changing world, and we have to take into account the direction of development in the society (OECD 2017). There is an ongoing discussion about the need for people with creativity and competence in how to handle changes in many countries including Norway. On the other hand, there seems to be less discussion about the need to emphasise students' learning of basic knowledge in the mathematical language algebra, needed in so many professions. I will argue that basic knowledge in algebra is more to be seen as complementary and necessary for being creative, rather than something opposing creativity. This may not be true for all types of societies, but at least for the highly developed technological society we have in many countries today. Without the language to develop technology and science, creativity is probably not very helpful. Algebra was probably not that important for so many fifty years ago as it is today. But taken into account the changes and challenges we are facing in a modern society (Ibid.), competence in algebra is essential and for that reason also an issue of importance from the perspectives of giving all students the possibility to pursue the education and job positions they want.

Failing to educate students to gain some fluency in the language of algebra may, for obvious reasons, have important consequences for the society, such as the shortage of people in a lot of professions and jobs. But the consequences for each individual student lacking this type of competence are no less serious. And it is especially thought provoking, that the low emphasis on the language algebra is most pronounced in the Nordic countries, well known for their emphasis on equal right for all citizens.

> During the last 50 years, the Nordic welfare state has been established as a unique model, with strong emphasis on equity of access to education of high level of quality. (Yang Hansen et al. 2014, p. 26)

The goal of equity of access to education of high quality may be more pronounced in the Nordic countries than in other countries, but many other countries around the world probably also support this goal.

Important characteristics of the Nordic welfare state, especially after the Second World War, have been free access to education and social mobility. After the war, education has in general been free of charge, at all levels, including college and university level. This has given people from all social classes in the society the possibility of pursuing all types of education. This means that it is no longer that important what type of background you have, economically or intellectually, and it has resulted in a large number of children from working class- and farmer class families having opportunities no one in their family ever had before. Based on the type of education people took, it was possible for people to join more or less any type of profession, and through this gain an influence in the society that their parents and grandparent could only dream of. This social mobility is probably also an important reason for the political stability seen in the Nordic countries today (Grønmo 2015).

But it is difficult to give equal right for all to educations and professions in the society, unless the school takes into account the competence needed for further education and professions today and in the future. We know that technological, economical, and natural science competencies are important today, and they are likely to be even more important in the future (OECD 2017). We also know that the mathematical language algebra is what students need to pursue many types of educations and professions in the society. Today, the Nordic countries seem to put a lot of effort into learning students *to use technology*, but little effort it seems to giving them the tools they need to be actively involved in *developing new technology*. This needs to be reflected upon from the perspective of students' equal right to education, and also from the perspective of social mobility and political stability. Also English-speaking countries seem to face problems by not emphasizing algebra throughout their school system, even if not to the same extent as in the Nordic countries.

In addition, since algebra is the language of generalization and the language of the relationships between quantities (Usiskin 1995) it is the basis for higher level abstract reasoning needed in all professions involving managerial decision making. So failing to develop algebraic competency among learners is denying them access to many occupations beyond science and technology, and is thus an obstacle to the human right of social mobility.

11.4 Reasons for Low Emphasis on Algebra

Analyses of different profiles in mathematics education around the world show notable differences between countries in how much emphasis is put on algebra. A consistent result of these analyses is that the Nordic and the English-speaking countries do not emphasize students' learning of pure mathematics as algebra, in opposition to countries in East Europe and East Asia (Olsen and Grønmo 2006; Grønmo and Olsen 2006a, b; Grønmo 2010). Why this is the case is an interesting question, probably related to development of curricula within individual countries, which again is influenced by the country's culture for school and education. Equal rights to education have been an important force for the development of school in many countries, and especially in the Nordic countries. Slogans as ‘mathematics for all’ have been part of this drive, as has the need for teaching all students the type of mathematics they need in daily life so as to be responsible, active citizens. The need for daily life mathematics has influenced discussion about content in mathematics in school. In particular, international comparative study PISA has highlighted the need for students to be able to use mathematics from such a perspective.

