About the Maximum Cardinality of the Digital Cover of a Curve with a Given Length

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Abstract. We prove that the number of pixels -with pixels as unit lattice squares- of the digitization of a curve Γ of Euclidean length l is less than $3\lfloor \frac{l}{\sqrt{2}} \rfloor + 4$ which improves by a ratio of $\frac{4\sqrt{2}}{3}$ the previous known bound in $4\lfloor l \rfloor$ [3]. This new bound is the exact maximum that can be reached. Moreover, we prove that for a given number of squares n , the Minimal Length Covering Curves of n squares are polygonal curves with integer vertices, an appropriate number of diagonal steps and 0, 1 or 2 vertical or horizontal steps. It allows to express the functions $N(l)$, the maximum number of squares that can be crossed by a curve of length l , and $L(n)$, the minimal length necessary to cross n squares. Extensions of these results are discussed with other distances, in higher dimensions and with other digitization schemes.

Keywords: length, cover, digitization, curve, complexity, cardinality.

1 Introduction

1.1 Multiresolution and Fixed Resolution Bounds for the Size of the Digital Representation of a Shape

The questions that we investigate in this paper came to our attention as we were trying to bound the size of the multiresolution representation, with quadrees or octrees, of a real 2D or 3D shape. Such a bound is known in the case of a 2D shape [3]. This bound on the number of quads of the representation of a shape is related with the maximal number of squares of size $\frac{1}{2^n}$ that a curve can cross. It means that the bound in a multiresolution framework comes from a bound, which is computed at a fixed resolution. The fixed resolution bound used in [3] is $NumberOfPixels(\Gamma) \leq 4 \lceil l_2(\Gamma) \rceil$ where Γ is a rectifiable curve of Euclidean length $l_2(\Gamma)$ and where pixels are a set of unit squares tiling the plane. This bound comes simply from the fact that a piece of curve of length 1 can not cross 4 new unit squares. Is this bound tight? Not exactly. And -even if it is not the purpose of this paper- the reader should keep in mind that a better bound for this number of pixels (crossed by a curve of given length) implies better bounds for the size of the representation of a shape in a multiresolution framework (the construction of the multiresolution bound is described in [3]). As far as we know

and except in previous references, this task to bound the number of squares crossed by a curve of given length is original. Our goal is to close the problem by computing the tightest possible bound, namely the exact maximum $N(l)$ of the number of pixels of the digitization of a curve Γ in function of its euclidean length l (Fig. 1). The result that we provide is better from a factor $2\sqrt{2}$ with respect to the previous known result in $6 \lceil l_2(\Gamma) \rceil$.

Fig. 1. The functions $N(l)$ and $L(n)$ giving respectively the maximum number of squares in the cover of a curve of length l and the minimal length necessary to cover n squares. Expressions are given in Theorem 1. Some examples of optimal curves (a, b,..., j) corresponding to the saillant points of the staircase function are drawn on the right.

1.2 Notations and Problem Statement

We start to work with closed squares $[i, i + 1] \times [j, j + 1]$ as pixels where i and j are integers. Given any real subset $P \subset \mathbb{R}^2$, the set S of squares which contains at least one point of P is usually called the cover of P (see Fig. 2). We denote it by $\text{cover}(P)$. We are interested in the cardinality $|\text{cover}(\Gamma)|$ of the cover of a rectifiable curve $\Gamma : [0, 1] \rightarrow \mathbb{R}^2$ with respect to its Euclidean length l (we recall that a rectifiable curve has a finite length which is defined as the upper bound of the lengths of the polygonal lines having ordered vertices on the curve). As an alternative terminology, we consider the number of squares crossed by the curve Γ . Other digitization schemes will be considered in the following but the results that we will provide for them are just consequences of the ones that we have with closed squares.

Fig. 2. A curve Γ and its cover $\text{cover}(\Gamma)$ namely the set of squares crossed by Γ either by one of their edges, or by one of their vertices

