Geometrical Characterization of the Uniqueness Regions Under Special Sets of Three Directions in Discrete Tomography

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Abstract. The faithful reconstruction of an unknown object from projections, i.e., from measurements of its density function along a finite set of discrete directions, is a challenging task. Some theoretical results prevent, in general, both to perform the reconstruction sufficiently fast, and, even worse, to be sure to obtain, as output, the unknown starting object. In order to reduce the number of possible solutions, one tries to exploit some a priori knowledge. In the present paper we assume to know the size of a lattice grid $\mathcal{A}$ containing the object to be reconstructed. Instead of looking for uniqueness in the whole grid $\mathcal{A}$, we want to address the problem from a *local* point of view. More precisely, given a limited number of directions, we aim in showing, first of all, which is the subregion of $\mathcal{A}$ where pixels are uniquely reconstructible, and then in finding where the reconstruction can be performed quickly (in linear time). In previous works we have characterized the shape of the region of uniqueness (ROU) for any pair of directions. In this paper we show that the results can be extended to special sets of three directions, by splitting them in three different pairs. Moreover, we show that such a procedure cannot be employed for general triples of directions. We also provide applications concerning the obtained characterization of the ROU, and further experiments which underline some regularities in the shape of the ROU corresponding to sets of three not-yet-considered directions.

Keywords: Discrete tomography · Lattice grid · Projection · Uniqueness region

1 Introduction

The retrieval of some characteristics of the internal structure of an object without direct inspection is one of the tasks of computerized tomography. Most of the data that one can achieve come out from a quantitative analysis of the spatial densities of the object, obtained by using bands of parallel high energy beams

along several directions that intersect the object itself. The attenuations of the rays are proportional to the densities of the intersected materials. The most relevant side effect that such a repeated scanning process may have on the object is the gradual changing of its internal structure, so that the object before and after the process is no longer the same. Understandably, it is desirable to obtain the maximum information about the internal structure of the object, delivering the minimum radiation dose. In the case where we consider the unknown structure as subject to a discretization process, we fall within the scope of the discrete tomography (for an overview of the topic see [12]). In this framework, an object turns out to be a finite set of points in the integer lattice, each of them having its own density value. So, an object can be modeled as a $2D$ or $3D$ matrix having integer entries, and whose dimensions are those of the minimal rectangle bounding the object, and it can be visualized in a grey-scale pixel image where each grey level corresponds to a different entry of the matrix.

For each discrete direction u , the quantitative analysis of the object's densities obtained along u can be modeled as an integer vector, say the vector of projections, summing up the values of the pixels that lie on each line parallel to u .

Practical applications require the *faithful* reconstruction of the whole unknown discrete object from projections along a set S of fixed lattice directions that are known a priori and that depend both on the accuracy and on the physics of the scanning machinery.

For only a few lattice directions, this problem is usually referred to as *Reconstruction Problem*, and, in general, it is not quickly solvable (here quickly means in polynomial time with respect to the dimensions of the object itself) even in the case of three directions [7]. However, this result, as many similar ones in the field, relies on the possibility of solving unknown images of arbitrarily large dimensions.

Aware of these theoretical results, the physics of the scanning machinery again plays a relevant role in bounding the area where the analyzed object can lie, allowing, at least, the faithful retrieval of a subset of its pixels. As a consequence, a suitable selection of the projections' directions may allow the reconstruction of the whole object. In [5] we have characterized the shape of subsets of pixels in a generic area, whose values are uniquely determined from two projections, without explicitly computing them. These regions are called Regions of Uniqueness, briefly ROU. Such a characterization relies on the fact that the projections' values along the two directions may recursively interact in order to jointly supply information about the same subsets of pixels, till reaching their full knowledge. Furthermore, we define an algorithm to determine the shape of the ROU that performs faster than computing the density values of its internal points.

In this paper, we start to extend the above results to three directions of projections: some preliminary cases are considered and, for each of them, we define the shape of the related ROU as superposition of rectangular areas whose dimensions are strictly related to those of the directions. Then, for the remaining cases, we provide experimental evidence of the presence of strong regularities in the boundary of the ROU, which is still related to the chosen directions. This allows us to foresee the possibility to fully generalize the ROU characterization to three directions.

