Chapter 2

Charged Particles in Near-Earth Space

In this chapter we discuss the concepts that govern the motion of charged particles in the geomagnetic field and the principles how they stay trapped in the radiation belts. The basic particle orbit theory can be found in most plasma physics textbooks. We partly follow the presentation in Koskinen (2011). A more detailed discussion can be found in Roederer and Zhang (2014). A classic treatment of adiabatic motion of charged particles is Northrop (1963).

The Lorentz force and Maxwell's equations are summarized in Appendix A.1 where we also introduce the key concepts and notations of basic electrodynamics used in the book.

2.1 Guiding Center Approximation

The equation of motion of a particle with charge q , mass m and velocity $\mathbf{v}$ under the *Lorentz force* due to the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields is

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (2.1)$$

where $\mathbf{p} = \gamma m \mathbf{v}$ is the relativistic momentum and $\gamma = (1 - v^2/c^2)^{-1/2}$ the *Lorentz factor*. As discussed in Appendix A.1 we do not use terms “rest mass” or “relativistic mass”. The mass of an electron is $m_e = 511 \text{ keV } c^{-2}$ and of a proton $m_p = 931 \text{ MeV } c^{-2}$.

Integration of the equation of motion in realistic magnetic field configurations must be done numerically. Numerical computation of even a fairly large number of particle orbits is not a problem for present day computers but to get a mental picture of particle motion we need an analytically tractable approach. Such is the *guiding center approximation* introduced by Hannes Alfvén in the 1940s (Alfvén 1950).

Let us start, for simplicity, from non-relativistic particles and consider a homogeneous static magnetic field in the direction of the z -coordinate with zero electric field. The equation of motion of a charged particle

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \quad (2.2)$$

describes a helical orbit with constant speed along the magnetic field and circular motion around the magnetic field line with the *angular frequency*

$$\omega_c = \frac{|q|B}{m} . \quad (2.3)$$

We call ω_c *gyro frequency* (also terms *cyclotron frequency* or *Larmor frequency* are frequently used). In the literature ω_c sometimes includes the sign of the charge q . In this book we write the gyro frequency as a positive quantity $\omega_c = |q|B/m$ and indicate the sign explicitly. The corresponding *oscillation frequencies* $f_{c\alpha} = \omega_{c\alpha}/(2\pi)$ of electrons and protons are

$$\begin{aligned} f_{ce}(\text{Hz}) &\approx 28 B(\text{nT}) \\ f_{cp}(\text{Hz}) &\approx 1.5 \times 10^{-2} B(\text{nT}) . \end{aligned}$$

The period of the gyro motion is

$$\tau_L = \frac{2\pi}{\omega_c} \quad (2.4)$$

and the radius of the circular motion perpendicular to the magnetic field

$$r_L = \frac{v_{\perp}}{\omega_c} = \frac{mv_{\perp}}{|q|B} , \quad (2.5)$$

where $v_{\perp} = \sqrt{v_x^2 + v_y^2}$ is the velocity perpendicular to the magnetic field. r_L is called the *gyro radius* (*cyclotron radius*, *Larmor radius*). Looking along (against) the magnetic field, the particle rotating clockwise (anticlockwise) has a negative charge. In plasma physics this is the convention of *right-handedness*.

This way we have decomposed the motion into two elements: constant speed $v_{\parallel}$ along the magnetic field and circular velocity $v_{\perp}$ perpendicular to the field. The sum of these components is a helical motion with the *pitch angle* α defined as

$$\begin{aligned} \tan \alpha &= v_{\perp}/v_{\parallel} \\ \Rightarrow \alpha &= \arcsin(v_{\perp}/v) = \arccos(v_{\parallel}/v) . \end{aligned} \quad (2.6)$$

Alfvén pointed out that this decomposition is convenient even in temporally and spatially varying fields if the variations are small compared to the gyro motion and that the field does not change much with the particle motion along the magnetic field during one gyro period. This is the *guiding center approximation*. The center of the gyro motion is the *guiding center* (GC) and we call the frame of reference where $v_{\parallel} = 0$ is the *guiding center system* (GCS).

In the GCS the charge gives rise to a current $I = q/\tau_L$ along its circular path with associated *magnetic moment*

$$\mu = I\pi r_L^2 = \frac{1}{2} \frac{q^2 r_L^2 B}{m} = \frac{1}{2} \frac{m v_{\perp}^2}{B} = \frac{W_{\perp}}{B}. \quad (2.7)$$

Here we have introduced “perpendicular energy” $W_{\perp}$ to refer to the kinetic energy related to the velocity perpendicular to the magnetic field. Similarly, we define “parallel energy” $W_{\parallel} = (1/2)mv_{\parallel}^2$. The total energy of the particle is $W = W_{\parallel} + W_{\perp}$. With the word “energy” we refer to the kinetic energy, written relativistically (Appendix A.1) as

$$W = mc^2(\gamma - 1). \quad (2.8)$$

The magnetic moment is actually a vector $(q/2) \mathbf{r}_L \times \mathbf{v}_{\perp}$, where the gyro radius is a vector $\mathbf{r}_L$ pointing from the guiding center to the particle. The magnetic moment of both negatively and positively charged particles is opposite to the ambient magnetic field. Thus charged particles tend to weaken the background magnetic field and the plasma consisting of free charges resembles a *diamagnetic* medium. This is a useful concept when we discuss electromagnetic waves in the cold plasma approximation (Chap. 4).

For relativistic particles the gyro frequency and gyro radius are obtained simply replacing m by γm . In the formulas later in the text ω_c refers to the non-relativistic frequency and γ is introduced explicitly, when needed. Of course, when calculating the gyro periods and gyro radii, γ must be included. Furthermore, the magnetic moment of relativistic particles must be expressed in terms of momentum ($p = \gamma m v$) as

$$\mu = \frac{p_{\perp}^2}{2mB}. \quad (2.9)$$

Note that here the constant mass m (not γm) is introduced in the denominator to give the familiar magnetic moment at the non-relativistic limit. We will return to the relativistic magnetic moment in Sect. 2.4.1.

2.2 Drift Motion

Adding a background electric field or letting the magnetic field be inhomogeneous, as it always is in the magnetosphere, modifies the path of the charged particles. If the effect remains small enough during one gyro period, the guiding center approximation is a useful tool to describe the motion.

2.2.1 $\mathbf{E} \times \mathbf{B}$ Drift

We start by adding a constant electric field perpendicular to the constant magnetic field. The equation of motion is again straightforward to solve. The guiding center is found to drift perpendicular to both electric and magnetic fields with the velocity

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \quad (2.10)$$

This is called the *electric drift* (or $\mathbf{E} \times \mathbf{B}$ drift). The drift velocity is independent of the charge, mass and energy of the particle. All charged particles move with the same velocity and thus the $\mathbf{E} \times \mathbf{B}$ drift does not give rise to electric current. An example of the drift is the magnetospheric convection (Sec. 1.3.1) transporting particles from the tail plasma sheet toward the Earth.

The $\mathbf{E} \times \mathbf{B}$ drift can also be found by making a non-relativistic ($\gamma \approx 1$) Lorentz transformation to the frame co-moving with the guiding center

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \quad (2.11)$$

In this frame $\mathbf{E}' = 0 \Rightarrow \mathbf{E} = -\mathbf{v} \times \mathbf{B}$, from which we find the solution (2.10) for $\mathbf{v}$. Note that a possible electric field component parallel to $\mathbf{B}$ cannot be eliminated by coordinate transformation because $\mathbf{E} \cdot \mathbf{B}$ is Lorentz invariant.