It follows from definition that the cover of a single point can have 1, 2 or 4 squares, 4 in the case of an integer point (i, j) . We can notice that an horizontal path of integer length $l = k$ with integer vertices crosses $4 + 2\lfloor l \rfloor$ squares while a diagonal path of length $l = k\sqrt{2}$ crosses exactly $4 + 3\frac{l}{\sqrt{2}}$ squares (see for instance Fig. 3) which is more for $k \geq 2$. Is it the maximum cardinality of the cover of a curve with a given length? This question is solved in Theorem 1 but let us introduce before some notations: We denote by $N(l)$ the maximum number of squares n that a curve Γ of length l can cross and by $L(n)$ the minimal length l necessary to cross n squares (existence of the minimum is proved in Lemma 2). As the function $N(l)$ takes discrete values as input, it is a staircase while $L(n)$ is the discrete function obtained by inverting the axes (see Fig. 1). Our main objective is to provide the expressions of $N(l)$ and $L(n)$. This result is obtained in two steps, a first lemma stating the existence of Minimum Length Covering Curves of a given *set* of squares (Sec. 2) and a second lemma about Minimum Length Covering Curves of a given *number* of squares (Sec. 3). Then, the main theorem provides the expressions of $N(l)$ and $L(n)$ as well as the shape of the optimal curves. We end with some extensions of these results (Sec. 4).

2 Minimum Length Covering Curves of a Set of Squares

Let us consider a finite set of squares S (not necessarily connected). We are interested in the lengths l of the curves Γ which crosses S namely with inclusion $S \subset \text{cover}(\Gamma)$. The set of the possible lengths l is an interval since if there exists a curve of length l crossing S , then for any $\epsilon \geq 0$, we can build a new curve Γ' crossing S with a length $L + \epsilon$ just by adding a segment of length ϵ . But is this interval of the form $[L, +\infty[$ or $]L, +\infty[$? In other words, do these curves Γ have a minimal length or is it a lower bound? Lemma 1 states the existence of curves of minimal length. We call them *Minimal Length Covering Curves -MLCC* for short- of S (see Fig. 4).

Before going further, notice here that we have just the inclusion $S \subset \text{cover}(\Gamma)$ and not equality. It makes a difference with polygonal curves known in the specific framework of closed digital curves as Minimum Length Polygon [1] [2] but the principle of length minimality is the same.

Fig. 3. Theorem 1 proves that these curves are optimal in the sense that no shorter curve has a cover with so much squares (or crosses so much squares)

Lemma 1. *Given a finite set of squares S , among all continuous curves Γ that cross all the squares of S ($S \subset \text{cover}(\Gamma)$), there exists at least one with a minimal length (called Minimal Length Covering Curves of S). MLCC are polygonal curves with vertices on the edges of the squares of S .*

Fig. 4. A finite set of squares S and its Minimal Length Covering Curve Γ . Notice here that we have just the inclusion $S \subset \text{cover}(\Gamma)$ and not equality. We can also notice that MLCC can have non integral vertices.

Proof. The lemma states two things: first the existence of the Minimal Length Covering Curves, and secondly a structural property of these curves. Let us start with the second point: the structure of MLCC.

Assume that a MLCC is not a polygonal curve, then there exists in its cover a square in which it is not polygonal. The curve goes in this square by a point x on its boundary and goes outside of it by another point y . The shortest path to go from x to y is -of course- the segment $[xy]$. It means that by replacing the arc of Γ from x to y by this segment, we reduce its length, which contradicts the hypothesis of length minimality (see Fig. 5).

It follows that the curve is polygonal. It follows also from previous remark that it cannot have vertices in the interior of the squares. The vertices are necessarily on the edges of the squares.

Fig. 5. A curve of minimal length crossing a set of pixel is necessarily polygonal. Otherwise, shorter curves would exist.

It remains to prove the first point of the lemma: the existence of a Minimum Length Covering Curve of S , in other words the fact that the lower bound of the lengths of the curves crossing S is a minimum. We consider a sequence of curves Γ_i verifying $S \subset \text{cover}(\Gamma_i)$ with a length L_i that converges to the lower bound of the lengths l of the curves crossing the n squares of S . Our goal is now to build a MLCC of S , namely a curve Γ of length l covering S .

Fig. 6. A set of (red) squares S , a curve Γ and the corresponding ordering of the squares (the ambiguity between the indices 1 and 2 is removed with the rule *east* before *west*). The curve is decomposed into 5 pieces $\Gamma_i^{s_k}$ in the red squares (in the square of index 1, this piece is reduced to a point) and 4 pieces $\Gamma_i^{\text{link}(s_k \rightarrow s_{k+1})}$ from the square s_k to s_{k+1} (the second piece is also reduced to a point).

