Exploiting Air-Pressure to Map Floorplans on Point Sets

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Abstract. We prove a conjecture of Ackerman, Barequet and Pinter. Every floorplan with n segments can be embedded on every set of n points in generic position. The construction makes use of area universal floorplans also known as area universal rectangular layouts.

The notion of area used in our context depends on a nonuniform density function. We, therefore, have to generalize the theory of area universal floorplans to this situation. The method is then used to prove a result about accommodating points in floorplans that is slightly more general than the conjecture of Ackerman et al.

1 Introduction

In our context a *floorplan* is a partition of a rectangle into a finite set of interior-disjoint rectangles. A floorplan is *generic* if it has no cross, i.e., no point where four rectangles of the partition meet. A *segment* of a floorplan is a maximal nondegenerate interval that belongs to the union of the boundaries of the rectangles. Segments are either horizontal or vertical. The segments of a generic floorplan are internally disjoint. Two floorplans F and F' are *weakly equivalent* if there exist bijections $\phi : S_H(F) \rightarrow S_H(F')$ and $\psi : S_V(F) \rightarrow S_V(F')$ between their horizontal and vertical segments such that segment s has an endpoint on segment t in F iff $\phi(s)$ has an endpoint on $\psi(t)$. A set P of points in $\mathbb{R}^2$ is *generic* if no two points from P have the same x or y coordinate. Section 2 provides a more comprehensive overview of definitions and notions related to floorplans.

Fig. 1. A generic set of six points and a generic floorplan with six segments

Let P be a set of n points in a rectangle R and let F be a generic floorplan with n internal segments. A *cover map* from F to P is a floorplan F' weakly equivalent to F with outer rectangle R such that every internal segment of F' contains exactly one point from P . Figure 2 shows an example.

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Fig. 2. Two cover maps from the floorplan of Fig. 1.b to the point set of Fig. 1.a

In this paper we answer a question of Ackerman et al. [1] by proving Theorem 1. The proof of the theorem and some variants and generalizations is the subject of Section 4.

Theorem 1. *If P is a generic set of n points and F is a generic floorplan with n internal segments, then there is a cover map from F to P .*

The proof is based on results about area representations of floorplans. The following theorem is known, it has been proven with quite different methods, see [14], [11], [5].

Theorem 2. *Let F be a floorplan with rectangles $R_1, \dots, R_{n+1}$, let A be a rectangle and let $w : \{1, \dots, n+1\} \rightarrow \mathbb{R}_+$ be a weight function with $\sum_i w(i) = \text{area}(A)$. There exists a unique floorplan F' contained in A that is weakly equivalent to F such that the area of the rectangle $R'_i = \phi(R_i)$ is exactly $w(i)$.*

In Section 3 we prove the generalization of Theorem 2 that will be needed for the proof of Theorem 1. In the generalized theorem (Theorem 3) the weight of a rectangle is measured as integral over some density function instead of the area.

2 Floorplans and Graphs

A *floorplan* is a dissection of a rectangle into a finite set of interiorly disjoint rectangles. From a given floorplan F we can obtain several graphs and additional structure. We introduce two of these and use them to define notions of equivalence for floorplans.

The Rectangular Dual. Let $R(F)$ be the set of rectangles of a floorplan F . It is convenient to include the enclosing rectangle in the set $R(F)$. The *rectangular dual* of F is the graph $G^*(F)$ with vertex set $R(F)$ and edges joining pairs of rectangles that share a boundary segment. Usually the notion of a rectangular dual is used in the other direction, i.e., it is assumed that a planar graph G is given and the quest is for a floorplan F such that $G = G^*(F)$. It is convenient to extend a floorplan F with four rectangles that frame F as shown in Figure 3. In the dual of an extended floorplan F_+ we omit the vertex that corresponds to the enclosing rectangle. With this twist in the definition of the dual we get: The dual $G_+^*(F)$ of the extended floorplan of a generic F is a 4-connected inner triangulation of a 4-gon. Indeed this is the characterization of the duals of extended generic floorplans. Buchsbaum et al. [3] and Felsner [6] provide many pointers to the literature related to floorplans and rectangular duals.

