



CHAPTER

1

René Descartes explaining his work to Queen Christina of Sweden. Vector spaces are a generalization of the description of a plane using two coordinates, as published by Descartes in 1637.

Vector Spaces

Linear algebra is the study of linear maps on finite-dimensional vector spaces. Eventually we will learn what all these terms mean. In this chapter we will define vector spaces and discuss their elementary properties.

In linear algebra, better theorems and more insight emerge if complex numbers are investigated along with real numbers. Thus we will begin by introducing the complex numbers and their basic properties.

We will generalize the examples of a plane and ordinary space to \mathbf{R}^n and \mathbf{C}^n , which we then will generalize to the notion of a vector space. The elementary properties of a vector space will already seem familiar to you.

Then our next topic will be subspaces, which play a role for vector spaces analogous to the role played by subsets for sets. Finally, we will look at sums of subspaces (analogous to unions of subsets) and direct sums of subspaces (analogous to unions of disjoint sets).

LEARNING OBJECTIVES FOR THIS CHAPTER

- basic properties of the complex numbers
- \mathbf{R}^n and \mathbf{C}^n
- vector spaces
- subspaces
- sums and direct sums of subspaces

1.A \mathbf{R}^n and \mathbf{C}^n

Complex Numbers

You should already be familiar with basic properties of the set \mathbf{R} of real numbers. Complex numbers were invented so that we can take square roots of negative numbers. The idea is to assume we have a square root of -1 , denoted i , that obeys the usual rules of arithmetic. Here are the formal definitions:

1.1 Definition *complex numbers*

- A **complex number** is an ordered pair (a, b) , where $a, b \in \mathbf{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by \mathbf{C} :

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$

- **Addition and multiplication** on \mathbf{C} are defined by

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i, \\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i;\end{aligned}$$

here $a, b, c, d \in \mathbf{R}$.

If $a \in \mathbf{R}$, we identify $a + 0i$ with the real number a . Thus we can think of \mathbf{R} as a subset of \mathbf{C} . We also usually write $0 + bi$ as just bi , and we usually write $0 + 1i$ as just i .

The symbol i was first used to denote $\sqrt{-1}$ by Swiss mathematician Leonhard Euler in 1777.

Using multiplication as defined above, you should verify that $i^2 = -1$. Do not memorize the formula for the product of two complex numbers; you can always rederive it by recalling that $i^2 = -1$ and then using the usual rules of arithmetic (as given by 1.3).

1.2 Example Evaluate $(2 + 3i)(4 + 5i)$.

Solution

$$\begin{aligned}(2 + 3i)(4 + 5i) &= 2 \cdot 4 + 2 \cdot (5i) + (3i) \cdot 4 + (3i)(5i) \\ &= 8 + 10i + 12i - 15 \\ &= -7 + 22i\end{aligned}$$

1.3 Properties of complex arithmetic

commutativity

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha \text{ for all } \alpha, \beta \in \mathbf{C};$$

associativity

$$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \text{ and } (\alpha\beta)\lambda = \alpha(\beta\lambda) \text{ for all } \alpha, \beta, \lambda \in \mathbf{C};$$

identities

$$\lambda + 0 = \lambda \text{ and } \lambda 1 = \lambda \text{ for all } \lambda \in \mathbf{C};$$

additive inverse

for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$;

distributive property

$$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \text{ for all } \lambda, \alpha, \beta \in \mathbf{C}.$$

The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication. The next example shows how commutativity of complex multiplication is proved. Proofs of the other properties above are left as exercises.

1.4 Example Show that $\alpha\beta = \beta\alpha$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution Suppose $\alpha = a + bi$ and $\beta = c + di$, where $a, b, c, d \in \mathbf{R}$. Then the definition of multiplication of complex numbers shows that

$$\begin{aligned} \alpha\beta &= (a + bi)(c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

and

$$\begin{aligned} \beta\alpha &= (c + di)(a + bi) \\ &= (ca - db) + (cb + da)i. \end{aligned}$$

The equations above and the commutativity of multiplication and addition of real numbers show that $\alpha\beta = \beta\alpha$.

1.5 Definition $-\alpha$, subtraction, $1/\alpha$, division

Let $\alpha, \beta \in \mathbf{C}$.

- Let $-\alpha$ denote the additive inverse of α . Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0.$$

- **Subtraction** on \mathbf{C} is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

- For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α . Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1.$$

- **Division** on \mathbf{C} is defined by

$$\beta/\alpha = \beta(1/\alpha).$$

So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation:

1.6 Notation \mathbf{F}

Throughout this book, \mathbf{F} stands for either \mathbf{R} or \mathbf{C} .

The letter \mathbf{F} is used because \mathbf{R} and \mathbf{C} are examples of what are called **fields**.