This coordinate transformation is possible for weak enough perpendicular forces $\mathbf{F}_\perp$ resulting in a general expression for the drift velocity

$$\mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}. \quad (2.12)$$

Here “weak enough” means $F/qB \ll c$. For example, for the $\mathbf{E} \times \mathbf{B}$ drift the ratio E/B must be much smaller than speed of light. Otherwise the guiding center approximation cannot be used.

2.2.2 Gradient and Curvature Drifts

The charged particles in the radiation belts move in a nearly dipolar magnetic field of the Earth that is curved and has gradients perpendicular to and along the field. Consider a spatially inhomogeneous magnetic field, assuming that the perpendicular and field-aligned components of the gradients are small within one gyration of the particle

$$|\nabla_{\perp} B| \ll B/r_L ; |\nabla_{\parallel} B| \ll (\omega_c/v_{\parallel})B .$$

Under these circumstances it is possible to use perturbation approach to solve the equation of motion (Northrop 1963). The validity of these conditions does not depend on the field geometry alone but also on the energy and mass of the particle whose motion is to be calculated.

As only weak spatial inhomogeneities of the field are assumed, we can make Taylor expansion of the external magnetic field around the particle's GC. The details of the calculation can be found in most advanced plasma physics textbooks (e.g., Koskinen 2011). The solution of the equation of motion can be expressed as a sum of the unperturbed gyro motion and a small correction.

We start by neglecting the field line curvature. Keeping the first order terms and averaging over one gyro period the force reduces to

$$\mathbf{F} = -\mu \nabla B . \quad (2.13)$$

If the gradient has a magnetic field-aligned component, the force causes acceleration/deceleration of charged particles along the field

$$\frac{dv_{\parallel}}{dt} = -\frac{\mu}{m} \nabla_{\parallel} B = -\frac{\mu}{m} \frac{\partial B(s)}{\partial s} \mathbf{b} , \quad (2.14)$$

where s is the coordinate and $\mathbf{b}$ the unit vector along the magnetic field.

In the perpendicular direction we find a drift across the magnetic field in the same way as for the zero order drift. The drift must balance the perpendicular force term (2.12) implying

$$\mathbf{v}_G = \frac{\mu}{qB^2} \mathbf{B} \times (\nabla B) = \frac{W_{\perp}}{qB^3} \mathbf{B} \times (\nabla B) . \quad (2.15)$$

This is called the *gradient drift*. The drift is perpendicular to both the magnetic field and its gradient. It is a result of small changes of the gyro radius as the particle gyrates in the inhomogeneous magnetic field (Fig. 2.1).

The gradient drift depends both on the perpendicular energy and the charge of the particle. Because negatively and positively charged particles drift to opposite directions, the drift contributes to the electric current in the plasma.

Fig. 2.1 The small variation of the gyro radius due to the magnetic field gradient results in the gradient drift

In a curved magnetic field also the GC motion is curved and the reference frame attached to the GC is not inertial. Let us denote the GC velocity by $\mathbf{V}$. Now $\mathbf{V}_{\parallel}$ is not exactly equal to $\mathbf{v}_{\parallel}$ because $\mathbf{V}$ is determined at the GC whereas $\mathbf{v}$ is the velocity of the charge at the distance of $\mathbf{r}_L$ from the GC. The difference becomes significant for particles whose gyro radii approach the gradient scale length, i.e., near the limit where the GC approximation becomes invalid (for a more detailed discussion, see, Roederer and Zhang 2014).

Let us move to the frame co-moving with the GC. Let the orthogonal basis $\{\mathbf{e}_i\}$ define the coordinate axes and choose $\mathbf{e}_3 \parallel \mathbf{v}_{\parallel} \parallel \mathbf{B}$. Now $\mathbf{v} = \sum v_i \mathbf{e}_i$, and $\{\mathbf{e}_i\}$ rotates when its origin moves with the GC.

$$\frac{d\mathbf{v}}{dt} = \sum_i \left(\frac{dv_i}{dt} \mathbf{e}_i + v_i \frac{d\mathbf{e}_i}{dt} \right) = \sum_i \left(\frac{dv_i}{dt} \mathbf{e}_i + v_i (\mathbf{V}_{\parallel} \cdot \nabla) \mathbf{e}_i \right). \quad (2.16)$$

The term $\sum v_i (\mathbf{V}_{\parallel} \cdot \nabla) \mathbf{e}_i$ is due to the curvature and causes a centrifugal effect. Averaged over one Larmor cycle the curvature force is

$$\mathbf{F}_C = - \left\langle m \sum_i v_i (\mathbf{V}_{\parallel} \cdot \nabla) \mathbf{e}_i \right\rangle. \quad (2.17)$$

Due to the assumption of weak curvature $(\mathbf{V}_{\parallel} \cdot \nabla) \mathbf{e}_i$ can be approximated constant in every point along the gyro orbit. In the present approximation the perpendicular velocity components v_1 and v_2 oscillate sinusoidally and thus $\langle v_1 \mathbf{e}_1 \rangle = \langle v_2 \mathbf{e}_2 \rangle = 0$. Furthermore, during one gyro period, on average, $v_{\parallel} \approx V_{\parallel}$ and we get

$$\mathbf{F}_C = -m V_{\parallel}^2 (\mathbf{e}_3 \cdot \nabla) \mathbf{e}_3. \quad (2.18)$$

A little exercise in differential geometry yields

$$(\mathbf{e}_3 \cdot \nabla) \mathbf{e}_3 = \mathbf{R}_C / R_C^2, \quad (2.19)$$

where $\mathbf{R}_C$ is the *radius of curvature* vector, pointing inward. Now

$$\mathbf{F}_C = -m V_{\parallel}^2 \frac{\mathbf{R}_C}{R_C^2}. \quad (2.20)$$

Because $\mathbf{B} = B \mathbf{e}_3$,

$$(\mathbf{e}_3 \cdot \nabla) \mathbf{e}_3 = (\mathbf{B} \cdot \nabla \mathbf{B}) / B^2 \quad (2.21)$$

and we can write the resulting *curvature drift* velocity as

$$\mathbf{v}_C = \frac{-m V_{\parallel}^2}{q B^2} \frac{\mathbf{R}_C \times \mathbf{B}}{R_C^2} = \frac{m V_{\parallel}^2}{q B^4} \mathbf{B} \times (\mathbf{B} \cdot \nabla) \mathbf{B}. \quad (2.22)$$

Now we can again approximate $v_{\parallel} \approx V_{\parallel}$ and express the curvature drift in terms of the parallel energy of the particle $W_{\parallel} = (1/2) m v_{\parallel}^2$.

If there are no local currents ($\nabla \times \mathbf{B} = 0$), as in the case of a pure dipole field, the curvature drift simplifies to

$$\mathbf{v}_C = \frac{2 W_{\parallel}}{q B^3} \mathbf{B} \times \nabla B \quad (2.23)$$

and $\mathbf{v}_G$ and $\mathbf{v}_C$ can be combined as

$$\mathbf{v}_{GC} = \frac{W_{\perp} + 2 W_{\parallel}}{q B^3} \mathbf{B} \times \nabla B = \frac{W}{q B R_C} (1 + \cos^2 \alpha) \mathbf{n} \times \mathbf{t}, \quad (2.24)$$

where $\mathbf{t} \parallel \mathbf{B}$ and $\mathbf{n} \parallel \mathbf{R}_C$ are unit vectors. The drift velocities are straightforward to write relativistically replacing m by γm .