The Transversal Structure. The transversal structure (also known as regular edge labeling) associated to a floorplan F is an orientation and coloring of the edges of the extended dual $G_+^*(F)$.

Fig. 3. A floorplan F , the extended floorplan F_+ , and the duals $G^*(F)$ and $G_+^*(F)$

Let G be an inner triangulation of a 4-gon with outer vertices s, a, t, b in counterclockwise order. A *transversal structure* for G is an orientation and 2-coloring of the inner edges of G such that two local conditions hold: 1) All edges incident to s, a, t and b are blue outgoing, red outgoing, blue ingoing, and red ingoing, respectively. 2) The edges incident to an inner vertex v come in clockwise order in four nonempty blocks consisting solely of red ingoing, blue ingoing, red outgoing, and blue outgoing edges, respectively.

Transversal structures have been studied in [9], [10], and in [12]. A proof of the following proposition can e.g. be found in [6].

Proposition 1. *Every transversal structure of an inner triangulation G of a 4-gon with outer vertices s, a, t, b is induced by a floorplan F with $G = G_+^*(F)$ and each floorplan F induces a transversal structure on $G_+^*(F)$.*

Figure 4 indicates the correspondence between floorplans and transversal structures.

Fig. 4. The two local conditions and an example of a transversal structure together with a corresponding floorplan

The Segment Contact Graph. Recall that we call a floorplan *generic* if it has no cross, i.e., no point where four rectangles meet. A *segment* of a floorplan is a maximal non-degenerate interval that belongs to the union of the boundaries of the rectangles. In the generic case intersections between segments only occur between horizontal and vertical segments and they involve an endpoint of one of the segments, i.e., they are contacts. If a floorplan has a cross at point p we can break one of the two segments that contain p into two to get a system of interiorly disjoint segments.

The *segment contact graph* $G_{\text{seg}}(F)$ of a floorplan F is the bipartite planar graph whose vertices are the segments of F and edges correspond to contacts between segments. From Figure 5 we see that $G_{\text{seg}}(F)$ is indeed planar and that the faces of $G_{\text{seg}}(F)$ are in bijection with the rectangles of F and are uniformly of degree 4. Therefore $G_{\text{seg}}(F)$ is a maximal bipartite planar graph, i.e., a quadrangulation.

Fig. 5. A floorplan F and two drawings of its segment contact graph $G_{\text{seg}}(F)$

The Separating Decomposition. The separating decomposition associated to a floorplan is an orientation and coloring of the edges of the segment contact graph.

Let Q be a quadrangulation, we call the color classes of the bipartition white and black and name the two black vertices on the outer face s and t . A *separating decomposition* of Q is an orientation and coloring of the edges of Q with colors red and blue such that two conditions hold: 1) All edges incident to s are ingoing red and all edges incident to t are ingoing blue. 2) Every vertex $v \neq s, t$ is incident to a nonempty interval of red edges and a nonempty interval of blue edges. If v is white, then, in clockwise order, the first edge in the interval of a color is outgoing and all the other edges of the interval are incoming. If v is black, the outgoing edge is the last one in its color in clockwise order (see the left part of Figure 6).

Separating decompositions have been studied in [4], [8], and [7]. To us they are of interest because of the following proposition proved e.g. in [6].

Proposition 2. A floorplan F induces a separating decomposition on its segment contact graph $G_{\text{seg}}(F)$ and every separating decomposition of a planar quadrangulation Q corresponds to a floorplan F with $Q = G_{\text{seg}}(F)$.

The right part of Figure 6 indicates the correspondence between floorplans and separating-decompositions.

Fig. 6. The rule for black and white vertices. A separating decomposition S and a floorplan F corresponding to S .

2.1 Notions of Equivalence for Floorplans

Definition 1. Two floorplans are weakly equivalent if they induce the same separating decomposition.

Definition 2. Two floorplans are strongly equivalent if they induce the same transversal structure.