Thus if we prove a theorem involving \mathbf{F} , we will know that it holds when \mathbf{F} is replaced with \mathbf{R} and when \mathbf{F} is replaced with \mathbf{C} .

Elements of \mathbf{F} are called **scalars**. The word “scalar”, a fancy word for “number”, is often used when we want to emphasize that an object is a number, as opposed to a vector (vectors will be defined soon).

For $\alpha \in \mathbf{F}$ and m a positive integer, we define α^m to denote the product of α with itself m times:

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}}.$$

Clearly $(\alpha^m)^n = \alpha^{mn}$ and $(\alpha\beta)^m = \alpha^m \beta^m$ for all $\alpha, \beta \in \mathbf{F}$ and all positive integers m, n .

Lists

Before defining \mathbf{R}^n and \mathbf{C}^n , we look at two important examples.

1.7 Example \mathbf{R}^2 and \mathbf{R}^3

- The set \mathbf{R}^2 , which you can think of as a plane, is the set of all ordered pairs of real numbers:

$$\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}.$$

- The set \mathbf{R}^3 , which you can think of as ordinary space, is the set of all ordered triples of real numbers:

$$\mathbf{R}^3 = \{(x, y, z) : x, y, z \in \mathbf{R}\}.$$

To generalize \mathbf{R}^2 and \mathbf{R}^3 to higher dimensions, we first need to discuss the concept of lists.

1.8 Definition *list, length*

Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \dots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

Thus a list of length 2 is an ordered pair, and a list of length 3 is an ordered triple.

Many mathematicians call a list of length n an n -tuple.

Sometimes we will use the word **list** without specifying its length. Remember, however, that by definition each list has a finite length that is a nonnegative integer. Thus an object that looks like

$$(x_1, x_2, \dots),$$

which might be said to have infinite length, is not a list.

A list of length 0 looks like this: $()$. We consider such an object to be a list so that some of our theorems will not have trivial exceptions.

Lists differ from sets in two ways: in lists, order matters and repetitions have meaning; in sets, order and repetitions are irrelevant.

1.9 Example *lists versus sets*

- The lists $(3, 5)$ and $(5, 3)$ are not equal, but the sets $\{3, 5\}$ and $\{5, 3\}$ are equal.
- The lists $(4, 4)$ and $(4, 4, 4)$ are not equal (they do not have the same length), although the sets $\{4, 4\}$ and $\{4, 4, 4\}$ both equal the set $\{4\}$.

 \mathbf{F}^n

To define the higher-dimensional analogues of \mathbf{R}^2 and \mathbf{R}^3 , we will simply replace \mathbf{R} with \mathbf{F} (which equals \mathbf{R} or \mathbf{C}) and replace the 2 or 3 with an arbitrary positive integer. Specifically, fix a positive integer n for the rest of this section.

1.10 Definition \mathbf{F}^n

\mathbf{F}^n is the set of all lists of length n of elements of \mathbf{F} :

$$\mathbf{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \dots, n\}.$$

For $(x_1, \dots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \dots, n\}$, we say that x_j is the j^{th} *coordinate* of (x_1, \dots, x_n) .

If $\mathbf{F} = \mathbf{R}$ and n equals 2 or 3, then this definition of \mathbf{F}^n agrees with our previous notions of \mathbf{R}^2 and \mathbf{R}^3 .

1.11 Example \mathbf{C}^4 is the set of all lists of four complex numbers:

$$\mathbf{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbf{C}\}.$$

*For an amusing account of how \mathbf{R}^3 would be perceived by creatures living in \mathbf{R}^2 , read **Flatland: A Romance of Many Dimensions**, by Edwin A. Abbott. This novel, published in 1884, may help you imagine a physical space of four or more dimensions.*

If $n \geq 4$, we cannot visualize \mathbf{R}^n as a physical object. Similarly, \mathbf{C}^1 can be thought of as a plane, but for $n \geq 2$, the human brain cannot provide a full image of \mathbf{C}^n . However, even if n is large, we can perform algebraic manipulations in \mathbf{F}^n as easily as in \mathbf{R}^2 or \mathbf{R}^3 . For example, addition in \mathbf{F}^n is defined as follows:

1.12 **Definition** *addition in \mathbf{F}^n*

Addition in \mathbf{F}^n is defined by adding corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

Often the mathematics of \mathbf{F}^n becomes cleaner if we use a single letter to denote a list of n numbers, without explicitly writing the coordinates. For example, the result below is stated with x and y in \mathbf{F}^n even though the proof requires the more cumbersome notation of (x_1, \dots, x_n) and (y_1, \dots, y_n) .

1.13 **Commutativity of addition in \mathbf{F}^n**

If $x, y \in \mathbf{F}^n$, then $x + y = y + x$.