As a consequence of the gradient and curvature of the near-Earth geomagnetic field charged high-energy particles drift around the Earth, electrons to the east and positively charged particles to the west, resulting in net westward current. The motion of low-energy particles is dominated by the $\mathbf{E} \times \mathbf{B}$ -drift.

The perturbation approach can be continued to higher orders. The recipe is the same as above. First determine the force due to the higher order perturbation and calculate the drift velocity to balance its effect. Mathematically this procedure results in an *asymptotic* expansion of the guiding center position as a function of time (for a detailed discussion, see, Northrop 1963).

2.3 Drifts in the Magnetospheric Electric Field

In magnetospheric physics a commonly used frame of reference is the *Geocentric Solar Magnetospheric* coordinate system (GSM). In the GSM system the X -axis points toward the Sun and the Earth's dipole axis is in the XZ -plane. Z points approximately northward and Y is opposite to the Earth's orbital motion around the Sun. As the Earth rotates, the XZ -plane flaps about the X -axis such that the dipole remains in the XZ -plane but can be tilted from the Z -direction maximally 34° , which is the sum of the tilts of the Earth's rotation axis ($\approx 23^\circ$) from the ecliptic plane and of the dipole axis ($\approx 11^\circ$) from the rotation axis.

The large-scale magnetospheric plasma flow in the GSM system gives rise to the electric field $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$. If the magnetic field is time-independent, the electric field is curl-free and can be expressed as the gradient of a scalar potential $\mathbf{E} = -\nabla\varphi$. During rapid changes of the magnetic field, such as geomagnetic storms, also the inductive electric field given by Faraday's law $\partial\mathbf{B}/\partial t = -\nabla \times \mathbf{E}$ must be taken into account.

Let us, for simplicity, consider the motion of relatively low-energy plasma sheet and plasmaspheric particles in the GSM equatorial plane. Assume further that the magnetospheric magnetic field is perpendicular to the equatorial plane pointing upward (Z -direction). Thus the sunward advection in the plasma sheet corresponds to a dawn-to-dusk (Y -direction) pointing electric field $E_0 \mathbf{e}_y$, which we here assume to be constant. Let r be the distance from the center of the Earth and ϕ the angle from the direction of the Sun. Then the electric field, called in this context convection electric field, is given by

$$\mathbf{E}_{conv} = -\nabla(-E_0 r \sin \phi) \quad (2.25)$$

and its potential is

$$\varphi_{conv} = -E_0 r \sin \phi. \quad (2.26)$$

The Earth and its atmosphere rotate in the GSM frame. The corotation extends in the equatorial plane roughly up to the plasmopause. The angular velocity toward the east is $\Omega_E = 2\pi/24\text{ h}$. Assuming, again for simplicity, a perfect corotation, the plasma velocity in the GSM frame is

$$\mathbf{V}_{rot} = \Omega_E r \mathbf{e}_\phi, \quad (2.27)$$

where $\mathbf{e}_\phi$ is the azimuthal unit vector pointing toward the east. Consequently the motion induced electric field is $\mathbf{E}_{rot} = -\mathbf{V} \times \mathbf{B}$, the potential of which is

$$\varphi_{rot} = \frac{-\Omega_E k_0}{r} = \frac{-\Omega_E B_0 R_E^3}{r}. \quad (2.28)$$

Fig. 2.2 Equipotential lines of convection and corotation electric fields in the equatorial plane. The local time directions are given at the faces of the panels. (From Koskinen 2011, reprinted by permission from SpringerNature)

Here $k_0 = 8 \times 10^{15} \text{ T m}^3$ is the Earth's dipole moment and B_0 the dipole field on the surface of the Earth at the equatorial plane ($B_0 \approx 30 \mu\text{T}$). The convection and corotation electric fields are illustrated in Fig. 2.2. The total field is the superposition $\mathbf{E}_{conv} + \mathbf{E}_{rot}$.

In addition to the large scale electric field in their frame of reference, the plasma particles are also under the influence of magnetic field gradient and curvature forces. Let us consider particles that move in the equatorial plane of the dipole ($\mathbf{v}_{\parallel} = 0$ or equivalently $\alpha = 90^\circ$). For these particles the curvature effect is zero and the total drift velocity including the $\mathbf{E} \times \mathbf{B}$ and gradient drifts is

$$\mathbf{v}_D = \frac{1}{B^2} \left[\mathbf{E}_{conv} + \mathbf{E}_{rot} - \nabla \left(\frac{\mu B}{q} \right) \right] \times \mathbf{B} = \frac{1}{B^2} \mathbf{B} \times \nabla \varphi_{eff}, \quad (2.29)$$

where μ is the magnetic moment of the particles. The effective potential is

$$\varphi_{eff} = -E_0 r \sin \phi - \frac{\Omega_E B_0 R_E^3}{r} + \frac{\mu B_0 R_E^3}{q r^3}. \quad (2.30)$$

In the time-independent potential field the particles move along stream lines of constant φ_{eff} . These stream lines depend on the charge and energy of the particles through their magnetic moments. For low-energy particles, whose perpendicular velocity in the moving frame is small ($\mathbf{v}_{\perp} \approx 0 \Rightarrow \mu \approx 0$), the streamlines are equipotential lines of the combined convection and corotation fields (Fig. 2.3). In this approximation the motion is a pure $\mathbf{E} \times \mathbf{B}$ -drift and all particles move with the same velocity.

Fig. 2.3 Orbits of low-energy particles ($\mu \approx 0$) in the equatorial plane assuming $E_0 = 0.3 \text{ mV m}^{-1}$. The distance between consecutive points is 10 min. (From Koskinen 2011, reprinted by permission from Springer Nature)

Figure 2.3 illustrates the formation of a *separatrix* that separates the low-energy corotating plasmaspheric plasma from the sunward cold plasma motion in the plasma sheet. In this single-particle model the separatrix is the plasmapause. The separatrix has an electric neutral point ($E = 0$) at the distance

$$r = \sqrt{\frac{\Omega_E B_0 R_E^3}{E_0}} \quad (2.31)$$

in the direction of 18 MLT. While this model of the plasmasphere is a strong simplification, it explains qualitatively why the plasmasphere is compressed during enhanced magnetospheric activity (Fig. 1.4) and the formation of the plasmaspheric bulge, or plume, in the evening sector (Fig. 1.5). The enhanced energy input enhances the convection velocity and thus the dawn-to-dusk electric field, whereas the rotation of the Earth remains constant and the corotation electric field is always the same. Consequently, the separatrix is pushed toward the Earth when the convection enhances, i.e., E_0 in (2.30) increases.

The real plasma has a finite temperature and the plasma drifting from the plasma sheet is much warmer than the plasmasphere. Thus the gradient drift term in (2.30) must be taken into account. Consequently, the shape, density and actual location of the plasmaspheric plume is determined by the actual plasma environment. The observed plume is widened and often pushed toward the earlier local time (Fig. 1.5). Furthermore, the plasmapause reacts to the changing electric field with some delay, which can lead to detachment of the plume from the plasmasphere and its subsequent disappearance to the dayside magnetopause.

In the finite-temperature plasma sheet, the magnetic field gradient separates the motion of positive and negative charges (remember that we consider here equatorial particles only, for which the curvature drift is zero). To illustrate this effect consider particles whose magnetic moment is so strong that it supersedes the effect of corotation electric field. Now the effective potential, again in the equatorial plane only, is

$$\varphi_{\text{eff}} = -E_0 r \sin \phi + \frac{\mu B_0 R_E^3}{q r^3} . \quad (2.32)$$

The $r \sin \phi$ -dependence implies that far from the Earth the particles follow the convection electric field, but closer in they become dominated by the magnetic drifts. This way the dipole field shields the cold plasmasphere from the hot plasma sheet.