The paper is organized as follows. In Sect. 2 the notation is set, together with the main definitions and a summary of the algorithm proposed in [5]. In Sect. 3 the obtained results are proven, while Sect. 4 deals with some experimental results. Section 5 addresses some further investigation and concludes the paper.

2 Definition and Known Results

Assume the image we aim to reconstruct is confined in a finite $m \times n$ grid $\mathcal{A} = \{[z_1, z_2] \in \mathbb{Z} \mid 0 \leq z_1 < m, 0 \leq z_2 < n\}$. We agree to identify each pixel of the image with its bottom-left corner, of coordinates $[i, j]$. We define the image itself as a map $g : \mathcal{A} \rightarrow \mathbb{Z}$, such that an integer number, corresponding to a color, is assigned to each pixel. By an abuse of notation, we will say that we reconstruct the grid instead of the contained image.

Directions are pairs (a, b) of coprime integers; the horizontal and vertical directions are $(1, 0)$ and $(0, 1)$, respectively. We say that a set $S = \{(a_k, b_k) \mid k = 1, \dots, d\}$ of d lattice directions is *valid* for the grid $\mathcal{A}$ if

$$\sum_{k=1}^d |a_k| < m, \quad \sum_{k=1}^d |b_k| < n.$$

We assume that m, n are large enough to guarantee that the various sets S we will consider are valid. A *lattice line* through $[i, j]$ with direction $u = (a, b)$ is the set

$$\{[i + ka, j + kb] \mid k \in \mathbb{Z}\}.$$

The *projections* along the direction u are the sums of the values of pixels on each line with direction u which meets the grid, collected in a pre-assigned order.

This is the usual way of collecting projections in the so-called grid model, which is largely employed also in other approaches to the tomographic problem, such as, for example, in the case of the Mojette transform (see for instance [8, 13]). However, the Mojette reconstruction algorithms usually move from the so-called Katz criterion ([11]), where it is required that

$$\sum_{k=1}^d |a_k| \geq m \quad \text{or} \quad \sum_{k=1}^d |b_k| \geq n.$$

When the Katz criterion is satisfied, one can ensure that the null space of the transform is empty, which means that no ghost exists, so that reconstruction is uniquely determined. This motivates the research of suitable reconstruction algorithms (see, for instance [14]). Differently, when working with sets of valid directions, uniqueness is not automatically guaranteed, so that looking for uniqueness conditions becomes a meaningful tomographic problem. This relies on the fact that, when valid directions are considered, the tomographic problem is in general ill-posed, due to the presence of *switching components*, also known as (*weakly*)

bad configurations. These consist of pairs of sets (Z, W) having the same projections along the given directions. From a bad configuration (Z, W) ghosts can be easily obtained, by giving opposite sign to the weights of the pixels of Z and W , and by setting all the other pixels to value zero. Recent results concerning ghosts in Discrete Tomography can be found in [1]. A grid was proven to be uniquely determined by a set S of directions if and only if it contains no bad configurations along the directions in S (see [6]). It is true, for instance, in the case of non-valid sets of directions.

Therefore, our aim is to reconstruct the grid $\mathcal{A}$, or suitable sub-regions of $\mathcal{A}$ from its projections along a set S of valid lattice directions. Such an approach goes back to [10], where the authors characterized all the possible ghosts contained in the null space of the transform in term of polynomial factorization.

As concerns valid sets, there exist theoretical results stating that we do not need to choose large sets S to achieve uniqueness of reconstruction: as shown in [9] and then completely characterized in [2], four suitably chosen valid directions are enough to get uniqueness in a binary grid (namely, when the image is defined as $g : \mathcal{A} \rightarrow \{0, 1\}$). This result has been extended to higher dimensions in [3]. In many cases, however, we cannot choose the set of employed directions, due to some physical or mechanical constraints. This led us to change perspective and ask a related question: Given a set of arbitrarily chosen directions, are there anyway parts of the grid which can be uniquely reconstructed? The answer is immediate: uniquely determined pixels are the ones which cannot be part of a switching component, so they are confined where switching components cannot be constructed, namely, in the corners of the grid. We then move to another question: Which pixels, among the uniquely determined ones, referred to as (*algebraic*) *region of uniqueness* (algebraic ROU), can be *quickly* reconstructed? In this case, quick means in linear time (we recall that the reconstruction of the whole grid is NP-hard for more than two directions). This question involves the definition of (*geometric*) *region of uniqueness* (geometric ROU), which has already been presented in [4, 5].