Proof Suppose $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then

$$\begin{aligned} x + y &= (x_1, \dots, x_n) + (y_1, \dots, y_n) \\ &= (x_1 + y_1, \dots, x_n + y_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \\ &= (y_1, \dots, y_n) + (x_1, \dots, x_n) \\ &= y + x, \end{aligned}$$

where the second and fourth equalities above hold because of the definition of addition in \mathbf{F}^n and the third equality holds because of the usual commutativity of addition in \mathbf{F} . ■

If a single letter is used to denote an element of \mathbf{F}^n , then the same letter with appropriate subscripts is often used

when coordinates must be displayed. For example, if $x \in \mathbf{F}^n$, then letting x equal (x_1, \dots, x_n) is good notation, as shown in the proof above. Even better, work with just x and avoid explicit coordinates when possible.

The symbol ■ means “end of the proof”.

1.14 **Definition** 0

Let 0 denote the list of length n whose coordinates are all 0 :

$$0 = (0, \dots, 0).$$

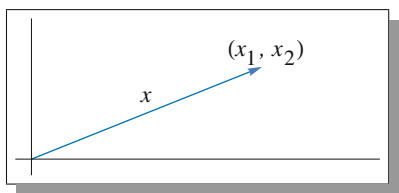
Here we are using the symbol 0 in two different ways—on the left side of the equation in 1.14, the symbol 0 denotes a list of length n , whereas on the right side, each 0 denotes a number. This potentially confusing practice actually causes no problems because the context always makes clear what is intended.

1.15 Example Consider the statement that 0 is an additive identity for \mathbf{F}^n :

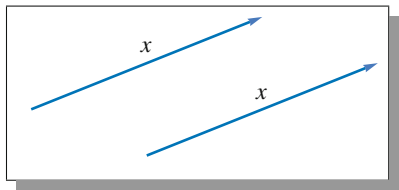
$$x + 0 = x \quad \text{for all } x \in \mathbf{F}^n.$$

Is the 0 above the number 0 or the list 0 ?

Solution Here 0 is a list, because we have not defined the sum of an element of \mathbf{F}^n (namely, x) and the number 0 .



Elements of \mathbf{R}^2 can be thought of as points or as vectors.



A vector.

*Mathematical models of the economy can have thousands of variables, say x_1, \dots, x_{5000} , which means that we must operate in \mathbf{R}^{5000} . Such a space cannot be dealt with geometrically. However, the algebraic approach works well. Thus our subject is called **linear algebra**.*

A picture can aid our intuition. We will draw pictures in \mathbf{R}^2 because we can sketch this space on 2-dimensional surfaces such as paper and blackboards. A typical element of \mathbf{R}^2 is a point $x = (x_1, x_2)$. Sometimes we think of x not as a point but as an arrow starting at the origin and ending at (x_1, x_2) , as shown here. When we think of x as an arrow, we refer to it as a **vector**.

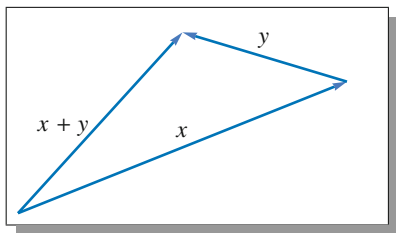
When we think of vectors in \mathbf{R}^2 as arrows, we can move an arrow parallel to itself (not changing its length or direction) and still think of it as the same vector. With that viewpoint, you will often gain better understanding by dispensing with the coordinate axes and the explicit coordinates and just thinking of the vector, as shown here.

Whenever we use pictures in \mathbf{R}^2 or use the somewhat vague language of points and vectors, remember that these are just aids to our understanding, not substitutes for the actual mathematics that we will develop. Although we cannot draw good pictures in high-dimensional spaces, the elements of these spaces are as rigorously defined as elements of \mathbf{R}^2 .

For example, $(2, -3, 17, \pi, \sqrt{2})$ is an element of \mathbf{R}^5 , and we may casually refer to it as a point in \mathbf{R}^5 or a vector in \mathbf{R}^5 without worrying about whether the geometry of \mathbf{R}^5 has any physical meaning.

Recall that we defined the sum of two elements of \mathbf{F}^n to be the element of \mathbf{F}^n obtained by adding corresponding coordinates; see 1.12. As we will now see, addition has a simple geometric interpretation in the special case of \mathbf{R}^2 .

Suppose we have two vectors x and y in \mathbf{R}^2 that we want to add. Move the vector y parallel to itself so that its initial point coincides with the end point of the vector x , as shown here. The sum $x + y$ then equals the vector whose initial point equals the initial point of x and whose end point equals the end point of the vector y , as shown here.



The sum of two vectors.

