Variation of the Structural Dynamic Characteristics of the Great Pyramid with the Limestone Properties

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Abstract. The great pyramid has existed for several millennia and persisted several natural disasters. Within this study the structural dynamic characteristics of the great pyramid were studied through modelling the great pyramid using finite element software. An Eigen-value modal analysis was performed generating the natural periods of vibration and the corresponding modes of vibration. These natural periods were compared to the dominant periods of earthquakes in order to judge the degree of the response to loading and whether it is resonant or not. The effect of variation in the properties of the limestone used in the pyramid construction was included in the study by studying a wide range of values of the modulus of elasticity. The findings of the study were found to be valid and representative.

Keywords: Structural dynamics · Earthquake engineering · Pyramid · Modal analysis · Dynamic analysis · Limestone

1 Introduction

The great pyramid located in Giza, Egypt is one of the ancient Seven Wonders of the World. Based on markings within one of the chambers inside the pyramid, Egyptologists believe that this vast structure was constructed by King Khufu (Cheops) of the fourth ancient Egyptian dynasty to function as his tomb. The major attribute that made this structure one of the ancient seven wonders of the world is its large size as it had a height of 149.5 m and a squared base with a side that is 230.4 m long on its construction between the years 2580 and 2560 B.C. It is worth to note that the sides of the pyramid were found to be accurate to the nearest mm which also raised a lot of questions between researchers on the construction methods used at that era to construct such a vast structure with that high degree of precision (Wikimedia Foundation Inc. 2016).

Except for the granite king's chamber, the vast majority of the pyramid structure was made of limestone that is believed to be majorly acquired from the Tura area near Helwan which is located on the other side of the Nile south of the area modernly called greater Cairo. The limestone blocks varied in dimensions according to their locations within the structure with the largest blocks located in the center. The outer lining of the pyramid was made of polished white limestone that was destroyed by a massive earthquake in 1303 AD (Clarke and Engelbach 1991). Unfortunately there was no way

to quantify the intensity of such an earthquake at that time to compare it to current day earthquake events however it is quite interesting to know that the major destruction by that event was only in the casing stones not in the body mass itself. It is also worth to note that although there are a lot of theories about how the great pyramid was constructed (Wikimedia Foundation Inc. 2016; Clarke and Engelbach 1991) however no researcher so far studied how this monument survived natural disasters along the past thousands of years. The current study at hand fills this gap in research.

The current study is based on analyzing the great pyramid dynamically. This study involved performing a free-vibration modal analysis on the pyramid by modelling it numerically using a finite element analysis software. The results of such analysis will be the natural periods of the structure that will enable us to Fig. 1 whether the response of the pyramid to earthquake and wind loads is expected to be resonant or not. However what makes any structural dynamic analysis of such a structure challenging is the mere fact that the material properties of limestone could significantly vary and this applies most to the modulus of elasticity and the strengths of this material (Ahmed 2015; Al-Shayea 2004; Martinez et al. 2012; Palchik and Hatzor 2002). Hence, the modal analysis is performed for a wide range of values of the modulus of elasticity covering the range of values reported by (Ahmed 2015; Al-Shayea 2004; Martinez et al. 2012; Palchik and Hatzor 2002).

Fig. 1. The three dimensional finite element model of the great pyramid.

2 Materials and Methods

2.1 Finite Element Model

The pyramid was modeled on the finite element software SAP2000 (Computers and Structures Inc. 2016) using three dimensional eight-node solid elements as shown in Fig. 1. The total number of solid elements was 256 connected using 345 nodes. This choice of solid elements targeted representing the mass distribution and stiffness within the pyramid structure in the most accurate manner as if such structure was modeled using one dimensional or two dimensional elements there would have been a high loss of accuracy within at least one dimension however the three dimensional elements guarantee the highest possible accuracy in results.