Definition 1. *The (geometric) ROU associated to a set S of directions is the set of pixels Q such that*

- (a) *Q belongs to a line, with direction in S , intersecting the grid just in Q , or*
- (b) *Q lies on a line, with direction in S , whose further intersections with $\mathcal{A}$ belong to the previously determined ROU.*

For $|S| = 2$, the ROU has been completely characterized in [5]. We now aim to extend this result to sets of three directions.

Let $a, b, c, d, e, f > 0$ and consider the directions $(-a, b), (-c, d), (-e, f)$; we consider directions with negative slope in order to argue on the configuration in the bottom-left and in the upper-right corners of the grid $\mathcal{A}$. Since the configuration is symmetric in the two corners, we focus on the bottom-left one. We want to compute the ROU associated to the three directions $(-a, b), (-c, d), (-e, f)$. To do so, we exploit the Double Euclidean Division Algorithm (DEDA) shown in [5], that we summarize here for the reader's convenience.

Given a pair $(-a, b), (-c, d)$ of lattice directions, DEDA computes the shape of the ROU in the bottom-left corner of the grid. It executes the Euclidean division between a and c , and b and d , in parallel, checking at each step if one of the remainders is zero, or if the two quotients are not equal. If one of these happens, DEDA stops and returns the shape of the ROU as a list of horizontal and vertical steps. Such a zigzag path starts from the bottom-right pixel of the ROU and ends at its upper-left pixel. If the algorithm does not stop at the initial step, then of some the pixels in the algebraic ROU do not belong to the geometric ROU, and we say that there is *erosion* in the configuration.

Example 1. Consider the pair of directions $(-25, 17), (-11, 7)$. DEDA computes

	<i>horizontal</i>	<i>vertical</i>
<i>level 0</i>	$25 = 2 \cdot 11 + 3,$	$17 = 2 \cdot 7 + 3,$
<i>level 1</i>	$11 = 3 \cdot 3 + 2,$	$7 = 2 \cdot 3 + 1.$

At level 1 the two quotients are not equal, so DEDA stops. The final output, whose zigzag path is $\mathcal{W} = (3, 3, 3, 3, 1, 5, 10, 14, 3, 3, 3, 3, 1, 5)$, is represented in Fig. 1.

Fig. 1. The ROU associated to the pair $(-25, 17), (-11, 7)$ (depicted in the bottom-left corner of the grid).

In this paper we want to describe the shape of the ROU for three directions in some particular cases, when, for any of the three possible pairs, there is no erosion. In [4, 5], the zigzag path associated to the absence of erosion for directions $(-a, b), (-c, d)$ has been proven to be

- for $a > c, b < d$ ([4, Corollary 1]):

$$\mathcal{W} = (b, a, d, c) \quad (1)$$

- if $a > c, b > d$ and moreover the quotients of the division between a and c , and b and d , are different or one of their remainders equals zero ([4, Theorem 3] and [5, Remark 11]):

$$\mathcal{W} = (d, c, b - d, a - c, d, c). \quad (2)$$

In case of erosion, the zigzag path becomes more fragmented; its detailed construction can be found in [5]. In what follows, we will study some special cases by setting $b > d > f$ and $b \geq d + f$. The strict inequalities among the first components a, c, e will determine the shape of the ROU in each case. In the following section we will refer indifferently to the absence of erosion and the corresponding zigzag path.

We will use the notation $R([i, j], [k, l])$ to refer to the rectangle whose bottom-right and upper-left corners are $[i, j]$ and $[k, l]$, respectively. Moreover, we set $ROU(u_1, \dots, u_r)$ to denote the region of uniqueness associated to the directions $u_1, \dots, u_r$ and having its bottom-left corner in the origin of the grid.