Such a daily life perspective is important, but it is also interesting to see to what extent this perspective has been especially influential in the Nordic and the English-speaking countries in opposition to countries in East Europe and East Asia. Based on data from PISA 2003 Olsen and Grønmo (2006) developed a classification system for analyzing this. All items in PISA were re-classified according to how close they were to “real world” or “daily life” mathematics as a way to further understand of the differences between the different profiles found in mathematics education. Their findings revealed that the profiles of the Nordic and the English-speaking countries were mainly accounted for by this variable, and that this variable had a higher degree of explanatory power than the aspects described in the framework of PISA when it came to understanding the clustering of countries in different profiles. The profile for the Nordic countries was strongly characterized by relatively high performance on items involving some sort of real world mathematics, and the same was true for the English-speaking countries. The East Asian and East European countries, however, achieved relatively lower on items categorized as some sort of real world mathematics. This result, consistent with Grønmo et al.’s (2004) findings, made it reasonable to conclude that real-world mathematics has been *a driving force* for school mathematics in the Nordic and in the English-speaking countries, in contrast to countries in East Europe and East Asia. Other researchers have also pointed out that an emphasis on everyday applications of mathematics has been an important driving force underlying changes in curriculum over the last decades (Mosvold 2009). The needs of mathematics for pupils in their daily lives have received more curricular attention than before, while more formal aspects of mathematics, such as algebra, have been reduced. From the mid-1980s, there has been a lot of discussion about the tendency to give more attention to daily life mathematics; see for example De Lange (1996) and Kilpatrick et al. (2005).

The findings presented here are consistent with those of other researchers in mathematics education, who have suggested that the mathematics curricula in the Nordic countries, as well as in the English-speaking countries, have been heavily influenced by an emphasis on real world mathematics and a daily life perspective on mathematics in compulsory school (Niss 1996; De Lange 1996; Gardiner 2004).

Olsen and Grønmo (2006) also found that there was a tendency for the pupils in the Nordic countries to perform relatively better on easier items, and with a non-significant tendency to achieve lower on items requiring accuracy in calculations. For the English-speaking countries the need for calculation was a significant factor indicating low relative achievement. This suggests that accuracy in calculations is not seen as that important in the Nordic and the English-speaking countries. To what extent an increased focus on daily life mathematics in Norway over the last decades has resulted in little attention being given to accuracy in calculation has also been discussed in several articles based on data from PISA and TIMSS (Bergem et al. 2005; Grønmo 2005; Grønmo and Olsen 2006a, b; Olsen and Grønmo 2006). It seems that it is not only algebra that is not emphasizes in some countries, but also emphasis on accuracy in calculation in arithmetic's seems to have been low.

Olsen and Grønmo (2006) concluded that on average the pupils in the Nordic countries performed relatively better on items with a realistic context, on items which included some sort of graphical material, on low difficulty items, and on items which did not include explicit algebraic expressions. Pupils in the Nordic countries also performed relatively better on items relating to probabilities and statistics in a daily life context (classified as Uncertainty in the PISA framework), and on items that tended to be of a more qualitative type which did not require any accuracy in performance of calculations. These results may also influence students' possibilities to learn algebra; since algebra may be seen as some type of generalization of arithmetic. Students' lack of competence in arithmetic is therefore likely to have a negative effect on students' learning of algebra (Brekke et al. 2000).

It is not only the content of curriculum that is important; it might be also the organization of the curriculum that might be problematic. In Norway are for example goals in the curriculum for grade 1 to grade 10 organized in three year blocs, while it is the teacher, the local school or community who decide what is to be emphasized each year (Utdanningsdirektoratet 2006). This seems to be one reason for why algebra is not emphasized in Norway. Algebra seems to be taken late, with teachers referring to algebra as abstract and therefore difficult to teach and difficult for students to learn. It might be abstract, which is part of why it is so powerful and useful in solving problems in so many different situations. It might also be argued in opposition to this, that since it is abstract, it is important to start learning it early so it can mature over time. Ordinary language, at least in written form, is also abstract; nevertheless, we start early in school so all shall be able to learn it.

Another important issue when it comes to what is emphasised in school is teacher education. The international comparative study TEDS-M 2008 of teacher education showed that the Nordic and the English-speaking countries do not emphasis algebra for their student teachers (Tatto et al. 2012; Grønmo and Onstad 2012; Blömeke et al. 2013). If teachers do not feel they are very competent in algebra themselves, it is understandable why they do not emphasis this mathematical content very much, or at least postpone it as long as possible according to what has to be done according to the curriculum.

You do not need very much algebra in daily life, but you do need it in many educations and professions. Researchers in Finland have warned about the problematic issue that even though Finland is a high achieving country in PISA, this does not reveal the whole truth about mathematics in their schools. Since PISA emphasis daily life mathematics, and not the type of mathematics needed for further education and professions, this study does not give the total picture of mathematics in their school.