Let us introduce properly the order of the squares of S crossed by Γ_i (see Fig. 6). We define it as the sequence of the squares in the order where the curve arrives in them, but some precisions are required: the curve can reach an integer point or follow an edge with the consequence that in some cases, we may have some ambiguities for the choice of the next square. We can avoid these small difficulties just by using in this case a rule such that *north* before *south* and in a second time *east* before *west*. It provides a decomposition of the curve $\Gamma_i = \bigcup_k (\Gamma_i^{s_k} \cup \Gamma_i^{\text{link}(s_k \rightarrow s_{k+1})})$ where $\Gamma_i^{s_k}$ is a continuous arc of Γ_i in the square s_k of S and where $\Gamma_i^{\text{link}(s_k \rightarrow s_{k+1})}$ is a continuous arc going from the square s_k to the square s_{k+1} (in the case of neighboring squares in S , these arcs may be reduced to points). We can also define the point x_k of entry of the curve Γ_i in its k th square s_k and y_k its k th point of exit (in the case where the initial point starts in a square of S as in Fig. 6, we choose it as x_0 and the same for the end point). We can already notice that without loss of generality, the piece of curve

Γ_i^k can be replaced by the segment going from x_k to y_k in the square of index k while the piece $\Gamma_i^{\text{link}(s_k \rightarrow s_{k+1})}$ can be replaced by a segment going from y_k to x_{k+1} . It just decreases the length of the curve with the consequence that the property of convergence of the lengths of these curves is preserved. Moreover, we still respect the local topology of the curve.

There is nevertheless a possible unwanted property: the sequence of the squares s_k may be infinite. It is easy to see that the curve cannot cross an infinite number of different squares, since its length is bounded but the sequence may be infinite due to repetitions (if there are some kinds of accumulations). In this case, we are going to replace the curve Γ_i by a shorter curve Γ'_i with, this time, a finite sequence of squares. As it is shorter, the convergence of the length $L_2(\Gamma'_i)$ is again preserved. How do we build Γ'_i ? Just by cutting useless loops:

If Γ_i crosses a square s , it may cross it several other times for indices $k, k', k'' \dots$ to have a path going to another square of S which has not been crossed before. It means that if the square s is crossed more than n times (we recall that n is the cardinality of the set S), it makes some loops and some of these loops are useless because they don't cross "new" squares of S (new in the sense that they were never crossed before). These useless loops going outside from s are replaced in a new curve Γ'_i by short segments in s without losing any square of S . Then after this cut, Γ'_i is shorter than Γ_i and it remains at most n loops for each square and thus n repetitions. Hence, the sequence of squares of S crossed by the curve Γ'_i is finite and bounded by n^2 .

We know now that for all curves Γ_i , our sequences of the squares contain at most n^2 squares of S . It follows that the set of the orders of the squares s_i^k of S crossed by the curves Γ_i is finite and one of these orders appears an infinite number of times in the sequences of the orders of the curves Γ_i . We consider now the subsequence of curves Γ_{i_j} with this order: they have all exactly the same sequence of squares s_k for $1 \leq k \leq n'$ (with $n' \leq n^2$). We can now focus on the points of entry $x_{j,k}$ and exit $y_{j,k}$ for the curve Γ_i in the square s_k . Then we have $2n'$ sequences of points of index j . As they move in compact sets (they belong to the boundary of a square) and the product of a finite number (here $2n' \leq 2n^2$) of compacts is still compact, we can extract from these $2n'$ sequences a unique sequence such that $\lim_{j \rightarrow +\infty} x_{j,k} = x_k$ and $\lim_{j \rightarrow +\infty} y_{j,k} = y_k$. It just remains to lie the points x_k to y_k and y_k to x_{k+1} by a segment in order to obtain a new curve Γ . What can we say about the length of Γ ? The length of Γ or Γ_{i_j} is the sum of the lengths of all their pieces in the squares s_k and between the squares:

$$L(\Gamma_{i_j}) = \sum_{k=0}^{n'} d(x_{j,k}, y_{i,k}) + \sum_{k=0}^{n'-1} d(y_{j,k}, x_{i+1,k})$$

$$L(\Gamma) = \sum_{k=0}^{n'} d(x_k, y_k) + \sum_{k=0}^{n'-1} d(y_k, x_k).$$

It follows by continuity of the distance $\lim_{j \rightarrow +\infty} L(\Gamma_{i_j}) = L(\Gamma)$. It proves that the length of the curve Γ is the lower bound and thus: we have built a curve Γ covering S of minimal length.

In many cases, the vertices of the MLCC are not only on the edges but on the integer points of the grid. Nevertheless, in some particular cases, it can occur that a vertex of the minimal curve is not an integer point but on one of the edges (see an example Fig. 6).