2.2 Material Properties

In order for the dynamic analysis to present realistic results, the material properties of the Tura limestone from which the great pyramid was constructed had to be accurately modeled. Some studies were performed to explore about the static and dynamic properties of limestone however none of them focused on dynamically analyzing the pyramid structure itself. The static modulus of elasticity of Bina limestone and Nekarot limestone ranged between 24800 MPa and 60450 MPa according to a study performed by (Palchik and Hatzor 2002) while Poisson's ratio was ranging between 0.23 and 0.27 according to the same study. On the other hand, the static and dynamic moduli of elasticity were studied by (Al-Shayea 2004); within this study different testing methods were used and the static modulus of elasticity was found to be ranging between 21000 MPa and 66882 MPa for thirteen different types of limestone while their dynamic moduli of elasticity ranged between 23793 MPa and 70895 MPa. A more recent study by (Martinez et al. 2012) involved a comparison between the static and dynamic moduli of elasticity. This comparison showed that the static moduli of elasticity of limestones ranged between 20000 MPa and 73000 MPa while the dynamic moduli of elasticity ranged between 30000 MPa and 85000 MPa. However, no studies were found that reported the specific value of the modulus of elasticity of the Tura limestone that was used to construct the great pyramid. Hence, the current study had to take a wide range of moduli of elasticity into account by performing a sensitivity study in which the moduli of elasticity ranged between 20000 MPa and 80000 MPa that covered the range of values reported by (Ahmed 2015; Al-Shayea 2004; Martinez et al. 2012; Palchik and Hatzor 2002). On the other hand a constant value of Poisson's ratio of 0.25 constituting the average of the values reported by (Palchik and Hatzor 2002) as the standard deviation of this number was about 0.01 which is significantly small and implies that using an average value could be accurate enough.

On the other hand a recent study was performed by (Ahmed 2015) in which the average density of the Tura limestone was found to be 1850 kg/m^3 with a standard deviation of 7 kg/m^3 which is significantly small when compared to the average value. Hence a constant value of limestone density of 1850 kg/m^3 is used within the current study as the variation within the values of this property could be reasonably neglected.

3 Modal Analysis

The modal analysis was performed in order to determine the modes of vibration of the pyramid and the corresponding natural period of each. As shown in Table 1, the modal analysis was repeated seven times for seven different values of Young's modulus ranging between 20000 MPa and 80000 MPa. This study of the sensitivity of the natural periods to changes in the modulus of elasticity was performed due to the fact that there is no specific value of the modulus of elasticity of the Tura limestone of which the pyramid was originally constructed.

Table 1. Variation of the natural periods for different modes with the change in modulus of elasticity.

Mode #	Modal participation factors			Natural period (s)						
	X	Y	Z	E = 20 GPa	E = 30 GPa	E = 40 GPa	E = 50 GPa	E = 60 GPa	E = 70 GPa	E = 80 GPa
1	0.320	0.006	0.000	0.196	0.161	0.139	0.124	0.114	0.105	0.098
2	0.006	0.320	0.000	0.196	0.161	0.139	0.124	0.114	0.105	0.098
3	0.000	0.000	0.005	0.173	0.141	0.122	0.109	0.100	0.092	0.086
4	0.000	0.000	0.000	0.172	0.140	0.122	0.109	0.099	0.092	0.086
5	0.006	0.001	0.000	0.155	0.127	0.110	0.098	0.090	0.083	0.078
6	0.001	0.006	0.000	0.155	0.127	0.110	0.098	0.090	0.083	0.078
7	0.001	0.140	0.000	0.129	0.106	0.091	0.082	0.075	0.069	0.065
8	0.140	0.001	0.000	0.129	0.106	0.091	0.082	0.075	0.069	0.065
9	0.000	0.000	0.190	0.106	0.086	0.075	0.067	0.061	0.056	0.053
10	0.000	0.000	0.000	0.104	0.085	0.074	0.066	0.060	0.056	0.052
11	0.000	0.000	0.140	0.104	0.085	0.074	0.066	0.060	0.056	0.052
12	0.023	0.044	0.000	0.102	0.084	0.072	0.065	0.059	0.055	0.051

The results of the modal analysis shown in Table 1 are ordered in a descending order with the first mode corresponding to the largest natural period and the largest modal participation factor of 32% as a translation in the x direction and 0.6% as a translation in the y direction as shown in Table 1. Also it could be noticed that the second mode of vibration has exactly the same natural period of vibration however the mode itself is changed as the modal participation factor is 32% as a translation in the y direction and 0.6% as a translation in the x direction. This is attributed to the fact that the pyramid base is squared hence the first two translational modes are identical however, and as shown in Figs. 2 and 3 they are acting perpendicular to each other while having the same period as the stiffness and mass in the x direction are exactly equal to those in the y direction hence it is expected to have the principal translational modes in each direction occurring at the same natural period. It could be also noticed that each of these two principal modes represents a perfect translation in one of the directions within the horizontal plane while the 0.6% modal participation in the perpendicular direction is very small and could have even been smaller or ideally zero if a finer mesh was used in the analysis. This could also be seen occurring for modes five

Fig. 2. First mode of vibration

Fig. 3. Second mode of vibration

and six as they both have exactly the same natural period of vibration with the fifth mode having a modal participation factor of 0.6% in the x direction and 0.1% in the y direction while the sixth mode has a modal participation factor of 0.1% in the x direction and 0.6% in the y direction and are perpendicular to each other as shown in Figs. 6 and 7. A similar behavior could also be seen when comparing the seventh mode to the eighth mode as obvious in Figs. 8 and 9.