In the introduction we said that two floorplans F and F' are weakly equivalent if there exist bijections $\phi : S_H(F) \rightarrow S_H(F')$ and $\phi : S_V(F) \rightarrow S_V(F')$ between their horizontal and vertical segments such that segment s has an endpoint on segment t in F iff $\phi(s)$ has an endpoint on $\phi(t)$. We claim that this yields the same equivalence classes as Definition 1. Clearly, if F and F' induce the same separating decomposition then they are weakly equivalent in the above sense. For the converse first observe that the segment contact graphs of F and F' are isomorphic, i.e., $G_{\text{seg}}(F) = G_{\text{seg}}(F')$. Now define an orientation Q_F on $Q = G_{\text{seg}}(F)$ by orienting s to t iff segment s has an endpoint on segment t . Let $Q_{F'}$ be the orientation defined on Q using the segments of F' . Observe that $Q_F = Q_{F'}$. Since a separating decomposition is uniquely determined by the underlying 2-orientation (see, [4] or [7]) we conclude that F and F' induce the same separating decomposition, i.e., $SD_F = SD_{F'}$. This implies that F and F' are weakly equivalent in the sense of Definition 1.

Eppstein et al. [5] use the term *layout* instead of floorplan. Their equivalent layouts correspond to strongly equivalent floorplans and order-equivalent layouts to weakly equivalent floorplans. Asinowski et al. [2] study independent notions of R -equivalence and S -equivalence for floorplans.

3 Realizing Weighted Floorplans via Air-Pressure

In this section we prove a generalization of Theorem 2 to situations where the “area” of a rectangle is replaced by the mass defined through a density distribution.

Let $\mu : [0, 1]^2 \rightarrow \mathbb{R}_+$ be a density function on the unit square whose total mass is 1, i.e., $\int_0^1 \int_0^1 \mu(x, y) dx dy = 1$. We assume that μ can be integrated over axis aligned rectangles and all fibers μ_x and μ_y can be integrated over intervals. Moreover, we require that integrating μ over a nondegenerate rectangle and fibers over nondegenerate intervals always yields nonzero values. The *mass* of an axis aligned rectangle $R \subseteq [0, 1]^2$ is defined as $m(R) = \iint_R \mu(x, y) dx dy$.

Theorem 3. *Let $\mu : [0, 1]^2 \rightarrow \mathbb{R}_+$ be a density function of total mass 1. If F is a floorplan with rectangles $R_1, \dots, R_{n+1}$ and $w : \{1, \dots, n+1\} \rightarrow \mathbb{R}_+$ a positive weight function with $\sum_{i=1}^{n+1} w(i) = 1$ then there exists a unique floorplan F' in the unit square that is weakly equivalent to F such that $m(R_i) = w(i)$ for each rectangle R_i .*

Our proof follows the air-pressure paradigm as proposed by Izumi, Takahashi and Kajitani [11]. We first describe the idea: Consider a realization of F in the unit square and compare the mass $m(R_i)$ to the intended mass $w(i)$. The quotient of these two values can be interpreted as the pressure inside the rectangle. Integrating this pressure along a side of the rectangle yields the force by which R_i is pushing against the segment that contains the side. The difference of pushing forces from both sides of a segment yields the effective force acting on the segment. The intuition is that shifting a segment in direction of the effective force yields a better balance of pressure in the rectangles. We will show that iterating such improvement steps drives the realization of F towards a situation with $m(R_i) = w(i)$ for all i , i.e., the procedure converges towards the floorplan F' whose existence we want to show.

In [11] the air-pressure paradigm was used for situations where the mass of a rectangle is its area. The authors observed fast convergence experimentally but they had no proof of convergence. Here we provide such a proof for the more general case of weights given by integrals over a density function.

A proof of Theorem 3 can also be given along the lines of the proof of Theorem 2 that has been given by Eppstein et al. in [5]. This approach has been taken by Schrenzenmaier [13]. The resulting proof is quite compact, however, it has the disadvantage of being purely existential. Schrenzenmaier also has a `java` implementation of the air-pressure approach that solves moderate size instances quickly.