Lemma 1 deals with a finite set of squares S . If S has now an infinite cardinality, then all covering curves are of infinite length since the diameter of S is itself infinite. MLCC are not defined in this case.

At last, we can notice that Lemma 1 holds in higher dimensions, with hypercubes instead of squares, and curves of dimension 1.

3 Minimum Length Covering Curves of n Squares

3.1 About Their Existence

Previous lemma proves the existence of a Minimum Length Covering Curve for a given finite set of squares S . Let us now relax the condition on S by assuming that we know only its cardinality. Hence, we provide a number of squares n and ask about the existence of a minimal length for the curves crossing n squares or more.

Lemma 2. *Given a natural integer n , there exists curves Γ of minimal length crossing n squares (for any curve P crossing n squares or more, $L(P) \geq L(\Gamma)$). We call them Minimal Length Covering Curve of n squares.*

Proof. Given a number of squares n (just the number), we can consider all the possible configurations of sets of n squares. We can reduce ourselves to 4-connected configurations, with the consequence that, up to a translation, there are only a finite number of such configurations. Thus, due to Lemma 1, the lower bound of the lengths of curves crossing n squares is in fact a minimum that is obtained for some polygonal curves with vertices on the edges of the squares.

3.2 MLCC of n Squares and Expressions of $L(n)$ and $N(l)$

We can at last express the functions $N(l)$ and $L(n)$ and provide the exact structure of Minimal Length Covering Curves of n squares.

Theorem 1. *Minimal Length Covering Curves of n squares are made of 0, 1 or 2 horizontal or vertical steps and an arbitrary number of diagonal steps (Fig. 1).*

- If $l \bmod \sqrt{2} < 2 \bmod \sqrt{2}$, then $N(l) = 3 \lfloor \frac{l}{\sqrt{2}} \rfloor + 4$ (optimal curves have $\lfloor \frac{l}{\sqrt{2}} \rfloor$ diagonal steps).

- If $2 \bmod \sqrt{2} \leq l \bmod \sqrt{2} < 1$ and $l > 1$, then $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 5$ (optimal curves have $\lfloor \frac{l}{\sqrt{2}} \rfloor - 1$ diagonal steps and 2 horizontal or vertical steps). If $2 - \sqrt{2} \leq l < 1$, we have still $N(l) = 4$ (the first step of the staircase is broken).
- If $1 \leq l \bmod \sqrt{2}$, then $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 6$ (optimal curves have $\lfloor \frac{l}{\sqrt{2}} \rfloor$ diagonal steps and 1 horizontal or vertical step).

Conversely, function $L(n)$ is:

- $L(1) = L(2) = L(3) = L(4) = 0$, $L(5) = 1$.
- If $n \bmod 3 = 0$ (and $n \geq 6$), then $L(n) = 1 + (\frac{n}{3} - 2)\sqrt{2}$.
- If $n \bmod 3 = 1$ (and $n \geq 6$), then $L(n) = (\lfloor \frac{n}{3} \rfloor - 1)\sqrt{2}$.
- If $n \bmod 3 = 2$ (and $n \geq 6$), then $L(n) = 2 + (\lfloor \frac{n}{3} \rfloor - 2)\sqrt{2}$.

Proof. The first step is to prove that a MLCC of n squares is necessarily a polygonal curve with integer vertices. Let Γ be a MLCC of n squares. It is of course a MLCC of its cover $\text{cover}(\Gamma)$. We assume that Γ has a non integral vertex, and we are going to build another curve Γ' with the same length crossing at least the same number n of squares. Then Γ' optimality will have consequences on the structure of Γ .

If we consider that the first or the last vertices are not integral, it is easy to see that it contradicts length minimality (there is only the particular case of a single horizontal or vertical segment but even in this case, it is clear that such segments with non integral vertices are not MLCC of n squares). Thus the extremities of a MLCC are integral points. Now, let us consider a non integral intermediary vertex v of Γ on the edge e between two squares s_i and s_{i+1} and obtain a contradiction. If we consider the quadrants of its edges, only one configuration among four is compatible with length minimality (Fig. 7).

Fig. 7. If we consider a non integral vertex of Minimum Length Covering Curve, we can imagine four cases a), b), c) or d) according to the quadrant of the direction of the next edge towards the initial one. Among these four cases represented here, only one is possible since the three others contradict the minimality of the length of the curve.