In the next definition, the 0 on the right side of the displayed equation below is the list $0 \in \mathbf{F}^n$.

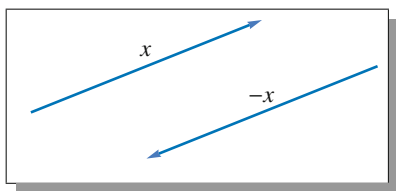
1.16 Definition *additive inverse in \mathbf{F}^n*

For $x \in \mathbf{F}^n$, the *additive inverse* of x , denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \dots, x_n)$, then $-x = (-x_1, \dots, -x_n)$.

For a vector $x \in \mathbf{R}^2$, the additive inverse $-x$ is the vector parallel to x and with the same length as x but pointing in the opposite direction. The figure here illustrates this way of thinking about the additive inverse in \mathbf{R}^2 .



A vector and its additive inverse.

Having dealt with addition in \mathbf{F}^n , we now turn to multiplication. We could define a multiplication in \mathbf{F}^n in a similar fashion, starting with two elements of \mathbf{F}^n and getting another element of \mathbf{F}^n by multiplying corresponding coordinates. Experience shows that this definition is not useful for our purposes. Another type of multiplication, called scalar multiplication, will be central to our subject. Specifically, we need to define what it means to multiply an element of \mathbf{F}^n by an element of \mathbf{F} .

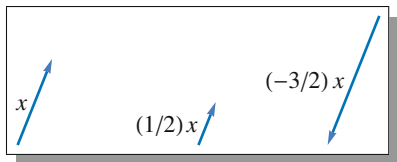
1.17 Definition scalar multiplication in \mathbf{F}^n

The **product** of a number λ and a vector in \mathbf{F}^n is computed by multiplying each coordinate of the vector by λ :

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n);$$

here $\lambda \in \mathbf{F}$ and $(x_1, \dots, x_n) \in \mathbf{F}^n$.

In scalar multiplication, we multiply together a scalar and a vector, getting a vector. You may be familiar with the dot product in \mathbf{R}^2 or \mathbf{R}^3 , in which we multiply together two vectors and get a scalar. Generalizations of the dot product will become important when we study inner products in Chapter 6.



Scalar multiplication.

Scalar multiplication has a nice geometric interpretation in \mathbf{R}^2 . If λ is a positive number and x is a vector in \mathbf{R}^2 , then λx is the vector that points in the same direction as x and whose length is λ times the length of x . In other words, to get λx , we shrink or stretch x by a factor of λ , depending on whether $\lambda < 1$ or $\lambda > 1$.

If λ is a negative number and x is a vector in \mathbf{R}^2 , then λx is the vector that points in the direction opposite to that of x and whose length is $|\lambda|$ times the length of x , as shown here.

Digression on Fields

A **field** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all the properties listed in 1.3. Thus \mathbf{R} and \mathbf{C} are fields, as is the set of rational numbers along with the usual operations of addition and multiplication. Another example of a field is the set $\{0, 1\}$ with the usual operations of addition and multiplication except that $1 + 1$ is defined to equal 0.

In this book we will not need to deal with fields other than \mathbf{R} and \mathbf{C} . However, many of the definitions, theorems, and proofs in linear algebra that work for both \mathbf{R} and \mathbf{C} also work without change for arbitrary fields. If you prefer to do so, throughout Chapters 1, 2, and 3 you can think of \mathbf{F} as denoting an arbitrary field instead of \mathbf{R} or \mathbf{C} , except that some of the examples and exercises require that for each positive integer n we have $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} \neq 0$.

EXERCISES 1.A

- 1 Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a + bi) = c + di.$$

- 2 Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

- 3 Find two distinct square roots of i .
- 4 Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.
- 5 Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.
- 6 Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.
- 7 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.
- 8 Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.
- 9 Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.
- 10 Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

- 11 Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

- 12 Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.
- 13 Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.
- 14 Show that $1x = x$ for all $x \in \mathbf{F}^n$.
- 15 Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.
- 16 Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

1.B Definition of Vector Space

The motivation for the definition of a vector space comes from properties of addition and scalar multiplication in \mathbf{F}^n : Addition is commutative, associative, and has an identity. Every element has an additive inverse. Scalar multiplication is associative. Scalar multiplication by 1 acts as expected. Addition and scalar multiplication are connected by distributive properties.

We will define a vector space to be a set V with an addition and a scalar multiplication on V that satisfy the properties in the paragraph above.

1.18 Definition addition, scalar multiplication

- An **addition** on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Now we are ready to give the formal definition of a vector space.