Let $R_i = [x_l, x_r] \times [y_b, y_t]$ be a rectangle of F . Recall that the mass of R_i is $m(R_i) = \int_{x_l}^{x_r} \int_{y_b}^{y_t} \mu(x, y) dy dx$. The pressure $p(i)$ in R_i is the fraction of the intended mass $w(i)$ and the actual mass $m(R_i)$, i.e., $p(i) = \frac{w(i)}{m(R_i)}$. Let s be a segment of F and let R_i be one of the rectangles with a side in s . Let s be vertical with x -coordinate x_s and let $s \cap R_i$ span the interval $[y_b(i), y_t(i)]$. The (undirected) *force imposed on s by R_i* is the pressure $p(i)$ of R_i times the density dependent length of the intersection.

$$f(s, i) = \frac{w(i)}{m(R_i)} \int_{y_b(i)}^{y_t(i)} \mu(x_s, y) dy = p(i) \int_{y_b(i)}^{y_t(i)} \mu_{x_s}(y) dy.$$

The *force acting on s* is obtained as a sum of the directed forces imposed on s by incident rectangles.

$$f(s) = \sum_{R_i \text{ left of } s} f(s, i) - \sum_{R_i \text{ right of } s} f(s, i).$$

Symmetric definitions apply to horizontal segments.

Balance for Rectangles and Segments

A segment s is in *balance* if $f(s) = 0$. A rectangle R_i is in *balance* if $p(i) = 1$, i.e., if $w(i) = m(R_i)$.

Lemma 1. *If all rectangles R_i of F are in balance, then all segments are in balance.*

Proof. Since all rectangles are in balance we can eliminate the pressures from the definition of the $f(s, i)$. With this simplification we get for a vertical segment s

$$f(s) = \sum_{R_i \text{ left of } s} \int_{y_b(i)}^{y_t(i)} \mu_{x_s}(y) dy - \sum_{R_j \text{ right of } s} \int_{y_b(j)}^{y_t(j)} \mu_{x_s}(y) dy.$$

Hence $f(s) = M_s - M_s = 0$, where M_s is the integral of the fiber density μ_{x_s} along s . The symmetric argument applies to horizontal segments. $\square$

Interestingly, the converse of the lemma also holds.

Proposition 3. *If all segments of F are in balance, then all rectangles are in balance.*

Proof. Suppose that F balances all segments but not all rectangles. Choose some τ with $\min_i p(i) < \tau \leq \max_i p(i)$. Let T_τ be the union of all rectangles R_i whose pressure exceeds τ and let Γ_τ be the boundary of T_τ .

Claim. The boundary Γ_τ of T_τ contains no complete segment.

Suppose Γ_τ contains the vertical segment s such that T_τ is left of s . Let I be a nontrivial interval on s that is defined as intersection of a rectangle R_i that has its right edge on s and a rectangle R_j that has its left edge on s . The force acting on s along I is $p(i) \int_I \mu_{x_s}(y) dy - p(j) \int_I \mu_{x_s}(y) dy$. Since $\int_I \mu_{x_s}(y) dy > 0$ by our assumption on μ and $p(i) > p(j)$ by definition of T_τ the force is positive. This holds for every interval I on s , hence, the overall force $f(s)$ acting on s is also positive. This contradicts the assumption that s is in balance and completes the proof of the claim. $\triangle$

Let s_0 be any segment which contributes to Γ_τ . From the lemma we know that at some interior point of segment s_0 the boundary leaves s_0 and continues along another segment s_1 . Again, the boundary has to leave s_1 at some interior point to continue on s_2 . Because this procedure always follows the boundary of T which is a region defined by a union of rectangles in F the sequence of segments has to get back to segment s_0 , i.e., there is an k such that $s_k = s_0$.

From the definition of the separating decomposition SD_F corresponding to F we find that $s_0 \leftarrow s_1 \leftarrow s_2 \leftarrow \dots \leftarrow s_{k-1} \leftarrow s_0$ is a directed cycle in SD_F . The four segments of the enclosing square of F do not contribute to the boundary of T_τ , simply because they cannot belong to a directed cycle of SD_F .