3 Theoretical Results

Denote by $\mathcal{B}$ a minimal bad configuration associated to the set $S = \{u_1 = (-a, b), u_2 = (-c, d), u_3 = (-e, f)\}$ and placed such that its leftmost pixel is adjacent to the left side of the grid, and its rightmost pixel is in the lowest row of $\mathcal{A}$. $\mathcal{B}$ consists of eight pixels, whose coordinates are

$$\begin{aligned} B_1 &= [a + c + e, 0], B_2 = [a + c, f], B_3 = [a + e, d], B_4 = [a, d + f], \\ B_5 &= [c + e, b], B_6 = [c, b + f], B_7 = [e, b + d], B_8 = [0, b + d + f]. \end{aligned}$$

Note that the case where one direction is the sum of the other two cannot happen under the hypothesis of absence of erosion.

Since (algebraic) uniqueness of reconstruction is equivalent to the absence of bad configurations, we look for the ROU in the complement of the union of the rectangles having their bottom-left corner in the pixels of $\mathcal{B}$.

For two directions $(-a, b), (-c, d)$, we know from [5] that the absence of erosion corresponds to a ROU which occupies the maximum obtainable area, whose zigzag path is as in Eq. (1) or (2). We remark that, in the mentioned cases, the concepts of algebraic and geometric uniqueness coincide.

A set of pixels is said to be *horizontally and vertically convex* (briefly, hv-convex) if its rows and columns are connected sets. We recall a useful result; for the proof see [5].

Lemma 1 ([5], Theorem 10). *The ROU is hv-convex.*

Lemma 2. $ROU(u_1, u_2) \subseteq ROU(u_1, u_2, u_3)$.

Proof. If a pixel is uniquely determined by the pair of directions u_1, u_2 , then it is uniquely reconstructed also by u_1, u_2, u_3 . $\square$

The following theorem proves the first non-erosive case.

Theorem 1. *Let $S = \{u_1 = (-a, b), u_2 = (-c, d), u_3 = (-e, f)\}$ be a valid set of directions for $\mathcal{A}$, such that $a < c < e$ and $b > d > f$. The shape of the related ROU is described by the path (f, e, d, c, b, a) .*

Proof. By Lemma 2 we have that $\text{ROU}(u_1, u_2)$, defined by the path (d, c, b, a) , is part of $\text{ROU}(u_1, u_2, u_3)$. Consider the rectangle $R_1 = R([a + c + e - 1, 0], [a + c, h - 1])$, adjacent to the right side of $\text{ROU}(u_1, u_2)$ and of size $e \times h$, where $h = \min(f, d - f)$ (see Fig. 2(a)). When moved along u_3 , R_1 first falls in $\text{ROU}(u_1, u_2)$ and then outside the grid $\mathcal{A}$. This means that R_1 is part of $\text{ROU}(u_1, u_2, u_3)$.

Consider now a sub-rectangle of R_1 , say $R_2 = R([a + 2c - 1, 0], [a + c, h - 1])$, whose length is c . A translation along u_2 moves it in $R_3 = ([a + c - 1, d], [a, d + h - 1])$, adjacent to $\text{ROU}(u_1, u_2)$ (Fig. 2(b)), and then outside $\mathcal{A}$. Therefore, also R_3 is part of $\text{ROU}(u_1, u_2, u_3)$. The sub-rectangle $R_4 = R([2a - 1, d], [a, d + h - 1])$ of R_3 , of length a , when moved along u_1 , ends in $R_5 = R([a - 1, b + d], [0, b + d + h - 1])$ (Fig. 2(c)) and then outside $\mathcal{A}$, so also R_5 is included in $\text{ROU}(u_1, u_2, u_3)$. If $\min(f, d - f) = f$, no further pixel is added. If $\min(f, d - f) = d - f$, now a similar argument is applied to the rectangle $R([a + c + e - 1, d - f], [a + c, f - 1])$ in order to add it and its translates to $\text{ROU}(u_1, u_2, u_3)$.

The obtained ROU has profile $\mathcal{W} = (f, e, d, c, b, a)$ and cannot be further extended. This is true since the bad configuration $\mathcal{B}$ is adjacent to $\text{ROU}(u_1, u_2, u_3)$ (Fig. 2(c)) and, by hv-convexity, all pixels in the quarters having a pixel in $\mathcal{B}$ as bottom-left corner cannot be uniquely determined. $\square$

Fig. 2. (a) $\text{ROU}(u_1, u_2)$ (light grey) and R_1 (dark grey). (b) The ROU (light grey) and R_3 (dark grey). (c) The ROU (light grey), R_5 (dark grey) and $\mathcal{B}$ (black).