1.19 Definition vector space

A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

$$u + v = v + u \text{ for all } u, v \in V;$$

associativity

$$(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv) \text{ for all } u, v, w \in V \text{ and all } a, b \in \mathbf{F};$$

additive identity

$$\text{there exists an element } 0 \in V \text{ such that } v + 0 = v \text{ for all } v \in V;$$

additive inverse

$$\text{for every } v \in V, \text{ there exists } w \in V \text{ such that } v + w = 0;$$

multiplicative identity

$$1v = v \text{ for all } v \in V;$$

distributive properties

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv \text{ for all } a, b \in \mathbf{F} \text{ and all } u, v \in V.$$

The following geometric language sometimes aids our intuition.

1.20 Definition *vector, point*

Elements of a vector space are called *vectors* or *points*.

The scalar multiplication in a vector space depends on \mathbf{F} . Thus when we need to be precise, we will say that V is a *vector space over \mathbf{F}* instead of saying simply that V is a vector space. For example, \mathbf{R}^n is a vector space over \mathbf{R} , and \mathbf{C}^n is a vector space over \mathbf{C} .

1.21 Definition *real vector space, complex vector space*

- A vector space over \mathbf{R} is called a *real vector space*.
- A vector space over \mathbf{C} is called a *complex vector space*.

Usually the choice of \mathbf{F} is either obvious from the context or irrelevant. Thus we often assume that \mathbf{F} is lurking in the background without specifically mentioning it.

With the usual operations of addition and scalar multiplication, \mathbf{F}^n is a vector space over \mathbf{F} , as you should verify. The example of \mathbf{F}^n motivated our definition of vector space.

The simplest vector space contains only one point. In other words, $\{0\}$ is a vector space.

1.22 Example \mathbf{F}^∞ is defined to be the set of all sequences of elements of \mathbf{F} :

$$\mathbf{F}^\infty = \{(x_1, x_2, \dots) : x_j \in \mathbf{F} \text{ for } j = 1, 2, \dots\}.$$

Addition and scalar multiplication on \mathbf{F}^∞ are defined as expected:

$$\begin{aligned}(x_1, x_2, \dots) + (y_1, y_2, \dots) &= (x_1 + y_1, x_2 + y_2, \dots), \\ \lambda(x_1, x_2, \dots) &= (\lambda x_1, \lambda x_2, \dots).\end{aligned}$$

With these definitions, \mathbf{F}^∞ becomes a vector space over \mathbf{F} , as you should verify. The additive identity in this vector space is the sequence of all 0's.

Our next example of a vector space involves a set of functions.

1.23 **Notation** \mathbf{F}^S

- If S is a set, then \mathbf{F}^S denotes the set of functions from S to \mathbf{F} .
- For $f, g \in \mathbf{F}^S$, the **sum** $f + g \in \mathbf{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in S$.

- For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the **product** $\lambda f \in \mathbf{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

As an example of the notation above, if S is the interval $[0, 1]$ and $\mathbf{F} = \mathbf{R}$, then $\mathbf{R}^{[0,1]}$ is the set of real-valued functions on the interval $[0, 1]$.

You should verify all three bullet points in the next example.

1.24 **Example** \mathbf{F}^S is a vector space

- If S is a nonempty set, then \mathbf{F}^S (with the operations of addition and scalar multiplication as defined above) is a vector space over \mathbf{F} .
- The additive identity of \mathbf{F}^S is the function $0 : S \rightarrow \mathbf{F}$ defined by

$$0(x) = 0$$

for all $x \in S$.

- For $f \in \mathbf{F}^S$, the additive inverse of f is the function $-f : S \rightarrow \mathbf{F}$ defined by

$$(-f)(x) = -f(x)$$

for all $x \in S$.

The elements of the vector space $\mathbf{R}^{[0,1]}$ are real-valued functions on $[0, 1]$, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.

Our previous examples of vector spaces, \mathbf{F}^n and \mathbf{F}^∞ , are special cases of the vector space \mathbf{F}^S because a list of length n of numbers in \mathbf{F} can be thought of as a function from $\{1, 2, \dots, n\}$ to \mathbf{F} and a sequence of numbers in \mathbf{F} can be thought of as a function from the set of

positive integers to \mathbf{F} . In other words, we can think of \mathbf{F}^n as $\mathbf{F}^{\{1,2,\dots,n\}}$ and we can think of \mathbf{F}^∞ as $\mathbf{F}^{\{1,2,\dots\}}$.

Soon we will see further examples of vector spaces, but first we need to develop some of the elementary properties of vector spaces.

The definition of a vector space requires that it have an additive identity. The result below states that this identity is unique.

1.25 Unique additive identity

A vector space has a unique additive identity.

Proof Suppose 0 and $0'$ are both additive identities for some vector space V . Then

$$0' = 0' + 0 = 0 + 0' = 0,$$

where the first equality holds because 0 is an additive identity, the second equality comes from commutativity, and the third equality holds because $0'$ is an additive identity. Thus $0' = 0$, proving that V has only one additive identity. ■

Each element v in a vector space has an additive inverse, an element w in the vector space such that $v + w = 0$. The next result shows that each element in a vector space has only one additive inverse.