> This conflict can be explained by pointing out that the PISA survey measured only everyday mathematical knowledge, something which could be - and in the English version of the survey report explicitly is - called “mathematical literacy”; the kind of mathematics which is needed in high-school or vocational studies was not part of the survey. No doubt, everyday mathematical skills are valuable, but by no means enough. (Astala et al. 2005)

A central question is therefore how to find a balance between what students need in their daily life and their needs to pursue further education and professions in the society. Equally important is a question about how mathematics for daily life and professions best can be implemented in school, taking all levels in school into account. These questions are by no means easy to answer, but they are far too important not to be asked and reflected upon.

11.5 Pure and Applied Mathematics

The discussions about what should constitute mathematics in compulsory school may be understood in the light of the considerable efforts and use of resources to develop education for all citizens in Western societies (Ernest 1991; Skovsmose 1994). The relationship between pure and applied mathematics has been part of these discussions, even if it has been argued that the distinction between pure and applied mathematics may not be very well founded from a historical point of view. Some of the main contributors in mathematics, as Newton, Fermat, Descartes, and Gauss among others, would probably not have recognised the distinction being made today between pure and applied mathematics—indicating that mathematics should be taught as a whole (Kline 1972). However, in a discussion about what should be the content of mathematics in school, this distinction does seem to be relevant and fruitful, as illustrated in the former analyses of different profiles in mathematics education. It also seems relevant in discussions of curriculum and curriculum changes for mathematics. As some have argued, an increasing focus on applied mathematics seems to have resulted in too little attention given to what we may call pure mathematics. Gardiner (2004) has argued that to apply mathematics you need some competence in traditional pure mathematics, and that it is a misunderstanding that teaching applied mathematics is *an alternative* to teaching pure mathematics, even if some seem to believe that. Grønmo (2005) and Grønmo and Olsen (2006a, b) also pointed to problems created by underestimating the importance of pure mathematics and that only emphasizing applied mathematics may be one possible reason for the low performance of Norwegian pupils in studies as TIMSS and PISA, especially on items involving algebra.

Figure 11.1 presents a commonly accepted model of the relationship between pure and applied mathematics taken from an influential United States policy document on standards in mathematics (National Council of Teachers of Mathematics 1989). PISA uses a slightly different form of this model (OECD 2003, p. 38). The right-hand side of the figure represents the mathematical world (what we may refer to as pure mathematics)—an abstract world with well-defined symbols and rules.

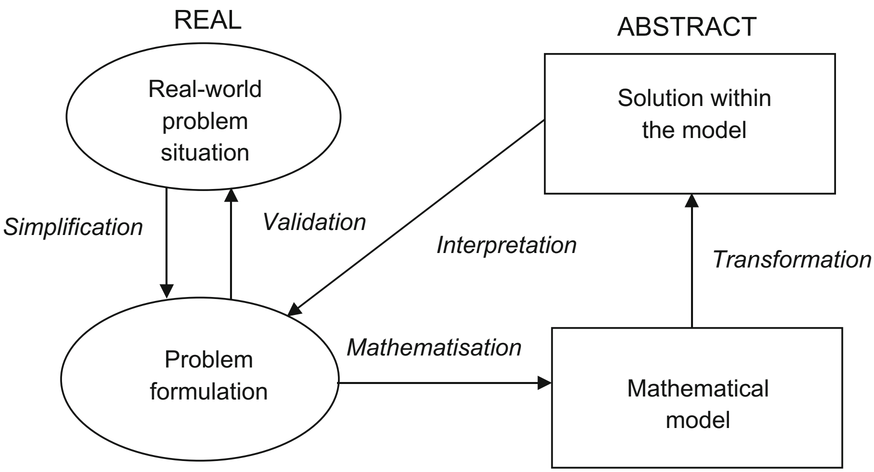

Fig. 11.1 The mathematisation cycle. *Source* National Council of Teachers of Mathematics (1989)

The left-hand side represents the real, concrete world, containing an infinite number of different contexts and situations. The context or situation presented may either be scientific or what might be called daily life. Working with pure mathematics, such as numbers or algebra out of any context, means working only on the right-hand side of the model. In applied mathematics, the starting point is intended to be a problem from the real world, which first has to be simplified, and then mathematized into a model representing the problem. School mathematics rarely starts with a real problem. What is presented as a problem for pupils has in almost every case already been simplified to make it accessible to them.