Recall the assumption that F balances all segments but not all rectangles. Let s be the vertical segment with maximal x -coordinate among all vertical segments that contribute to a boundary Γ_τ for some τ . From the choice of s it is clear that T_τ is to the left of s . Consider the segment $s' = s_{k-1}$ following $s = s_0$ in the cycle $s_0 \leftarrow \dots \leftarrow s_{k-1} \leftarrow s_0$ in SD_F corresponding to Γ_τ . Left from the contact point p of s and s' the segment s' is part of the boundary Γ_τ of T_τ . From the choice of τ and s it follows that to the right of p the rectangles on both sides of s' have the same pressure $p(i)$. Otherwise the right part of s' would belong to some boundary $\Gamma_{\tau'}$ and the vertical segment following s' on $\Gamma_{\tau'}$ is in contradiction to the choice of s .

Now consider $f(s')$ and split the contributions to this force at p . Left of p the pressure on the side of T_τ exceeds the pressure from the other side. Right of p the rectangles on both sides of p have the same pressure. This shows that in contradiction to the assumption $f(s') \neq 0$. This completes the proof of the proposition. $\square$

Balancing Segments and Optimizing the Entropy

Proposition 4. *If a segment s of F is unbalanced, then we can keep all the other segments at their position and shift s parallel to a position where it is in balance. The resulting floorplan F' is weakly equivalent to F .*

Proof. We consider the case of a vertical segment, the horizontal case is symmetric. Let x_s be the x -coordinate of s . With S_- and S_+ we denote the sets of rectangles in F that touch s from the left respectively from the right. Let R_l be the rectangle with a left boundary of maximal x -coordinate x_l in S_- and let R_r be the rectangle with a right boundary of minimal x -coordinate x_r in S_+ . Note that if t satisfies $x_l < t < x_r$ then segment s can be shifted parallel to the position $x_s = t$ and the resulting floorplan is weakly equivalent to F .

For $t \in (x_l, x_r)$ we define $h(t)$ as the force acting on s when the segment it is shifted to $x_s = t$. We observe:

- The pressure $p(i)$ depends continuously on t for all rectangles $R_i \in S_- \cup S_+$.
- The value of $\int_I \mu_t(y) dy$ is a continuous function of t for all intervals I .

Hence, $h(t)$ is a continuous function. With t approaching x_l from the right the area of R_l tends to zero. Hence, the mass $m(R_l)$ also tends to zero and the pressure $p(l)$ tends to infinity. Since $\int_{y_b(t)}^{y_t(l)} \mu_t(y) dy > 0$ we conclude that $h(t) \rightarrow +\infty$ with $t \rightarrow x_l$. A similar reasoning involving R_r shows that $h(t) \rightarrow -\infty$ with $t \rightarrow x_r$. It follows that there is some $t_0 \in (x_l, x_r)$ with $h(t_0) = 0$. Hence, if we shift s to the position $x_s = t_0$ the force acting on s vanishes and s is in balance. $\square$

Definition 3. The entropy of a rectangle R_i of F is defined as $-w(i) \log p(i)$. The entropy of the floorplan F is

$$E = \sum -w(i) \log p(i)$$

The proof of Theorem 3 will be completed after showing the following

- (1) The entropy E is always nonpositive.
- (2) $E = 0$ if and only if all rectangles R_i of F are in balance.
- (3) Shifting an unbalanced segment s into its balance position increases the entropy.
- (4) The process of repeatedly shifting unbalanced segments into their balance position makes F converge to a floorplan F' such that the entropy of F' is zero.
- (5) The solution is unique.

The first two of these statements are shown in the next lemma.

Lemma 2. The entropy E is always nonpositive and $E = 0$ if and only if all rectangles R_i of F are in balance.

Proof. We use that $p(i) > 0$ and hence $\log p(i) \geq (1 - \frac{1}{p(i)}) = (1 - \frac{m(R_i)}{w(i)})$. For the entropy of R_i we get $-w(i) \log p(i) \leq -w(i)(1 - \frac{m(R_i)}{w(i)}) = m(R_i) - w(i)$. This yields

$$E = \sum_i -w(i) \log p(i) \leq \sum_i m(R_i) - \sum_i w(i) = 1 - 1 = 0$$

The equality $E = 0$ is equivalent to equality for each summand. Hence $0 = \log p(i) = (1 - \frac{m(R_i)}{w(i)})$ and $m(R_i) = w(i)$ for all i . $\square$

Lemma 3. Shifting an unbalanced segment s into its balance position increases the entropy.