The positive and negative high-energy charges drift as illustrated in Fig. 2.4. They have different separatrices, called *Alfvén layers*. Because the plasma sheet is a finite particle source, a larger fraction of positive charges pass the Earth in the evening sector and a larger fraction of negative charges in the morning sector. This leads to piling of positive space charge in the evening sector and negative charge in the morning sector. The accumulated charges are discharged by magnetic field-aligned currents flowing to the ionosphere from the evening sector and from the ionosphere to the magnetosphere in the morning sector.

Figure 2.4 also gives a qualitative explanation for a mechanism how energetic particles can be lost from the plasmasphere during the main phase of strong magnetospheric storms, and on the other hand, how particles on open drift paths can become trapped again during the recovery phase. As the convective electric field rapidly grows during the main phase, the particles are pushed outward from the radiation belt in the dayside. The particles, whose trajectories are open, intercept

Fig. 2.4 Formation of Alfvén layers of energetic particles. (From Koskinen 2011, reprinted by permission from SpringerNature)

the magnetopause and are lost from the inner magnetosphere. At the same time more particles on the nightside $E \times B$ -drift deeper into the ring current and radiation belts. Once the activity ceases, the trapping boundary, i.e., the Alfvén layer, moves outward and thus particles that were originally on open drift paths past the Earth find themselves trapped into the expanding plasmasphere. Note, however, that the actual losses and enhancements of high-energy radiation belt particles depend on several other processes discussed in the subsequent chapters of this book.

2.4 Adiabatic Invariants

The trapped charged particles in the radiation belts perform three nearly periodic motions in the geomagnetic field. These quasi-periodic motions are related to *adiabatic invariants* in the terminology of Hamiltonian mechanics.

In physical systems symmetries correspond to conservation laws. For example, in classical mechanics rotational symmetry is associated with conservation of angular momentum. The angular momentum is invariant in a periodic motion around the axis of symmetry. If the motion is nearly-periodic, such as the gyro motion in the guiding center approximation, adiabatic invariants take the role of conserved quantities. The adiabatic invariants do not need to be the same as conserved quantities in the strictly periodic case and their conservation depends critically of the “slowness” of the variations.

In the framework of *Hamiltonian mechanics* it can be shown that if Q and P are the canonical coordinate and momentum of the system and if the motion is nearly periodic, the integral

$$I = \oint P dQ \quad (2.33)$$

over the (quasi-)period of Q is an adiabatic invariant, i.e., a conserved quantity. This statement requires, of course, a proof (see, e.g., Bellan 2006, or advanced classical mechanics textbooks).

A classic example of an adiabatic invariant is the *Lorentz–Einstein pendulum* in the gravitational field (g). Let the length of the pendulum (l) change so slowly that its frequency $\omega = \sqrt{g/l}$ does not change much during one swing. However, as work is done on the pendulum, the energy of the pendulum per unit mass

$$W = \frac{1}{2}l^2\dot{\theta}^2 + \frac{1}{2}gl\theta^2, \quad (2.34)$$

where θ is the angle from the vertical direction, is not constant. It is a straightforward exercise to show that in this case the conserved quantity is the ratio W/ω . This is closely analogous to the magnetic moment

$$\mu = \frac{W_{\perp}}{B} = \frac{q}{m} \frac{W_{\perp}}{\omega_c}, \quad (2.35)$$

which readily suggests that the magnetic moment is an adiabatic invariant, if the magnetic field changes slowly with respect to the gyro motion.

2.4.1 The First Adiabatic Invariant

A recommendable introduction to the Hamiltonian treatment of radiation belt particles is Ukhorskiy and Sitnov (2013). To show that the magnetic moment is an adiabatic invariant in the Hamiltonian framework recall from the classical electrodynamics that the momentum of a particle in an electromagnetic field is $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$, where $\mathbf{A}$ is the vector potential of the field. We can take the gyro radius vector $\mathbf{r}_L$ as the canonical coordinate in the plane perpendicular to the magnetic field. Then the corresponding canonical momentum is $\mathbf{p}_{\perp}$. Assuming that the guiding center approximation is valid, i.e., the gyro motion is nearly periodic, the integral

$$\begin{aligned} I &= \oint \mathbf{p}_{\perp} \cdot d\mathbf{r}_L = \oint m\mathbf{v}_{\perp} \cdot d\mathbf{r}_L + q \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \\ &= \int_0^{2\pi r_L} m v_{\perp} dl + q \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= 2\pi m v_{\perp} r_L - |q| B \pi r_L^2 = \frac{2\pi m}{|q|} \mu \end{aligned} \quad (2.36)$$

is an adiabatic invariant. In plasma physics the magnetic moment is called the *first adiabatic invariant*.

The physical dimension of μ is energy/magnetic field. Its SI-unit is J T^{-1} but in radiation belt physics the non-SI unit MeV G^{-1} is commonly used.

Note that also the magnetic flux enclosed within the gyro orbit

$$\Phi = B \pi r_L^2 = \frac{2\pi m}{q^2} \mu \quad (2.37)$$

is constant. This has important implication to flux conservation in macroscopic plasma physics, which is an expression of the concept of magnetic field lines being frozen-in to plasma motion.

To directly show, within Newtonian mechanics, that the magnetic moment, or any of the other invariants discussed below, is an adiabatic invariant is a cumbersome task. Plasma physics textbooks usually treat some special cases only. A fairly complete treatise can be found in Northrop (1963).

For the purpose of this book, it is instructive to look at the invariance of μ in a static magnetic field in the absence of electric field. Since there is no temporal variation of the magnetic field, the energy¹ of the particle $W = W_{\parallel} + W_{\perp}$ is constant, i.e.,

$$\frac{dW_{\parallel}}{dt} + \frac{dW_{\perp}}{dt} = 0. \quad (2.38)$$

As $W_{\perp} = \mu B$,

$$\frac{dW_{\perp}}{dt} = \mu \frac{dB}{dt} + \frac{d\mu}{dt} B. \quad (2.39)$$

Now $dB/dt = v_{\parallel} dB/ds$ is the change of the magnetic field along the GC path. The parallel force is accroding to (2.14)

$$m \frac{dv_{\parallel}}{dt} = -\mu \frac{dB}{ds}. \quad (2.40)$$

By multiplying this by $v_{\parallel} = ds/dt$ we get

$$\frac{dW_{\parallel}}{dt} = -\mu \frac{dB}{dt}. \quad (2.41)$$

Thus

$$\frac{dW_{\parallel}}{dt} + \frac{dW_{\perp}}{dt} = B \frac{d\mu}{dt} = 0, \quad (2.42)$$

illustrating that μ is an adiabatic invariant *in this special case*.

If the magnetic field changes slowly in time ($\partial/\partial t \ll \omega_c$), Faraday's law implies the presence of an inductive electric field along the gyro path of the particle, yielding acceleration

$$\frac{dW_{\perp}}{dt} = q(\mathbf{E} \cdot \mathbf{v}_{\perp}). \quad (2.43)$$

¹ Recall that in our notation W refers to kinetic energy unless stated otherwise.