For any type of applied mathematics, the pupils need to have some knowledge of pure mathematics to find a correct mathematical solution. Applied mathematics can therefore be seen as more complex than pure mathematics, if the same mathematics is involved in the two cases. Gardiner (2004) argues extensively that even if the ability to use mathematics to solve daily life problems is a main goal for school mathematics, this cannot be seen as an alternative to basic knowledge and skills in pure mathematics. It may rather underline the pupils' need for being able to orient them in the world of pure mathematics as a necessary prerequisite to solving real world problems.

PISA aims at embedding all items in a context as close to a real-life situation as possible, while most items in TIMSS are pure mathematical items with no context, or items with a simplified context, as has long been the tradition in school mathematics. TIMSS therefore gives extensive information about pupils' knowledge in pure mathematics—or what may be called traditional school mathematics—while PISA mainly displays pupils' competence in solving items in a daily life context with the use of some mathematical knowledge—what may be referred to as applied mathematics, and usually in a rather complex context (Wu 2009; Olsen and Grønmo 2006; Grønmo and Olsen 2006a, b).

Many countries have as a goal that, on leaving compulsory school, all pupils should have a type of competence that makes them well prepared to solve daily life problems using mathematics. This has been seen as important for active citizens in a modern society, and has by some been referred to as functional numeracy (Niss 1994, 2003; De Lange 1996). The aim of PISA is to test pupils in this type of mathematical competence, defined in the study as Mathematical Literacy.

Countries representing the East European profile performed relatively better in TIMSS than in PISA in 2003 (Grønmo and Olsen 2006a, b). This may indicate that most of the East European countries give little attention to the left-hand side of the mathematisation cycle. The general message that this example serves to communicate is that concentrating only on pure mathematics in school may not be the best if the aim is to foster pupils who are mathematically literate, pupils who can use mathematics to solve the daily life problems they are likely to be exposed to. In contrast to the East European countries, countries representing the East Asian profile as for example Japan, are high achieving in both TIMSS and PISA. This may indicate that pure mathematics is emphasized in the mathematics curriculum in East Asian countries as for example Japan, at the same time as attention is given to the full cycle of applied mathematics.

A European country such as the Netherlands, also high-achieving in both studies in 2003, revealed some clear differences from Japan on performance levels in different topics in TIMSS. Comparing achievement in Japan and the Netherlands in Grade 8, the countries achieved equally well in the topics number, measurement, and data, while there were clear differences between these countries in their achievement levels in algebra and geometry. This indicates that even high-achieving countries may have pronounced differences in what they emphasise in their curriculum. Algebra and geometry seem to be *much more in focus in Japan* than in the Netherlands. But when it comes to achieving well in mathematical literacy, as tested in PISA, the Netherlands is doing just as well as Japan. Grønmo and Olsen (2006a, b) took this as an indication that the "basics" of most importance for daily life mathematics, are the fundamental concepts of number and operations with numbers.

The achievement in algebra has been low for a long time in Norway (IEA 2017a), which have been pointed out in several national report based on data from TIMSS (Grønmo and Onstad 2009; Grønmo et al. 2010, 2012; as well as in articles based on TIMSS studies (Grønmo 2010; Grønmo and Onstad 2013a, b). It is also worth noticing that despite the fact that Norway measured a general improvement in mathematics in grade 8 in TIMSS from 2011 to TIMSS 2015, there was a significant decrease in achievement in algebra (Bergem et al. 2016).

Even if algebra is not the most important content for applying mathematics to solve daily life problems, this content knowledge is highly relevant for those going into studies and professions in need of more mathematical competence. The conclusions pointed out in the discussion of Fig. 11.1 about the need for basic knowledge and skills to be able to apply mathematics is just as relevant for applying algebra as it is for applying number in mathematics problem solving. This aspect received more attention in the TIMSS 2007 report (Grønmo and Onstad 2009), referring to the problems in Norway in recruiting pupils to educational programs and professions requiring knowledge in algebra. In connection to this, the report also posed a critical question about what the Norwegian compulsory school offers to their most talented pupils in mathematics, the pupils who are most likely to be recruited to studies and professions in need of this type of mathematical knowledge (Grønmo et al. 2014).

One consequence of a growing focus on applied mathematics may be that problems arise if too little attention is given to pure mathematics. If pupils lack elementary knowledge and skills with numbers, this is important also for their *possibility to learn algebra*. It has been pointed out that problems pupils have learning algebra in many cases are caused by a *too weak basis in arithmetic* (Brekke et al. 2000). And as already underlined, if talented pupils are not given the opportunity to learn basic concepts and skills in algebra, it will probably lead to later problems in recruiting them to studies and professions in need of such knowledge (Grønmo et al. 2016).