Proof. We consider a vertical segment s as in the proof of Proposition 4 and assume $f(s) > 0$. Let t_0 be the first zero of $h(t)$ right of x_s . For all $t \in [x_s, t_0]$ the force $h(t)$ acting on s is positive, i.e., pushing s to the right.

Let $E(t)$ be the entropy of the floorplan when s is shifted towards the position $x_s = t_0$. We consider $E(t)$ as a function of t .

Claim. $\frac{d}{dt} E(t) = h(t)$.

Only rectangles touching s change their contribution to $E(t)$. Let $R_i = [x_l, t] \times [y_1, y_2]$ be a rectangle on the left of s , i.e., $R_i \in S_-$, and t is the x -coordinate of the right side of R_i . Hence

$$\begin{aligned} \frac{d}{dt}(-w(i) \log p(i)) &= -w(i) \frac{1}{p(i)} p'(i) = -w(i) \frac{m(R_i)}{w(i)} \frac{d}{dt} \frac{w(i)}{m(R_i)} = \\ -w(i) \frac{m(R_i)}{w(i)} w(i) \frac{-m'(R_i)}{m^2(R_i)} &= \frac{w(i)}{m(R_i)} m'(R_i) = \frac{w(i)}{m(R_i)} \frac{d}{dt} \int_{x_l}^t \int_{y_1}^{y_2} \mu(x, y) dy dx = \\ \frac{w(i)}{m(R_i)} \int_{y_1}^{y_2} \mu_t(y) dy &= p(i) \int_{y_1}^{y_2} \mu_t(y) dy. \end{aligned}$$

When $R_i \in S_+$ the mass $m(R_i)$ is decreasing with t so that $m'(R_i)$ is negative and $\frac{d}{dt}(-w(i) \log p(i)) = -p(i) \int_{y_1}^{y_2} \mu_t(y) dy$. Summing this over all rectangles incident to s we obtain that $\frac{d}{dt} E(t) = h(t)$. This is the claim. $\triangle$

While shifting s from the initial position x_s to t_0 we have $h(t) > 0$. The claim implies that the derivative of the entropy is positive and, hence, the entropy is increasing. $\square$

We continue with item (4) from our program. To prove this, however, we have to add a condition to the process of balancing segments. The iteration has to be performed such that no unbalanced segment can be ignored. A rule is called *nonignoring* if it complies with this condition. Here are two examples of nonignoring selection rules: 1) Choose the segment for balancing uniformly at random from the set of unbalanced segments. 2) Always choose the segment so that the increase of the entropy is as large as possible.

Proposition 5. *Let $F_0, F_1, F_2, \dots$ be a sequence of floorplans where F_{i+1} is obtained from F_i by balancing an unbalanced segment from F_i . If the selection of segments is nonignoring, then there is a subsequence $G_0, G_1, \dots$ of floorplans that has a limit $G = \lim G_i$ and the entropy of the floorplan G is zero.*

Proof. Let $s_1, s_2, \dots, s_n$ be the inner segments of F . Floorplans that are weakly equivalent to F can be encoded by the coordinate vector of the segments. This vector z in $\mathbb{R}^n$ has the value $z(i) = x_s$ if s_i is a vertical segment and $z(i) = y_s$ if s_i is horizontal. A sequence of floorplans is converging if the corresponding coordinate vectors converge.