During one gyro period the particle gains energy

$$\Delta W_{\perp} = q \int_0^{2\pi/\omega_c} \mathbf{E} \cdot \mathbf{v}_{\perp} dt. \quad (2.44)$$

Due to the slow change we can replace the time integral by a line integral over one closed gyration and use Stokes' law

$$\Delta W_{\perp} = q \oint_C \mathbf{E} \cdot d\mathbf{l} = q \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -q \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}, \quad (2.45)$$

where $d\mathbf{S} = \mathbf{n} dS$, $\mathbf{n}$ is the normal vector of the surface S pointing to the direction determined in the right-hand sense by the positive circulation of the loop C . For small variations of the field $\partial \mathbf{B} / \partial t \approx \omega_c \Delta B / 2\pi$. Thus

$$\Delta W_{\perp} = \frac{1}{2} |q| \omega_c r_L^2 \Delta B = \mu \Delta B. \quad (2.46)$$

On the other hand

$$\Delta W_{\perp} = \mu \Delta B + B \Delta \mu, \quad (2.47)$$

implying $\Delta \mu = 0$. Thus for slow time variation μ is conserved although the inductive electric field accelerates the particle. This is analogous to the Lorentz-Einstein pendulum: The energy is not conserved but μ is.

In Chap. 6 we will discuss how temporal variations with a frequency near the gyro frequency of the particles can break the invariance of the magnetic moment and lead to acceleration, scattering and loss of particles through wave-particle interactions.

Generalization of the magnetic moment to relativistic particles must be done with some care. The relativistic conserved quantity is $p_{\perp}^2 / B$. In Eq. (2.9) we formulated the relativistic magnetic moment dividing this by the constant $2m$ (recall that m is the velocity-independent constant mass of the particle). Now

$$\mu_{rel} = \frac{p_{\perp}^2}{2mB} = \frac{\gamma^2 m^2 v_{\perp}^2}{2mB} = \gamma^2 \frac{mv_{\perp}^2}{2B} = \gamma^2 \mu_{nonrel}. \quad (2.48)$$

When reading the literature it is important to be alert whether the magnetic moment in an equation is the relativistic or non-relativistic quantity. Northrop (1963) pointed out that it is not easy to prove in a general magnetic configuration that this really is the correct generalization of the non-relativistic magnetic moment. Further complications arise if there is a parallel electric field that accelerates the particle and the total momentum is not constant.

Magnetic Mirror and Magnetic Bottle

The invariance of the magnetic moment leads us to important concepts in the motion of radiation belt particles: the magnetic mirror, the magnetic bottle and the loss-cone.

Assume first that the total kinetic energy W and the magnetic moment $\mu = W_{\perp}/B$ of a charged particle are conserved. Let the guiding center of the particle move along the magnetic field in the direction of a weak positive gradient of B . $W_{\perp}$ can increase until $W_{\parallel} \rightarrow 0$. The perpendicular velocity is $v_{\perp} = v \sin \alpha$ and we can write the magnetic moment as

$$\mu = \frac{mv^2 \sin^2 \alpha}{2B}. \quad (2.49)$$

On the other hand, as now $v^2 \propto W$ is also assumed to be constant, the pitch angles at two different magnetic field intensities are related as

$$\frac{\sin^2 \alpha_1}{\sin^2 \alpha_2} = \frac{B_1}{B_2}. \quad (2.50)$$

When $W_{\parallel} \rightarrow 0$, $\alpha \rightarrow 90^\circ$. The deceleration of the GC motion in an increasing magnetic field (2.14) is said to be due to the *mirror force* $\mathbf{F} = -\mu \nabla_{\parallel} B$, which finally turns the motion back toward the weakening B . The magnitude of the *mirror field* B_m at the turning point depends on the particle's pitch angle at a reference point B_0 . At the mirror field $\alpha_m = 90^\circ$ and we get

$$\sin^2 \alpha_0 = B_0/B_m. \quad (2.51)$$

As $B_m < \infty$, every mirror field is leaky. Particles having smaller pitch angles than α_0 in the field B_0 pass through the mirror. These particles are said to be in the *loss cone*. Two opposite mirrors form a *magnetic bottle*, which confines particles outside the loss cone of the weaker of the two magnetic mirrors. The quasi-dipolar near-Earth magnetic field is a giant inhomogeneous and temporally variable magnetic bottle trapping the radiation belt and ring current particles.

The mirror force does not need to be the only force affecting the parallel motion of the GC. If there is an electric field with a parallel component $\mathbf{E}_{\parallel}$ and/or the particle is in a gravitational field, the parallel equation of motion becomes

$$m \frac{dv_{\parallel}}{dt} = q\mathbf{E}_{\parallel} + m\mathbf{g}_{\parallel} - \mu \nabla_{\parallel} B. \quad (2.52)$$

If the non-magnetic forces can be derived from a potential $U(s)$ the equation of motion (2.14) is

$$m \frac{dv_{\parallel}}{dt} = -\frac{\partial}{\partial s}[U(s) + \mu B(s)]. \quad (2.53)$$

Now the GC moves in the effective potential $U(s) + \mu B(s)$. Work is done to the charged particle and thus its energy is not conserved but its magnetic moment is. In solar-terrestrial physics, examples of such potentials are the gravitational field on the Sun and parallel electric fields above discrete auroral arcs, of which the latter can have consequences to the radiation belts, affecting the width of the loss cone. Also plasma waves with a parallel electric field component contribute to the dynamics of radiation belt particles.

2.4.2 The Second Adiabatic Invariant

Assume that the energy is conserved, i.e., the speed v of the particle is constant. The *bounce motion* between the magnetic mirrors of a magnetic bottle is nearly periodic if the field does not change much during one *bounce period*, or *bounce time*, τ_b

$$\tau_b = 2 \int_{s_m}^{s'_m} \frac{ds}{v_{\parallel}(s)} = \frac{2}{v} \int_{s_m}^{s'_m} \frac{ds}{(1 - B(s)/B_m)^{1/2}}, \quad (2.54)$$

where s is the arc length along the GC orbit and s_m and s'_m are the coordinates of the mirror points. Here we have used Eq. (2.51) and $v_{\parallel}/v = \cos \alpha = \sqrt{1 - B(s)/B_m}$ to move the constant speed outside the integral, which now depends on the magnetic field configuration only. The bounce period is defined over the entire bounce motion back and forth.

The concept of bounce motion makes sense in the guiding center approximation if $\tau_b \gg \tau_L$. Thus the condition for the bounce motion to be nearly periodic is more restrictive than the condition for nearly periodic gyro motion

$$\tau_b \frac{dB/dt}{B} \ll 1. \quad (2.55)$$

If this condition is fulfilled, there is an associated adiabatic invariant. In nearly-periodic bounce motion the track of the GC does not enclose any magnetic flux and the canonical momentum reduces to $p_{\parallel}$. Now the canonical coordinate is the position of the GC along the magnetic field line. Consequently,

$$J = \oint p_{\parallel} ds \quad (2.56)$$

is an adiabatic invariant, generally known as the *second adiabatic invariant* or *longitudinal invariant*. The invariance of J is broken when the temporal variation of the magnetic field takes place in time scales comparable to or shorter than the bounce motion.

Both μ and J depend on particle momentum. As long as μ is conserved, the momentum can be eliminated from the second invariant by introducing a purely field-geometric quantity K

$$K = \frac{J}{\sqrt{8m\mu}} = I\sqrt{B_m} = \int_{s_m}^{s'_m} [B_m - B(s)]^{1/2} ds, \quad (2.57)$$

where m is the mass of the particle. The integral I is

$$I = \int_{s_m}^{s'_m} \left[1 - \frac{B(s)}{B_m} \right]^{1/2} ds \quad (2.58)$$

and $J = 2pI$. Since the invariance of J requires the invariance of μ , and m is the constant mass, J and K are physically equivalent. The physical dimension of K is square root of magnetic field times length, which in radiation belt studies is expressed in units of $R_E G^{1/2}$.