1.26 Unique additive inverse

Every element in a vector space has a unique additive inverse.

Proof Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v . Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Thus $w = w'$, as desired. ■

Because additive inverses are unique, the following notation now makes sense.

1.27 Notation $-v, w - v$

Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v ;
- $w - v$ is defined to be $w + (-v)$.

Almost all the results in this book involve some vector space. To avoid having to restate frequently that V is a vector space, we now make the necessary declaration once and for all:

1.28 Notation V

For the rest of the book, V denotes a vector space over \mathbf{F} .

In the next result, 0 denotes a scalar (the number $0 \in \mathbf{F}$) on the left side of the equation and a vector (the additive identity of V) on the right side of the equation.

1.29 The number 0 times a vector

$0v = 0$ for every $v \in V$.

Note that 1.29 asserts something about scalar multiplication and the additive identity of V . The only part of the definition of a vector space that connects scalar multiplication and vector addition is the distributive property. Thus the distributive property must be used in the proof of 1.29.

Proof For $v \in V$, we have

$$0v = (0 + 0)v = 0v + 0v.$$

Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$, as desired. ■

In the next result, 0 denotes the additive identity of V . Although their proofs are similar, 1.29 and 1.30 are not identical. More precisely, 1.29 states that the product of the scalar 0 and any vector equals the vector 0 , whereas 1.30 states that the product of any scalar and the vector 0 equals the vector 0 .

1.30 A number times the vector 0

$a0 = 0$ for every $a \in \mathbf{F}$.

Proof For $a \in \mathbf{F}$, we have

$$a0 = a(0 + 0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides of the equation above gives $0 = a0$, as desired. ■

Now we show that if an element of V is multiplied by the scalar -1 , then the result is the additive inverse of the element of V .

1.31 The number -1 times a vector

$(-1)v = -v$ for every $v \in V$.

Proof For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$

This equation says that $(-1)v$, when added to v , gives 0. Thus $(-1)v$ is the additive inverse of v , as desired. ■

EXERCISES 1.B

- 1 Prove that $-(-v) = v$ for every $v \in V$.
- 2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.
- 3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.
- 4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?
- 5 Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V . (The phrase “a condition can be replaced” in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

- 6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in \mathbf{R} . Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty, \\ \infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over \mathbf{R} ? Explain.

1.C Subspaces

By considering subspaces, we can greatly expand our examples of vector spaces.

1.32 Definition *subspace*

A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

1.33 Example $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbf{F}\}$ is a subspace of \mathbf{F}^3 .

*Some mathematicians use the term **linear subspace**, which means the same as subspace.*

The next result gives the easiest way to check whether a subset of a vector space is a subspace.

1.34 Conditions for a subspace

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

additive identity

$$0 \in U$$

closed under addition

$$u, w \in U \text{ implies } u + w \in U;$$

closed under scalar multiplication

$$a \in \mathbf{F} \text{ and } u \in U \text{ implies } au \in U.$$

The additive identity condition above could be replaced with the condition that U is nonempty (then taking $u \in U$, multiplying it by 0, and using the condition that U is closed under scalar multiplication would imply that $0 \in U$). However, if U is indeed a subspace of V , then the easiest way to show that U is nonempty is to show that $0 \in U$.

Proof If U is a subspace of V , then U satisfies the three conditions above by the definition of vector space.

Conversely, suppose U satisfies the three conditions above. The first condition above ensures that the additive identity of V is in U .

The second condition above ensures that addition makes sense on U . The third condition ensures that scalar multiplication makes sense on U .

If $u \in U$, then $-u$ [which equals $(-1)u$ by 1.31] is also in U by the third condition above. Hence every element of U has an additive inverse in U .

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V . Thus U is a vector space and hence is a subspace of V . ■

The three conditions in the result above usually enable us to determine quickly whether a given subset of V is a subspace of V . You should verify all the assertions in the next example.

1.35 Example subspaces

(a) If $b \in \mathbf{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of \mathbf{F}^4 if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on \mathbf{R} is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of \mathbf{C}^∞ .

Verifying some of the items above shows the linear structure underlying parts of calculus. For example, the second item above requires the result that the sum of two continuous functions is continuous. As another example, the fourth item above requires the result that for a constant c , the derivative of cf equals c times the derivative of f .

Clearly $\{0\}$ is the smallest subspace of V and V itself is the largest subspace of V . The empty set is not a subspace of V because a subspace must be a vector space and hence must contain at least one element, namely, an additive identity.