On the other hand, comparing modes nine, ten, eleven and twelve could show that these modes occur at periods very near to each other while having significantly different vertical modal participation factors representing different degrees of vertical vibrations which could be also seen when comparing the mode shapes in Figs. 10, 11, 12 and 13. However, this does not significantly affect the structure when it comes to vibrations due to earthquakes as the major components of the forces resulting from earthquakes are horizontal while vertical forces due to such events are significantly less in magnitude when compared to their horizontal components. However, such modes could have significant participation when it comes to wind forces that act perpendicular to the surface but are expected to have minor effects when compared to the dead load of the structure itself.

It is also worth to note that modes three, four, nine, ten and eleven have modal participation factors of zeros in the x and y directions, this is attributed to the fact that three of these five modes are vertical moments with twisting components and the other two are perfectly twisting modes of vibration which could be clearly noticed when comparing the mode shapes in Figs. 4, 5, 10, 11 and 12. However, twisting modes are not expected to cause significant effects in cases of the pyramid vibrating under lateral loads as the pyramid is symmetric about the two horizontal axes hence its center of gravity and its center of rigidity nearly coincide on each other causing nearly no twisting moments to act about the vertical axis of the pyramid.

As shown in Table 1, the period for the first mode of vibration ranged between 0.196 s (corresponding to a modulus of elasticity of 20000 MPa) and 0.098 s (corresponding to a modulus of elasticity of 80000 MPa). Figure 14 shows the variation of the principal natural period of the pyramid with the change in the modulus of elasticity. A regression analysis was performed in which the relation between the modulus of elasticity perfectly fitted a power regression in which the period was inversely related to the square root of the modulus of elasticity with a correlation coefficient of 1 reflecting

Fig. 4. Third mode of vibration

Fig. 5. Fourth mode of vibration

Fig. 6. Fifth mode of vibration

Fig. 7. Sixth mode of vibration

Fig. 8. Seventh mode of vibration

Fig. 9. Eighth mode of vibration

Fig. 10. Ninth mode of vibration

Fig. 11. Tenth mode of vibration

Fig. 12. Eleventh mode of vibration

Fig. 13. Twelfth mode of vibration

a high correlation. This proves the validity of the results as in any structural system the natural period of the structure is inversely proportional to the root of its stiffness which is directly related to the modulus of elasticity and hence the natural period of the structure must be inversely proportional to the square root of the modulus of elasticity. Hence, the model is proven to produce valid results when it comes to representing the structural dynamic characteristics of the great pyramid.

On the other hand the twelfth mode of vibration had a period that ranged between 0.102 s (corresponding to a modulus of elasticity of 20000 MPa) and 0.051 s (corresponding to a modulus of elasticity of 80000 MPa). As expected, the remaining modes had natural periods ranging between the 0.051 s and 0.196 s. These ranges of different natural periods corresponding to different modes of vibrations and different moduli of elasticity are all significantly less than the dominant periods of typical earthquakes and typical boundary winds which exceed 0.5 s (Tedesco et al. 1999). That implies that the great pyramid is not expected to resonantly vibrate due to earthquakes or winds and will mainly behave in a quasi-static manner when subjected to such loads with some partial participation of the background component of the response due to the dynamic

Fig. 14. Variation of the principal natural period with the modulus of elasticity.

load. This could explain why the great pyramid survived significantly strong earthquakes during thousands of years as none of these seismic events acted at a period that was equal to the natural periods of the structure hence no major dynamic magnification has occurred due to any of these events.

4 Conclusions

The following conclusions could be drawn from the performed study:

- The natural periods for the twelve modes of vibrations studied ranged between 0.051 s and 0.196 s for moduli of elasticity ranging between 20000 MPa and 80000 MPa.
- The natural periods were proven to be inversely related to the square root of the modulus of elasticity which confirms the validity of the results produced by the modal analysis.
- The natural periods of vibration for the various modes were significantly less than the range of the dominant periods of the earthquake ground motions implying that the vibration is not expected to be as significant as the background and quasi-static components of vibration explaining how the pyramid existed for thousands of years.

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