Consider the sequence of coordinate vectors $z_0, z_1, \dots$ of the given sequence of floorplans. Since each of the coordinates of these vectors is from the interval $(0, 1)$, there is a convergent subsequence. Let $G_0, G_1, \dots$ be the corresponding convergent sequence of floorplans and let e_i be the entropy of G_i . From Lemma 3 we know that the e_i are an increasing sequence of negative numbers. Assume that the sequence e_i converges to $-a \neq 0$. Consider the limit $G = \lim G_i$. Since the entropy of G is $-a < 0$ there is an unbalanced rectangle R_j in G (Lemma 2) and, hence, there is an unbalanced segment s in G (Proposition 3). Let Δ be the increase of the entropy that comes from balancing s in G . Now, for all i greater than a sufficiently large N the floorplan G_i is so close to G that balancing s in G_i implies an increase of entropy of at least $\Delta/2$. For all i greater than a sufficiently large M we also have $e_i > -a - \Delta/2$. It follows that the unbalanced segment s was not used for balancing in any G_i with $i \geq \max(M, N)$. This is in contradiction to the assumption that the process is nonignoring. $\square$

Actually, a stronger statement is true. The full sequence $F_0, F_1, F_2, \dots$ is also converging. To prove this we need the uniqueness from Proposition 6. In fact if G is the unique floorplan that is weakly equivalent to F and has $m(R_i) = w(i)$ for all i , then it follows from the continuity of the entropy that there is an $\varepsilon > 0$ such that all floorplans whose entropy is larger than $-\varepsilon$ have a coordinate vector that is δ_ε close to the coordinate vector of G . This implies that $\lim F_i = G$.

Proposition 6. *For every floorplan F with $n + 1$ rectangles and every positive weight function $w : \{1, \dots, n + 1\} \rightarrow \mathbb{R}_+$ with $\sum_i w(i) = 1$ there is a unique floorplan F' in the unit square that is weakly equivalent to F and has $m(R_i) = w(i)$ for all i .*

A proof of the proposition can be found in the paper by Eppstein et al. [5].

4 Accommodating Floorplans on Point Sets

Let P be a generic set of n points in a rectangle $\mathcal{R}$. Let F be a generic floorplan and S be a subset of the segments of F of size n . A *cover map* from (F, S) to P is a floorplan F' with outer rectangle $\mathcal{R}$ that is weakly equivalent to F such that every segment from $S' = \phi(S)$ contains a point from P . The following is a generalization of Theorem 1.

Theorem 4. *If P is a generic set of n points in a rectangle $\mathcal{R}$ and F is a generic floorplan with a prescribed subset S of the segments of size n , then there is a cover map F' from (F, S) to P .*

Proof. We use Theorem 3 as a tool for the proof. First we transform the point set P into a suitable density distribution $\mu = \mu_P$ inside $\mathcal{R}$. This density is defined as the sum of a uniform distribution μ_1 with $\mu_1(q) = 1/\text{area}(\mathcal{R})$ for all $q \in \mathcal{R}$ and a distribution μ_2 that represents the points of P . Choose some $\Delta > 0$ such that $\|p - p'\| > 3\Delta$ for all $p, p' \in P$. Define $\mu_2 = \sum_{p \in P} \mu_p$ where $\mu_p(q)$ takes the value $(\Delta^2 \pi)^{-1}$ on the disk $D_\Delta(p)$ of radius Δ around p and value 0 for q outside of this disk.

For a density ν defined on $\mathcal{R}$ and a rectangle $R \subseteq \mathcal{R}$ let $\nu(R)$ be the integral of the density ν over R . Since $\mu_1(\mathcal{R}) = 1$ and $\mu_p(\mathcal{R}) = 1$ for all $p \in P$ the total mass of $\mathcal{R}$ is $\mu(\mathcal{R}) = 1 + n$.

Transform the floorplan F into a floorplan F_S depending on the set S of segments that have to cover points of P : inflate every segment in S to form a thin rectangle. This description is not very formal but with a look at Figure 7 should make clear how to produce F_S from F and S . Let $\mathcal{S}$ be the set of new rectangles obtained by inflating segments from S .

Define weights for the rectangles of F_S as follows. If F_S has r rectangles we define $w(R) = 1 + 1/r$ if $R \in \mathcal{S}$ and $w(R) = 1/r$ for all the rectangles of F_S that came from rectangles of F . The total weight, $\sum_R w(R) = 1 + n$ equals the total mass $\mu(\mathcal{R})$.

The data $\mathcal{R}$ with μ and F_S with w constitute, up to scaling of $\mathcal{R}$ and w , a set of inputs for Theorem 3. From the conclusion of the theorem we obtain a floorplan F'_S weakly equivalent to F_S such that $m(R) = \iint_R \mu(x, y) dx dy = w(R)$ for all rectangles.