This only possible configuration allows to unfold the curve by a sequence of symmetries so that the cardinality of the cover of the images of each segment is preserved at each step (Fig. 8). At the end, as the curve is completely unfolded,

Fig. 8. The fact that only case from Fig. 7-d) can occur in a MLCC allows to unfold any MLCC with a sequence of symetries that preserve the cardinality of the covers of each edge. Hence the unfolded curve Γ' has the same length than Γ (MLCC of n squares), crosses at least the same number of squares -it is still a MLCC of n squares- and due to monotonicity, case from Fig. 7-d) cannot occur anymore.

we obtain a monotonic curve Γ' such that its cover has at least the same number of squares than Γ . As Γ' has the same length and crosses at least the same number of squares, it is still a MLCC of n squares. Its length is minimal and due to monicity, there can exist no more intermediary vertices such as in Fig. 7. It follows that all intermediary vertices of Γ' are integer points.

Let us now consider a segment of Γ' going for instance from $(0,0)$ to (a,b) where (a,b) are coprime positive integers. The initial point $(0,0)$ is in 4 squares. The last one too. And the segment crosses $a-1$ vertical edges and $b-1$ horizontal edges. As there is no integer point in the inner part of the segment, it makes $5+a+b$ squares. Now let us assume that $0 < b < a$ and compare with a horizontal segment $(a,0)$. It crosses $4+2a$ squares which is at least equal to the $5+a+b$ squares of the segment (a,b) . And $(a,0)$ is shorter. It follows that Γ' cannot have a segment (a,b) different from $(1,0)$, $(1,1)$, or $(0,1)$ because otherwise, by replacing the segment (a,b) by $(a,0)$, we obtain a shorter path crossing at least the same number of squares. If we come back to our initial MLCC Γ , it proves that a MLCC of n squares can contain only segments in horizontal, vertical and diagonal directions, and whatever the order of these steps, as soon as the cover has no loop, it does not change the number of squares. It follows that any MLCC of n squares can be unfolded in a polygonal curve with a horizontal segment $(a,0)$ and a diagonal one (b,b) . The length is $a+b\sqrt{2}$ and the number of squares is $4+2a+3b$. Knowing these structural properties, the MLCC of n squares are characterized by the minimization problem: minimize $a+b\sqrt{2}$ under constraints $n \leq 4+2a+3b$ with positive integer values a and b . This minimum provides the value of the function $L(n)$. The other way to consider the question is the converse one: the maximum of $4+2a+3b$ subject to constraint $a+b\sqrt{2} \leq l$, with a and b positive integers provides the value of the function $N(l)$ (see Fig. 9).

The critical values are given by integer points (a,b) with left triangle of slopes $-\frac{2}{3}$ and $-\frac{\sqrt{2}}{2}$ which does not contain any other integer point. It follows that only three values of a are possible: 0, 1 or 2 (if $a \geq 3$, then $(a',b') = (a-3, b+2)$ verifies also $N \leq 4+2a'+3b'$ with a better score i.e smaller length $a'+b'\sqrt{2} =$

Fig. 9. On the left, $L(n)$ is the minimum of $a + b\sqrt{2} \leq l$ under constraint $n \leq 4 + 2a + 3b$ with $(a, b) \in \mathbb{N}^2$. On the right, $N(l)$ is the maximum of $4 + 2a + 3b$ under constraint $a + b\sqrt{2} \leq l$. Due to the respective slopes of the extremal lines $-\frac{\sqrt{2}}{2}$ and $-\frac{2}{3}$, the optimal points have coordinates a equal to 0, 1 or 2 (the red points).

$a + b - 3 + 2\sqrt{2}$). Then the optimal curves are mainly diagonal with 0, 1 or 2 horizontal or vertical steps. Without more details here, it leads to the expressions of functions $L(n)$ and $N(l)$ expressed in Theorem 1.

4 Extensions of the Results

Let us give an overview of the consequences and extensions of Theorem 1.

4.1 With Other Kinds of Covers

In the case of some other kinds of covers, for instance by considering squares $[i, i + 1[\times [j, j + 1[$ whose union is a partition of the plane $\mathbb{R}^2$, Minimal Length Covering Curves no more exist since the lower bound is not a minimum anymore. Nevertheless, the difference for the functions $N(l)$ remains small : if the curve Γ is a MLCC of a set of closed squares S (as previously), then we just have to add a little loop around each integer point or each edge that it crosses to build a curve of length $L(\Gamma) + \epsilon$ which has exactly the same cover (see Fig. 10). It follows in this case

Theorem 2. *With the squares $[i, i + 1[\times [j, j + 1[$, we have*

If $l \bmod \sqrt{2} \leq 2 \bmod \sqrt{2}$, then $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 4$

If $2 \bmod \sqrt{2} < l \bmod \sqrt{2} \leq 1$, then $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 5$

If $1 < l \bmod \sqrt{2}$, then $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 6$.