Neither the bounce period nor the integral I can be expressed in a closed form. We can, however, find useful expressions for particles mirroring close to the minimum of a symmetric magnetic bottle, e.g., the dipole equator, where we can approximate the field by a parabola

$$B(s) \simeq B_0 + \frac{1}{2}a_0s^2. \quad (2.59)$$

Here $a_0 = \partial^2 B / \partial s^2$ evaluated at the equator, where the magnetic field is B_0 . Now the parallel equation of motion $m dv_{\parallel} / dt = -\mu \partial B / \partial s$ becomes

$$m \frac{d^2 s}{dt^2} = -\mu a_0 s. \quad (2.60)$$

This is the familiar equation of motion for a linear pendulum with the period

$$\tau_b = 2\pi \sqrt{\frac{m}{\mu a_0}} = 2\pi \sqrt{\frac{mB}{W_{\perp} a_0}} \approx \frac{2\pi\sqrt{2}}{v} \sqrt{\frac{B_0}{a_0}}, \quad (2.61)$$

where we have approximated $v \approx v_{\perp}$ and $B \approx B_0$, as we consider particles with the equatorial pitch angle $\alpha_{eq} \approx 90^\circ$.

Now we have also the expression for the bounce period at the limit $\alpha_{eq} \rightarrow 90^\circ$ where the integrand in (2.54) diverges at the same time as the path length goes to zero. The bounce time remains finite, even if there is no actual bounce motion. This is analogous to the linear mechanical pendulum at rest (here $v_{\parallel} = 0$). The period of the pendulum is well-defined whether it swings or not! For the bounce motion about the dipole equator (2.61) this means that the “equatorially mirroring” particles do not violate the guiding center hierarchy $\tau_b \gg \tau_L$. Once the equatorial pitch angle

deviates from 90° , if only slightly, the bounce motion takes the time τ_b because the particle spends most of its bounce time close to the mirror point.

For equatorially mirroring particles integrals J , K and I are trivially zero. Using the same expansion as before we can find analytical expression for I for particles mirroring close to the equator. Letting $s = 0$ at the equator the mirror points are found to be

$$s_m \approx \pm \sqrt{\frac{2B_0}{a_0}} \left(1 - \frac{B_0}{B_m}\right) \quad (2.62)$$

and the integral reduces to

$$I \simeq \frac{\pi}{\sqrt{2}} \sqrt{\frac{B_0}{a_0}} \cos^2 \alpha_{eq} . \quad (2.63)$$

This approximation is good if $B_m/B_0 \lesssim 1.1$, corresponding to particles whose equatorial pitch angles are in the range $75^\circ \lesssim \alpha_{eq} \leq 90^\circ$. B_m and a_0 depend on the magnetic field configuration. In the dipole field the integral is found to be (Roederer 1970)

$$I \simeq \frac{\pi}{3\sqrt{2}} L R_E \left(1 - \frac{k_0}{B_m L^3 R_E^3}\right) . \quad (2.64)$$

The bounce periods in the dipole field are calculated in Sect. 2.5.

2.4.3 The Third Adiabatic Invariant

Also the drift across the magnetic field may be nearly-periodic if the drift motion takes place around an axis. This is particularly important in the radiation belts where particles drift around the Earth in the quasi-dipolar magnetic field. The corresponding *third adiabatic invariant* is the magnetic flux Φ through the closed contour defined by the equatorial track of the GC given by

$$\Phi = \oint \mathbf{A} \cdot d\mathbf{l} , \quad (2.65)$$

where $\mathbf{A}$ is the vector potential of the field and $d\mathbf{l}$ is the arc element along the drift path of the GC, which now can be taken as the canonical momentum and coordinate. In order for Φ to be an adiabatic invariant, the *drift period* τ_d has to fulfil $\tau_d \gg \tau_b \gg \tau_L$. Consequently, the invariant is weaker than μ and J and much slower changes in the magnetic field can break the invariance of Φ .

Table 2.1 Summary of adiabatic invariants

Invariant	Velocity	Time-scale	Validity
Magnetic moment μ	Gyro motion $\mathbf{v}_\perp$	Gyro period $\tau_L = 2\pi/\omega_c$	$\tau \gg \tau_L$
Longitudinal-invariant J	Parallel velocity of GC $\mathbf{V}_\parallel$	Bounce period τ_b	$\tau \gg \tau_b \gg \tau_L$ and μ constant
Flux invariant Φ	Perp. velocity of GC $\mathbf{V}_\perp$	Drift period τ_d	$\tau \gg \tau_d \gg \tau_b \gg \tau_L$ and μ and J constant

Table 2.1 summarizes the functions $\{\mu, J, \Phi\}$ and the conditions for their adiabatic invariance.

In three-dimensional space there are at most three independent adiabatic invariants but the actual number can be smaller, even zero. In radiation belts μ is often a good invariant and J is invariant for particles that spend at least some time in the magnetic bottle of the nearly-dipolar field. Under steady conditions also Φ is constant, but its invariance can be broken by spatial inhomogeneities and ultra low frequency (ULF) oscillations of the magnetospheric magnetic field, as will be discussed in Chap. 6.

In the language of Hamiltonian mechanics the functions $\{\mu, J, \Phi\}$ defined in Eqs. (2.36), (2.56), and (2.65), whether invariant or not, form a set of *canonical action variables* or *action integrals* of the canonical electromagnetic momentum $\mathbf{p} + q\mathbf{A}$ over the periods of the canonical coordinates $\mathbf{s}_i$

$$J_i = \frac{1}{2\pi} \oint_i (\mathbf{p} + q\mathbf{A}) \cdot d\mathbf{s}_i \quad (2.66)$$

each of which having an associated *phase angle* φ_i : the gyro phase, bounce phase and drift phase. $\{J_i, \varphi_i\}$ is a convenient set of six independent variables, e.g., when discussing velocity distribution functions, also known as phase space densities (Sect. 3.5), of radiation belt particles.

Note that the set of action integrals and their adiabatic invariance depend on the symmetries of the magnetic field configuration. For example, in a two-dimensional picture of the current sheet in the Earth's magnetotail the motion of charged particles may be symmetric about the center of the sheet with an associated adiabatic invariant. Such a motion is known as *Speiser motion*. The violation of the invariance in this case leads to the chaotization of the particle motion with consequences to the stability of the current sheet (e.g., Büchner and Zelenyi 1989).

2.4.4 Betatron and Fermi Acceleration

When particles drift adiabatically in an inhomogeneous magnetic field, their energies and/or pitch angles are affected. Consider the rate of change of the kinetic

energy W of a charged particle in a general time-dependent magnetic field $\mathbf{B}$. The time derivative in a moving frame of reference is $d/dt = \partial/\partial t + \mathbf{V} \cdot \nabla$, where $\mathbf{V}$ is the velocity of the frame of reference, i.e., the velocity of the GC. In the GCS the energy equation is

$$\frac{dW_{GCS}}{dt} = \mu \frac{dB}{dt} = \mu \left(\frac{\partial B}{\partial t} + \mathbf{V}_\perp \cdot \nabla_\perp B + V_\parallel \frac{\partial B}{\partial s} \right). \quad (2.67)$$

This equation can be transformed to the energy equation in the frame of reference of the observer (OBS)

$$\begin{aligned} \frac{dW_{OBS}}{dt} &= \frac{dW_{GCS}}{dt} + \frac{d}{dt} \left(\frac{1}{2} m V_\parallel^2 \right) + \frac{d}{dt} \left(\frac{1}{2} m V_\perp^2 \right) \\ &= \mu \frac{\partial B}{\partial t} + q \mathbf{V} \cdot \mathbf{E}, \end{aligned} \quad (2.68)$$

where non-electromagnetic forces have been neglected.