Following [5], we generalize the theorem above to other cases without erosion.

Theorem 2. Let $S = \{u_1 = (-a, b), u_2 = (-c, d), u_3 = (-e, f)\}$ be a valid set of directions for $\mathcal{A}$, such that $b > d > f$ and $b \geq d + f$. Assume that the three directions do not erode pairwise. The path $\mathcal{W}$ delimiting the ROU is

- (a) for $a < c < e$: $\mathcal{W} = (f, e, d, c, b, a)$;
- (b) for $a < e < c$: $\mathcal{W} = (f, e, d - f, c - e, f, e, b, a)$;
- (c) for $c < a < e$: $\mathcal{W} = (f, e, d, c, b - d, a - c, d, c)$;
- (d) for $c < e < a$:
 - (d.1) if $a > c + e$: $\mathcal{W} = (f, e, d, c, b - d - f, a - c - e, f, e, d, c)$;

- (d.2) if $a \leq c + e$: $\mathcal{W} = (f, e, d, c, b - d, a - c, d, c)$;
- (e) for $e < a < c$: $\mathcal{W} = (f, e, d - f, c - e, f, e, b - f, a - e, f, e)$;
- (f) for $e < c < a$:
 - (f.1) if $a > c + e$: $\mathcal{W} = (f, e, d - f, c - e, f, e, b - d - f, a - c - e, f, e, d - f, c - e, f, e)$;
 - (f.2) if $a \leq c + e$, then $b > d + f$ and $\mathcal{W} = (f, e, d - f, c - e, f, e, b - d, a - c, d - f, c - e, f, e)$.

Proof. The case (a) has already been proven in Theorem 1. We omit details of the proofs of cases (b) – (f1) for brevity. These are quite similar and exploit the same strategy as in Theorem 1, even if they do not follow immediately, since the number of required steps in adding the various rectangles increases and changes from case to case. Concerning case (f.2), we need a special argument. Assume $e < c < a$, $a \leq c + e$, and suppose that $b = d + f$. Since $d > f$, then $d > \frac{b}{2}$, so the quotient between b and d is 1. In order to avoid erosion, the quotient between a and c has to be different from 1, or equal to 1 with remainder zero (namely, $a = c$). The last case is not possible, since $a > c$. Then $a \geq 2c$, but

$$c + e < 2c \leq a,$$

a contradiction. So the case (f.2) is possible only for $b > d + f$. In this case, the same argument as in Theorem 1 can be adapted to prove that $\mathcal{W} = (f, e, d - f, c - e, f, e, b - d, a - c, d - f, c - e, f, e)$. $\square$

In all the mentioned cases, the configuration is maximal for each pair of directions. This means that we take the whole area below the minimal bad configuration $\mathcal{B}$, whose profile corresponds to the aforementioned statements, in the various cases.

We will provide in the next section the profiles that come out from the non-erosive cases of the previous theorems.

4 Experimental Results

In this section, we first determine the ROU associated to given triples, and then argue on synthetic data.

Several triples are considered; their corresponding ROU is depicted in Fig. 3. Each triple corresponds to a case treated in Theorem 2:

- (i) $S = \{(-7, 13), (-8, 5), (-9, 4)\}$, corresponding to case (a) (Fig. 3(a));
- (ii) $S = \{(-5, 19), (-14, 13), (-7, 5)\}$, corresponding to case (b) (Fig. 3(b));
- (iii) $S = \{(-5, 9), (-2, 5), (-7, 2)\}$, corresponding to case (c) (Fig. 3(c));
- (iv) $S = \{(-7, 9), (-2, 5), (-3, 2)\}$, corresponding to case (d.1) (Fig. 3(d));
- (v) $S = \{(-8, 9), (-2, 5), (-7, 2)\}$, corresponding to case (d.2) (Fig. 3(e));
- (vi) $S = \{(-5, 9), (-7, 6), (-3, 2)\}$, corresponding to case (e) (Fig. 3(f));
- (vii) $S = \{(-12, 13), (-6, 7), (-3, 4)\}$, corresponding to case (f.1) (Fig. 3(g));
- (viii) $S = \{(-9, 10), (-7, 5), (-5, 2)\}$, corresponding to case (f.2) (Fig. 3(h)).