4.2 With Closed Curves

If we restrict ourselves to closed curves, we have clearly $N_{\text{closed}}(l) \leq N(l)$ while $L_{\text{closed}}(n) \geq L(n)$. We can also notice that a square in the diagonal direction

Fig. 10. Given a curve Γ with its cover S in the sens of closed squares $[i, i+1] \times [j, j+1]$, for any $\epsilon > 0$, we can build new curves Γ' of length $L(\Gamma') \leq L(\Gamma) + \epsilon$ with the same cover S but now in the sens of semi-open squares $[i, i+1[\times [j, j+1[$

with sides of length $k\sqrt{2}$ and integer vertices is a closed curve of length $l = 4k\sqrt{2}$ which crosses exactly $12k = 3\lfloor \frac{l}{\sqrt{2}} \rfloor \leq 12\lfloor \frac{l}{4\sqrt{2}} \rfloor$ squares. It follows

$$12\lfloor \frac{l}{4\sqrt{2}} \rfloor \leq N_{closed}(l) \leq N(l) \leq 6 + 12\lfloor \frac{l}{4\sqrt{2}} \rfloor.$$

It means in particular that the difference between the values of the functions for closed and general curves remains bounded by a small constant.

4.3 In Dimension d

The results presented here could be generalized in dimension d : the first intuition is to consider that the MLCC of n hypercubes in $\mathbb{R}^d$ will follow paths in the "hyper"-diagonal direction of the hypercubes. A path of k hyperdiagonal steps has a length $k\sqrt{d}$ for a number of hypercubes equal to $2^d + (2^d - 1)k$. The ratio is $\frac{2^d - 1}{\sqrt{d}}$. This should be compared with a path which remains for instance in the hyperplane $x_d = 0$ and an hyperdiagonal direction in this plane. Such a path crosses $2(2^{d-1} + (2^{d-1} - 1)k)$ hypercubes (the coefficient 2 comes from the hypercubes centered with $x_d = + - \frac{1}{2}$) for k steps of length $\sqrt{d-1}$. The ratio is $2\frac{2^{d-1} - 1}{\sqrt{d-1}}$. The comparison between these two ratios shows that the second one is greater than the first one except for $d = 2$. It follows that some MLCC in dimension d are in fact MLCC drawn in dimension 2 with all other coordinates equal to integers. If we notice $N^d(l)$ the maximal number of hypercubes crossed by a curve of length l in dimension d , then $N^d(l) \geq 2^{d-2}N^2(l)$ which makes more hypercubes than the $2^d + \frac{2^d - 1}{\sqrt{d}}l$ hypercubes crossed by an hyperdiagonal path.

4.4 With Other Distances

With L_1 norm instead of Euclidean norm for the length of the curve, the expressions of $N(l)$ and $L(n)$ are quite straightforward. We just have to notice that following the curve, new squares appear each time that a line $x = i$ or $y = j$ is

crossed. The L_1 distance necessary to cross two consecutive lines is 1. It follows $N(l) = 4 + 2\lfloor l \rfloor$ and $L(n) = \lfloor \frac{n-3}{2} \rfloor$ (for $n > 2$). MLCC of n squares are just polygonal curves with horizontal and vertical segments.

5 Conclusion

In this article, we have given a precise expression of two functions, $N(l)$ and $L(n)$, which respectively calculates the maximum number of squares in the cover of a curve of length l , and the minimal length to cover n squares. These results were obtained by constructing MLCC, curves of minimal length for a given number of squares n . We have also extracted some possible extensions of this work, with other distances, higher dimensions or particular classes of curves.

In our study, we aim at using these results in a multiresolution scheme, based on quadrees or octrees. Like in this paper, we have to determine the values of the functions $N(l)$ and $L(n)$, guided this time by appropriate tree traversal strategies.

The next future work is to consider *surfaces* instead of curves in any arbitrary dimension $d > 2$. However, it is no more possible to bound the number of hypercubes covered by a surface in function of its area since there exists surfaces of area as small as necessary crossing all the hypercubes of compact domain. It makes the MACS (Minimum Area Covering Surfaces) problem of n hypercubes completely different. Several ideas could be followed by considering specific classes of surfaces or another characteristic of the surface that may be related again with curves length.

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