The first term in the right hand side of (2.68) describes the gyro betatron effect on charged particles, in which the changing magnetic field at the position of the GC leads to *gyro betatron acceleration* of the charged particle through the inductive electric field. The increase in the perpendicular velocity of the particle increases also the gradient drift velocity of the GC that depends on the perpendicular energy (2.15).

The second term describes the action of the electric field on the motion of the GC covering both magnetic field-aligned acceleration (if $E_\parallel \neq 0$) and another betatron effect, called *drift betatron acceleration*. The second term is zero if the velocity of the GC and electric field are perpendicular to each other, i.e., the drift betatron acceleration requires that the electric field has a component in the direction of particle's drift velocity. Therefore, the drifts due to the electric field do not contribute to drift betatron acceleration.

When the GC advects adiabatically in a quasi-static electric field toward an increasing field ($B_2 > B_1$), the invariance of μ implies

$$\frac{W_{\perp 2}}{W_{\perp 1}} = \frac{B_2}{B_1}. \quad (2.69)$$

Thus $W_{\perp 2} > W_{\perp 1}$. This leads to an anisotropic “pancake” velocity space distribution (Sect. 3.4) of particles transported by the convection electric field from the magnetotail to the inner magnetosphere. Pancake distributions drive two of the most important wave modes in radiation belt dynamics, the electromagnetic ion cyclotron waves and the whistler-mode chorus waves (Sect. 5.2). The betatron effect driven by the inductive electric field is discussed in Sect. 2.6 in the context of the drift shells in a quasi-dipolar magnetic field.

A special case of betatron acceleration is the drift of a particle in J -conserving bounce motion toward a region where the mirror field increases. This corresponds

to moving the mirror points closer to each other and, consequently, decreasing $\oint ds$. To compensate this $v_{\parallel}$ of the particle and thus its parallel energy $W_{\parallel}$ in the observer's frame must increase. This mechanism is known as *Fermi acceleration*. A mechanical analogue of Fermi acceleration is hitting a tennis ball with a racket. In the spectator's frame the ball is accelerated but in the racket's frame it just mirrors. The concept of Fermi acceleration is often used in the context of moving astrophysical shock fronts, the solar-terrestrial examples of which are interacting CME shocks or an ICME shock approaching the bow shock of the Earth.

2.5 Charged Particles in the Dipole Field

Figure 2.5 illustrates the motion of charged particles in the guiding center approximation in a dipole field introduced in Sect. 1.2.1.

The guiding center approximation can be applied in a static dipole field if the particle's gyro radius is much smaller than the curvature radius R_C of the field. In terms of the particle's *rigidity* $p_{\perp}/|q|$ the condition for the guiding center approximation is

$$r_L \left| \frac{\nabla_{\perp} B}{B} \right| = \frac{p_{\perp}}{|q| R_C B} \propto \frac{p_{\perp}}{|q| r_0 B} \ll 1, \quad (2.70)$$

i.e., the GC approximation is valid if

$$\frac{p_{\perp}}{|q|} \ll r_0 B, \quad (2.71)$$

Fig. 2.5 Motion of an electron in a dipole field

where r_0 is the Earth-centered distance to point where the field line crosses the dipole equator. The conversion of particle energy to rigidity is given by the relativistic relationship between momentum and kinetic energy (A.16)

$$p_{\perp}^2 = \frac{1}{c^2}(W^2 + 2mc^2W) \sin^2 \alpha. \quad (2.72)$$

Rigidity describes how the magnetic field affects a charged particle. A particle with a large rigidity is less affected by the ambient magnetic field than a particle with smaller rigidity. The rigidities of most cosmic rays are so large that they cannot be trapped by the Earth's dipole field and just move through the radiation belts being deflected by the magnetic field. The deflection depends on the rigidity of the particle and on the angle it arrives to the magnetosphere.

Some of the cosmic ray particles penetrate to the atmosphere, the most energetic ones even to the ground. In cosmic ray physics it is common to use the units familiar from elementary particle physics, setting $c = 1$. Thus particle mass, momentum and energy are all given in energy units, typically with GeV, which is of the order of proton mass $m_p = 0.931$ GeV. Consequently, the unit of rigidity is GV.

Let λ_m be the mirror latitude of a particle trapped in the dipole field and B_0 the magnetic field at the equatorial plane. The equatorial pitch angle of the particle is

$$\sin^2 \alpha_{eq} = \frac{B_0}{B(\lambda_m)} = \frac{\cos^6 \lambda_m}{(1 + 3 \sin^2 \lambda_m)^{1/2}}. \quad (2.73)$$

Denote the latitude, where the field line intersects the surface of the Earth, by λ_e . If $\lambda_e < \lambda_m$, the particle hits the Earth before mirroring and is lost from the bottle. Actually the loss takes place already in the upper atmosphere through collisions with atmospheric atoms and molecules. This can, of course, happen within a wide range of altitudes. In this book we mainly discuss particles with energies $\gtrsim 100$ keV, most of which deposit their energy at altitudes around 100 km or below. In any case 100 km is small as compared to $1 R_E$ and we can approximate the equatorial half-width of the loss cone by

$$\sin^2 \alpha_{eq,l} = L^{-3}(4 - 3/L)^{-1/2} = (4L^6 - 3L^5)^{-1/2}, \quad (2.74)$$

where L is the geocentric distance to the point where the magnetic field line crosses the dipole equator expressed in units of R_E , i.e., McIlwain's L -parameter (Sect. 1.2.1). The particle is in the loss cone if $\alpha_{eq} < \alpha_{eq,l}$.

As shown in Fig. 2.6, the half-width of the equatorial loss cone in the outer radiation belt is a few degrees and widens quickly along the field lines when the magnetic latitude exceeds 30° . In addition to the loss-cone width, the curves in the figure indicate the magnetic latitudes where the dipole field lines cross the surface of the Earth, e.g., $L = 4$ at $\lambda = 60^\circ$.

Fig. 2.6 The half-width of the atmospheric loss cone as a function of magnetic latitude for L -parameters 2–7

The bounce period in a dipolar bottle is

$$\begin{aligned}
 \tau_b &= 4 \int_0^{\lambda_m} \frac{ds}{v_{\parallel}} = 4 \int_0^{\lambda_m} \frac{ds}{d\lambda} \frac{d\lambda}{v_{\parallel}} \\
 &= \frac{4r_0}{v} \int_0^{\lambda_m} \frac{\cos \lambda (1 + 3 \sin^2 \lambda)^{1/2}}{1 - \sin^2 \alpha_{eq} (1 + 3 \sin^2 \lambda)^{1/2} / \cos^6 \lambda} d\lambda \\
 &= \frac{4r_0}{v} T(\alpha_{eq}).
 \end{aligned} \tag{2.75}$$

$T(\alpha_{eq})$ is known as the *bounce function*, which needs to be integrated numerically. At the limit $\lambda_m = 90^\circ$ ($\alpha_{eq} = 0$) the integral can be given in a closed form yielding $T(0) = (S/r_0)/2 = 1.38$, where S is the length of the field line (1.7). For nearly equatorially mirroring particles ($\alpha_{eq} \approx 90^\circ$), the dipole field can be approximated as a parabola (2.59), yielding $T(\pi/2) = (\pi/6)\sqrt{2} \approx 0.74$. Between these extremes ($0 < \alpha_{eq} < 90^\circ$) a good approximation is (Schulz and Lanzerotti 1974)

$$T(\alpha_{eq}) \approx 1.3802 - 0.3198 (\sin \alpha_{eq} + \sqrt{\sin \alpha_{eq}}). \tag{2.76}$$

For $\alpha_{eq} \gtrsim 40^\circ$ a little less accurate approximation

$$T(\alpha_{eq}) \approx 1.30 - 0.56 \sin \alpha_{eq} \quad (2.77)$$

is often good enough in practice (as used by, e.g., Roederer 1970; Lyons and Williams 1984). As $T(\alpha_{eq})$ is of the order of 1, $\tau_b \approx 4r_0/v$ is a pretty good approximation in back-of-the-envelope calculations.