11.6 How to Learn the Mathematical Language Algebra

There are different ways of learning a language, but to be fluent in a language, to be able to use it in a lot of different contexts, a good way is to experience using it in a lot of different situations, and to give students the opportunities to mature their competence over time. To learn algebra, as other languages, we have to take this into account. This indicates that since learning of algebra is essential for so many students in our societies, it is necessary to reflect this throughout the school system. Algebra is not only relevant for what we teach students in lower or upper secondary school, the basis for this language, as with spoken and written languages is laid much earlier. This paper has already pointed to how learning of arithmetic is an important basis for learning algebra (Brekke et al. 2000). But it is also necessary to discuss what part of algebra can be implemented even at lower levels in school.

The curriculum in a Nordic country as Norway put much more emphasize on algebra in the sixties and seventies than today. Textbooks showed that students were exposed to letters as X for a variable number already in grade 1 in the 50s and 60s, for students at the age of seven. The Norwegian curriculum from 1974 indicated that students should start learning about variables in a simple setting from the start of school, and that there should be a special focus on elementary algebra from grades 4–6, with consolidation and expansion of it in grade 7–9. This was a period that is referred to as modern mathematics, with a lot of emphasis on abstract mathematics at all levels in school. It is not a good solution to go back and copy this, because it probably went too far in the abstract and formal direction. But it seems troublesome that after that time there has been a long period with the opposite problem, too little emphasis on pure abstract mathematics as arithmetic and especially on algebra. Education, as many other things in life, is probably more about a balance between different aspects and goals. Some have referred to changes in school as jumping from one ditch to the other, or following the swing of a pendulum (Ernest 1989), but never finding the best balance between all the different and sometimes contradictory goals schools are supposed to handle.

If you want to improve something, a common way is to look at and learn from those who are good at it. However, the Nordic countries seems to have a strong tradition for mostly looking to each other since they share common values and ideology based on geography, culture and history (Bergem et al. 2016; Grønmo et al. 2016). For good reasons, the Nordic countries have pointed out that their societies are stable with a good social system for all citizens. But this is not the same as the Nordic countries being best in everything, nor a good reason for not looking to other countries in the world for improvement.

To do well in algebra, it seems more reasonable to look at some of the East-Asian countries, like Japan, Singapore or Hong Kong. These countries have a very different culture, but nevertheless, we can look at what they are doing, and pick up ideas about what is good and what is not, even if our cultural background is very different. Cultural similarities can be an advantage, but also a disadvantage if taken too far.

A country like Japan has, for example, produced some very interesting videos about their way of teaching algebra to middle school students (TIMSSVIDEO 1999). In this video, they use the differences between students' competences as a resource in their teaching, not a problem, activating their students in interesting discussions. It is well documented that East-Asian countries perform much better in algebra than Nordic and English-speaking countries. We have master students comparing textbooks in Norway and Singapore, especially looking at how they start to teach algebra (Karimzdeh 2014).

We also need to discuss at what level in school we should start teaching algebra to our students. Some countries start teaching formal algebra at early levels in school, while other countries prefer to formalize algebra much later. As already mentioned, it is also interesting to look back to the curriculum in our countries some decades ago, where algebra was also emphasized more than in our present curriculum. This was before the emphasis on daily life mathematics was supposed to, more or less; solve the problems in school mathematics (Gardiner 2004).

The school curriculum has to meet the needs of all students. Not only for those who will need algebra for an academic career, but also for those only needing daily life mathematics. This is the challenge we have to meet in our societies, based on students' needs and how to meet them in compulsory school, as well as in lower and upper secondary school. One source of answers is to look at world wide practices. Looking at countries with different types of educational systems and different cultures, to learn, to discuss and reflect upon their methods to find ways of improving our own system. We can probably learn from everybody, from different countries all over the world, at the same time as we keep in mind that just copying anybody is not the way to do it.

11.7 Summary and Further Research

As pointed out earlier in this paper (ref Sect. 11.5) numbers and algebra constitute important parts of pure mathematics. Working with numbers and algebra out of any context are pure mathematics, while using numbers or algebra to solve a problem in daily life or in professional life are applied mathematics. It is also underlined that to apply any type of mathematics, the students need to be competent in the type of pure mathematics they are supposed to apply (Gardiner 2004). It is especially for their professional life, higher education and work, students need algebra. To little emphasize on pure mathematics as algebra in school is therefore problematic from the point of view to give all students' equal opportunities to pursue the education or professions they want.