The subspaces of \mathbf{R}^2 are precisely $\{0\}$, \mathbf{R}^2 , and all lines in \mathbf{R}^2 through the origin. The subspaces of \mathbf{R}^3 are precisely $\{0\}$, \mathbf{R}^3 , all lines in \mathbf{R}^3 through the origin, and all planes in \mathbf{R}^3 through the origin. To prove that all these objects are indeed subspaces is easy—the hard part is to show that they are the only subspaces of \mathbf{R}^2 and \mathbf{R}^3 . That task will be easier after we introduce some additional tools in the next chapter.

Sums of Subspaces

The union of subspaces is rarely a subspace (see Exercise 12), which is why we usually work with sums rather than unions.

When dealing with vector spaces, we are usually interested only in subspaces, as opposed to arbitrary subsets. The notion of the sum of subspaces will be useful.

1.36 Definition *sum of subsets*

Suppose U_1, \dots, U_m are subsets of V . The **sum** of U_1, \dots, U_m , denoted $U_1 + \dots + U_m$, is the set of all possible sums of elements of U_1, \dots, U_m . More precisely,

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}.$$

Let's look at some examples of sums of subspaces.

1.37 Example Suppose U is the set of all elements of \mathbf{F}^3 whose second and third coordinates equal 0, and W is the set of all elements of \mathbf{F}^3 whose first and third coordinates equal 0:

$$U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, y, 0) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$$

Then

$$U + W = \{(x, y, 0) : x, y \in \mathbf{F}\},$$

as you should verify.

1.38 Example Suppose that $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$ and $W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$. Then

$$U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\},$$

as you should verify.

The next result states that the sum of subspaces is a subspace, and is in fact the smallest subspace containing all the summands.

1.39 Sum of subspaces is the smallest containing subspace

Suppose U_1, \dots, U_m are subspaces of V . Then $U_1 + \dots + U_m$ is the smallest subspace of V containing U_1, \dots, U_m .

Proof It is easy to see that $0 \in U_1 + \cdots + U_m$ and that $U_1 + \cdots + U_m$ is closed under addition and scalar multiplication. Thus 1.34 implies that $U_1 + \cdots + U_m$ is a subspace of V .

Clearly U_1, \dots, U_m are all contained in $U_1 + \cdots + U_m$ (to see this, consider sums $u_1 + \cdots + u_m$ where all except one of the u 's are 0). Conversely, every subspace of V containing U_1, \dots, U_m contains $U_1 + \cdots + U_m$ (because subspaces must contain all finite sums of their elements). Thus $U_1 + \cdots + U_m$ is the smallest subspace of V containing U_1, \dots, U_m . ■

Sums of subspaces in the theory of vector spaces are analogous to unions of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.

Direct Sums

Suppose U_1, \dots, U_m are subspaces of V . Every element of $U_1 + \cdots + U_m$ can be written in the form

$$u_1 + \cdots + u_m,$$

where each u_j is in U_j . We will be especially interested in cases where each vector in $U_1 + \cdots + U_m$ can be represented in the form above in only one way. This situation is so important that we give it a special name: direct sum.

1.40 Definition *direct sum*

Suppose U_1, \dots, U_m are subspaces of V .

- The sum $U_1 + \cdots + U_m$ is called a **direct sum** if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j .
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the \oplus notation serving as an indication that this is a direct sum.

1.41 Example Suppose U is the subspace of \mathbf{F}^3 of those vectors whose last coordinate equals 0, and W is the subspace of \mathbf{F}^3 of those vectors whose first two coordinates equal 0:

$$U = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\}.$$

Then $\mathbf{F}^3 = U \oplus W$, as you should verify.

1.42 Example Suppose U_j is the subspace of \mathbf{F}^n of those vectors whose coordinates are all 0, except possibly in the j^{th} slot (thus, for example, $U_2 = \{(0, x, 0, \dots, 0) \in \mathbf{F}^n : x \in \mathbf{F}\}$). Then

$$\mathbf{F}^n = U_1 \oplus \cdots \oplus U_n,$$

as you should verify.

Sometimes nonexamples add to our understanding as much as examples.

1.43 Example Let

$$U_1 = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\},$$

$$U_2 = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\},$$

$$U_3 = \{(0, y, y) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$$

Show that $U_1 + U_2 + U_3$ is not a direct sum.

Solution Clearly $\mathbf{F}^3 = U_1 + U_2 + U_3$, because every vector $(x, y, z) \in \mathbf{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0),$$

where the first vector on the right side is in U_1 , the second vector is in U_2 , and the third vector is in U_3 .

However, \mathbf{F}^3 does not equal the direct sum of U_1, U_2, U_3 , because the vector $(0, 0, 0)$ can be written in two different ways as a sum $u_1 + u_2 + u_3$, with each u_j in U_j . Specifically, we have

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1)$$

and, of course,

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0),$$

where the first vector on the right side of each equation above is in U_1 , the second vector is in U_2 , and the third vector is in U_3 .