The definition of the weight function w and the density μ is so that F'_S should be close to a cover map from (F, S) to P : Only the rectangles $R \in \mathcal{S}$ that have been constructed by inflating segments may contain a disk $D_\Delta(p)$ and each of these rectangles

Fig. 7. Floorplans F with as prescribed subset S of segments (bold) and the floorplan F_S obtained by doubling the segments of S

may contain at most one of the disks. This suggests a correspondence $S \leftrightarrow P$. However, a rectangle $R \in S$ may use parts of several discs to accumulate mass. To find a correspondence between S and P we define a bipartite graph G whose vertices are the points in P and the rectangles in S :

- A pair (p, R) is an edge of G iff $R \cap D_\Delta(p) \neq \emptyset$ in F'_S .

The proof of the theorem will be completed by proving two claims: 1) G admits a perfect matching. 2) From F'_S and a perfect matching M in G we can produce a floorplan F' that realizes the cover map from (F, S) to P .

For the first of the claims we check Hall's matching condition: Consider a subset A of S . Since F_S is realizing the prescribed weights we have $m(A) = \mu(A) = \sum_{R \in A} \mu(R) = \sum_{R \in A} w(R) = |A|(1 + 1/r)$. Since $\mu_1(A) < 1$ and $\mu_p(A) \leq 1$ for all $p \in P$ there must be at least $|A|$ points $p \in P$ with $\mu_p(A) > 0$, these are the points that have an edge to a rectangle from A in G . We have thus shown that every set A of inflated segments is incident to at least $|A|$ points in G , hence, there is an injective mapping $\alpha : S \rightarrow P$ such that $R \cap D_\Delta(\alpha(R)) \neq \emptyset$ in F'_S for all $R \in S$.

To prove the second claim we have to construct a floorplan F' that realizes the cover map from (F, S) to P : Let s be a segment in S and let R_s be the rectangle in F'_S that corresponds to s . If s is horizontal we define s' to be the unique maximal horizontal segment in R_s whose y -coordinate is as close to the y -coordinate of the point $\alpha(R_s)$ as possible. For vertical segments we focus on the x -coordinate. For segments s of F that do not belong to S set $s' = s$. The collection $\{s' : s \text{ segment in } F\}$ of segments may fail to be a floorplan, see e.g. Figure 8.b. However, if s_1 and s_2 are segments of F such that s_1 has one of its endpoints on s_2 and $s_2 \in S$ then we can extend s'_1 into R_{s_2} to recover the contact with s'_2 . By doing this for all qualifying pairs s'_1, s'_2 we get a floorplan, see Figure 8.c. This floorplan is weakly equivalent to F' but there may still be segments of S that do not quite cover the assigned point. By construction the distance from a segment to its assigned point is at most Δ . Since Δ is small compared to the distances of points in P we can shift all segments orthogonally to make them cover their assigned points. Again this may spoil the floorplan property, see e.g. Figure 8.d. However, enlarging or shortening of segments by an amount of at most Δ at the ends finally yields the floorplan F' that realizes the cover from (F, S) to P . $\square$

The topic of [1] was the study of the number $Z(P)$ of rectangulations of a generic point set P . In our terminology this is the total number of cover maps from floorplans with n inner segments to a generic point set P with n points. Theorem 4 implies that this number is at least as large as the number of weak equivalence classes of floorplans.

Fig. 8. a) A solution F'_S for the instance from Fig. 1. The arrows indicate a matching α . b) Segments $s \in S$ shifted to their optimal position in R_s . c) Enlarged segments recover the contacts. d) Some segments s are moved outside R_s to cover the corresponding points $\alpha(R_s)$. Small final adjustments (clipping and enlarging) yield F' .

This is the Baxter number B_{n+1} which is known to be of order $\Theta(8^{n+1}/(n+1)^4)$. In [1] an upper bound for $Z(P)$ of order $O(20^n/n^4)$ is shown. To improve this bound remains an intriguing problem.

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