Fig. 3. The zigzag profile of the ROU determined by sets S of three directions. (a) $S = \{(-7, 13), (-8, 5), (-9, 4)\}$, corresponding to case (a) of Theorem 2. (b) $S = \{(-5, 19), (-14, 13), (-7, 5)\}$, corresponding to case (b) of Theorem 2. (c) $S = \{(-5, 9), (-2, 5), (-7, 2)\}$, corresponding to case (c). (d) $S = \{(-7, 9), (-2, 5), (-3, 2)\}$, corresponding to case (d.1). (e) $S = \{(-8, 9), (-2, 5), (-7, 2)\}$, corresponding to case (d.2). (f) $S = \{(-5, 9), (-7, 6), (-3, 2)\}$, corresponding to case (e). (g) $S = \{(-12, 13), (-6, 7), (-3, 4)\}$, corresponding to case (f.1). (h) $S = \{(-9, 10), (-7, 5), (-5, 2)\}$, corresponding to case (f.2).

The characterization provided in Theorem 2 can also be exploited to investigate possible Regions Of Interest (ROI), selected in advance. More precisely, by varying the choice of the triples, one can try to partially, or even completely, include the ROI in the ROU. This could be useful in real applications, since reconstruction in the ROU is provided in linear time, due to the uniqueness property. Therefore, one could try to adapt the ROU in order to match some ROI of the image to be reconstructed. As an example, we have considered an (80×80) -sized random phantom, consisting of 50 different colors (or grey levels), and, for a selected triangular ROI, we have explored the different portions that can be reconstructed for each one of the six triples $(-a, b), (-c, d), (-e, f)$ obtained by fixing b, d, f and considering all the possible permutations of a, c, e . Results are shown in Fig. 4. Figure 5 shows the results concerning the same six triples when applied to a (50×50) -sized random phantom, consisting of 20 different colors (or grey levels).

Fig. 4. Perfect and linear time reconstruction of different portions of a triangular ROI. Upper line: Original Phantom. Middle line: The ROU determined by the triples $(-5, 12), (-7, 6), (-13, 3)$ (left), $(-5, 12), (-13, 6), (-7, 3)$ (middle), $(-7, 12), (-5, 6), (-13, 3)$ (right). Bottom line: The ROU determined by the triples $(-13, 12), (-5, 6), (-7, 3)$ (left), $(-7, 12), (-13, 6), (-5, 3)$ (middle), $(-13, 12), (-7, 6), (-5, 3)$ (right) (Color figure online).

Fig. 5. The same results as in Fig. 4, for a (50×50) -sized random phantom consisting of 20 different colors. Some extra-pixels (included in the white squares) outside the ROU can sometimes be reconstructed (Color figure online).

Fig. 6. The ROU associated to the triple $(-11, 8), (-7, 5), (-2, 3)$.

Note that, in this case, for some triples also a few extra-pixels not belonging to the ROU (included in white squares) can be reconstructed. This is obtained by adding to the reconstruction procedure the information that the number of employed grey levels is estimated by the number of grey levels included in the ROU. In this case, if for instance a line L has more than one intersection with $\mathcal{A} \setminus \text{ROU}$, and the projection along L equals zero, then each pixel on L gets zero value. This activates an iterative procedure which possibly allows to gain further pixels not included in the ROU.

5 Conclusions

In this paper we started extending the characterization of the region of uniqueness (ROU), obtained in [5] for pairs of directions, to sets of three directions. We have studied some special cases, when the so-called erosion does not occur, and described the profile of the ROU in the eight possible configurations. This is just a preliminary step, in view of a complete characterization of the shape of the ROU for three directions. In future work we will have to deal with profiles whose path is still not understood (see for instance Fig. 6, where the ROU has been computed by a liner-time program, just by applying Definition 1). We are confident about the fact that the shape of the ROU can be explained in terms of numerical relations among quotients and remainders of the entries of the employed directions.

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