The symbol \oplus , which is a plus sign inside a circle, serves as a reminder that we are dealing with a special type of sum of subspaces—each element in the direct sum can be represented only one way as a sum of elements from the specified subspaces.

The definition of direct sum requires that every vector in the sum have a unique representation as an appropriate sum. The next result shows that when deciding whether a sum of subspaces is a direct sum, we need only consider whether 0 can be uniquely written as an appropriate sum.

1.44 Condition for a direct sum

Suppose U_1, \dots, U_m are subspaces of V . Then $U_1 + \dots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \dots + u_m$, where each u_j is in U_j , is by taking each u_j equal to 0.

Proof First suppose $U_1 + \dots + U_m$ is a direct sum. Then the definition of direct sum implies that the only way to write 0 as a sum $u_1 + \dots + u_m$, where each u_j is in U_j , is by taking each u_j equal to 0.

Now suppose that the only way to write 0 as a sum $u_1 + \dots + u_m$, where each u_j is in U_j , is by taking each u_j equal to 0. To show that $U_1 + \dots + U_m$ is a direct sum, let $v \in U_1 + \dots + U_m$. We can write

$$v = u_1 + \dots + u_m$$

for some $u_1 \in U_1, \dots, u_m \in U_m$. To show that this representation is unique, suppose we also have

$$v = v_1 + \dots + v_m,$$

where $v_1 \in U_1, \dots, v_m \in U_m$. Subtracting these two equations, we have

$$0 = (u_1 - v_1) + \dots + (u_m - v_m).$$

Because $u_1 - v_1 \in U_1, \dots, u_m - v_m \in U_m$, the equation above implies that each $u_j - v_j$ equals 0. Thus $u_1 = v_1, \dots, u_m = v_m$, as desired. ■

The next result gives a simple condition for testing which pairs of subspaces give a direct sum.

1.45 Direct sum of two subspaces

Suppose U and W are subspaces of V . Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof First suppose that $U + W$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$, where $v \in U$ and $-v \in W$. By the unique representation of 0 as the sum of a vector in U and a vector in W , we have $v = 0$. Thus $U \cap W = \{0\}$, completing the proof in one direction.

To prove the other direction, now suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U, w \in W$, and

$$0 = u + w.$$

To complete the proof, we need only show that $u = w = 0$ (by 1.44). The equation above implies that $u = -w \in W$. Thus $u \in U \cap W$. Hence $u = 0$, which by the equation above implies that $w = 0$, completing the proof. ■

Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain 0. So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals $\{0\}$.

The result above deals only with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is not enough to test that each pair of the subspaces intersect only at 0. To see this, consider Example 1.43. In that nonexample of a direct sum, we have $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$.

EXERCISES 1.C

- 1 For each of the following subsets of \mathbf{F}^3 , determine whether it is a subspace of \mathbf{F}^3 :
 - (a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
 - (b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
 - (c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;
 - (d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$.
- 2 Verify all the assertions in Example 1.35.
- 3 Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.
- 4 Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.
- 5 Is \mathbf{R}^2 a subspace of the complex vector space \mathbf{C}^2 ?
- 6
 - (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of \mathbf{R}^3 ?
 - (b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of \mathbf{C}^3 ?
- 7 Give an example of a nonempty subset U of \mathbf{R}^2 such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of \mathbf{R}^2 .
- 8 Give an example of a nonempty subset U of \mathbf{R}^2 such that U is closed under scalar multiplication, but U is not a subspace of \mathbf{R}^2 .

- 9 A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called **periodic** if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from \mathbf{R} to \mathbf{R} a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.
- 10 Suppose U_1 and U_2 are subspaces of V . Prove that the intersection $U_1 \cap U_2$ is a subspace of V .
- 11 Prove that the intersection of every collection of subspaces of V is a subspace of V .
- 12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.
- 13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.
[This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace \mathbf{F} with a field containing only two elements.]
- 14 Verify the assertion in Example 1.38.
- 15 Suppose U is a subspace of V . What is $U + U$?
- 16 Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V , is $U + W = W + U$?
- 17 Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V , is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

- 18 Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?
- 19 Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

- 20 Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of \mathbf{F}^4 such that $\mathbf{F}^4 = U \oplus W$.

21 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of \mathbf{F}^5 such that $\mathbf{F}^5 = U \oplus W$.

22 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of \mathbf{F}^5 , none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

23 Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

24 A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let U_e denote the set of real-valued even functions on \mathbf{R} and let U_o denote the set of real-valued odd functions on \mathbf{R} . Show that $\mathbf{R}^{\mathbf{R}} = U_e \oplus U_o$.