A number of analyses based on data from several international large-scale studies have shown that there are great differences between countries when it comes to the role of algebra in school mathematics. While countries in East Asia and East Europe emphasize students' learning of pure mathematics as algebra, this is not the case in the Nordic or English speaking countries. This is likely to interfere with students' rights and opportunities to pursue the education they want, which is especially thought provoking given the common consensus, especially in the Nordic countries, when it comes to students' equal right to education at all levels in school. The fact that education in general is free of charge is important characteristic of the Nordic welfare state. This is based on a consensus about the importance of an equal rights to education for all citizens and at all educational levels.

To what extent a country offers students the opportunity to learn algebra may also interfere with the society's possibilities to recruit people into a number of professions needed in a modern society. This is the case for professions in technology as engineering and computer science, in natural sciences as physics, chemistry, biology, and in studies of economy. Educations or professions using mathematics as a tool need students with basic competence in the mathematical language algebra. We also have to take into account that this type of competence is likely to be more important as our society become more dependent of technology, to solve problems related to environmental problems or problems in economics. It has also been suggested that all higher level professionals need the high level thinking skills that rest on a basic understanding of algebra. Countries can, as many already do, hire people from other countries if their school system does not educate enough people with the competence needed. But not educating the working force needed in a country is not a good solution, neither for individuals in the country, or for the country as a whole. Especially for rich countries spending a lot of resources on education, this seems to be an unsatisfactory or bad solution.

This paper has referred to a number of analyses drawing the same conclusions about the problems arising from the low emphasis algebra in several countries, and also pointed out some problems and perspectives to be discussed in solving this. Also some solutions such as looking to other countries that are successful in teaching and learning algebra have been mentioned. However, we need more research on how this can be done in a good way. More research projects comparing countries with different profiles in mathematics education seems to be one possibility, as a way to learn from each other. Copying other countries ways of dealing with algebra may be tempting, but probably not the best way forward. In a complex field like education, differences in culture are likely to play a major role we must take into account if we want to improve our own educational system. Cooperation and discussions between countries with different educational systems and cultures is more likely to be successful, especially if we include it as part of research about education. Earlier in this paper, reference was made to a tendency in the Nordic countries to mostly compare and discuss our systems in relation to other countries close to us, geographically or culturally. This is probably true for many countries in the world. But just as our societies become more dependent on all types of technology, countries in the world will also be more dependent on cooperation on a broader range, in education as in other fields, than in former times. There have been more references lately to the fact that we share one world together, and that we have to solve and take care of it together, especially when it comes to environmental or economical problems. But this may be just as true when discussing education all over the world, an important factor for solving most problems we are facing.

This paper is heavily based on a number of secondary analyses of data from different type of international comparative studies, as TIMSS, PISA, TIMSS Advanced and TEDS-M. To be able to conduct such type of analyses it is important that countries participate in several of these studies. No study, no matter how good the quality of the study is, can give the best answer on how to improve a country's educational system. All the studies referred to in this paper have quite different frameworks for what they want to test, as well as quite different ways of developing items to cover their frameworks. For this paper, the differences between TIMSS and PISA have played an important role, and giving researchers the possibility to ask and answer more questions about the role of algebra and daily life mathematics than any of these studies could answer alone. Participation in both TIMSS for compulsory school and TIMSS Advanced last year of upper secondary school has also given countries the possibility to see how what is emphasized at one level in school seems to influence other levels in school. In that way, a country will get a much broader view of their own educational system, and better information about how to improve education in their country.

Researchers in Finland have warned about the problem Finland is facing because they mostly only participated in PISA:

> A proper mathematical basis is needed especially in technical and scientific, biology included. The PISA survey tells very little about this basis, which should already be created in comprehensive school. Therefore, it would be absolutely necessary that, in the future, Finland would participate also in international surveys which evaluate mathematical skills essential for further studies. (Astala et al. 2005, https://matematiikkalehtisolmu.fi/2005/erik/PisaEng.html)

International comparative studies have produced a large databank that can be used to answer a high number of interesting research questions. Especially by combining analyses of data from different international comparative studies will give researchers unique possibilities for better answers to their research questions. This type of further research needs to be emphasized to improve education all over the world, which is also likely to influence our chances of solving the problems we are facing today and will face in the future.

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