Finally, we investigate the drift time around the Earth. In the terrestrial dipole field (where $\nabla \times \mathbf{B} = 0$), ions drift to the west and electron to the east with the combined curvature and gradient drift velocity (2.24)

$$\begin{aligned} v_{GC} &= \frac{W}{qBR_C}(1 + \cos^2 \alpha) \\ &= \frac{3mv^2r_0^2 \cos^5 \lambda (1 + \sin^2 \lambda)}{2qk_0 (1 + 3 \sin^2 \lambda)^2} \left[2 - \sin^2 \alpha \frac{(1 + 3 \sin^2 \lambda)^{1/2}}{\cos^6 \lambda} \right], \end{aligned} \quad (2.78)$$

where the Eqs. (1.4) and (1.8) for the dipole field $B(\lambda)$ and curvature radius $R_C(\lambda)$ have been inserted.

In the drift motion around the Earth, v_{GC} is usually less interesting than the bounce-averaged azimuthal speed $\langle d\phi/dt \rangle = \langle v_{GC}/r \cos \lambda \rangle$ that gives the drift rate of the guiding center around the dipole axis. A straightforward calculation yields

$$\begin{aligned} \left\langle \frac{d\phi}{dt} \right\rangle &= \frac{4}{v\tau_b} \int_0^{\lambda_m} \frac{v_{GC}(\lambda)(1 + 3 \sin^2 \lambda)^{1/2}}{\cos^2 \lambda \cos \alpha(\lambda)} d\lambda \\ &\equiv \frac{3mv^2r_0}{2qk_0} g(\alpha_{eq}) = \frac{3mv^2R_E L}{2qk_0} g(\alpha_{eq}), \end{aligned} \quad (2.79)$$

where

$$g(\alpha_{eq}) = \frac{1}{T(\alpha_{eq})} \int_0^{\lambda_m} \frac{\cos^3 \lambda (1 + \sin^2 \lambda) [1 + \cos^2 \alpha(\lambda)]}{(1 + 3 \sin^2 \lambda)^{3/2} \cos \alpha(\lambda)} d\lambda. \quad (2.80)$$

Similarly to $T(\alpha_{eq})$ also $g(\alpha_{eq})$ is of the order of 1. For equatorial pitch angles larger than 30° $g(\alpha_{eq})$ can be approximated as

$$g(\alpha_{eq}) \approx 0.7 + 0.3 \sin(\alpha_{eq}) \quad (2.81)$$

yielding for non-relativistic equatorial particles ($\alpha_{eq} = 90^\circ$)

$$\left(\frac{d\phi}{dt} \right)_0 = \frac{3mv^2R_E L}{2qk_0} = \frac{3\mu}{qr_0^2}. \quad (2.82)$$

Fig. 2.7 Drift periods of radiation belt electrons with energies 100 keV, 1, 5, and 10 MeV as a function of L calculated from Eq. (2.83)

For relativistic equatorial particles this is modified to

$$\left(\frac{d\phi}{dt}\right)_0 = \frac{3mc^2 R_E L}{2qk_0} \gamma \beta^2 = \frac{3\mu_{rel}}{\gamma q r_0^2}, \quad (2.83)$$

where μ_{rel} is the relativistic first invariant $p_\perp^2/(2mB)$ (2.48) and $\beta = v/c$. The coefficient

$3mc^2 R_E/(2qk_0)$ is

- 0.035 degrees per second for electrons and
- 64.2 degrees per second for protons.

Note that the drift period decreases with increasing L . While this may, at the first sight, seem counterintuitive, recall that according to the expression for the gradient–curvature drift velocity in current-free magnetic field (2.24) the drift speed is proportional to $1/(BR_C)$. The dipole field scales as L^{-3} and the curvature radius as L . Thus $v_{GC} \propto L^2$, $(d\phi/dt)_0 = v_{GC}/L \propto L$ and $\tau_d \propto L^{-1}$.

In Fig. 2.7 drift periods of radiation belt electrons are plotted for selected energies in the range 100 keV–10 MeV at L -shells 2–8 and Table 2.2 gives examples of electron gyro, bounce and drift periods at $L = 2$, $L = 4$ and $L = 6$. The bounce times are calculated from (2.75) for $\alpha_{eq} = 80^\circ$ corresponding to the bounce

Table 2.2 Examples of approximate electron gyro, bounce and equatorial drift periods for selected energies. Note that the magnetic drift period of 10-keV electrons deep in the plasmasphere ($L = 2$) is longer than the corotation time (24 h). Thus their physical drift motion around the Earth is determined by the corotational electric field. Note also that the ultra-relativistic particles move practically with the speed of light, thus their bounce times at a given L -shell are almost identical

		10 keV	100 keV	1 MeV	5 MeV	10 MeV
$L = 2$	τ_L	9.71 μ s	11.4 μ s	28.1 μ s	103 μ s	196 μ s
	τ_b	0.64 s	0.23 s	0.14 s	0.13 s	0.13 s
	τ_d	44.2 h	3.65 h	36.9 min	8.09 min	4.19 min
$L = 4$	τ_L	0.08 ms	0.09 ms	0.26 ms	0.82 ms	1.57 ms
	τ_b	1.27 s	0.46 s	0.27 s	0.26 s	0.26 s
	τ_d	22.1 h	1.83 h	18.5 min	4.05 min	2.09 min
$L = 6$	τ_L	0.26 ms	0.31 ms	0.76 ms	2.88 ms	5.29 ms
	τ_b	1.91 s	0.69 s	0.41 s	0.38 s	0.38 s
	τ_d	12.3 h	1.22 h	12.3 min	2.70 min	1.40 min

function $T \approx 0.75$. The drift periods are calculated from (2.83) for particles with $\alpha_{eq} = 90^\circ$.

2.6 Drift Shells

The guiding center of a charged particle traces a *drift shell* when the particle drifts around the Earth (right-hand picture in Fig. 2.5). In a symmetric static magnetic field in the absence of external forces when μ and W are conserved, the drift shell can be uniquely identified by

$$\begin{aligned} I &= \text{constant} \\ B_m &= \text{constant} \end{aligned} \quad (2.84)$$

where I is the integral (2.58) and B_m the mirror field. In a dipolar configuration the magnetic field and the drift shells are symmetric around the dipole axis. The drift shells are in this case defined by constant L and commonly referred to as L -shells.

Beyond about $3\text{--}4 R_E$ temporal and spatial asymmetries of the magnetospheric magnetic field affect the formation and evolution of the drift shells. The geomagnetic field is compressed on the dayside by the Chapmann–Ferraro current (1.14), which together with the stretching of the nightside magnetic field causes azimuthal asymmetry. The asymmetry increases with increasing dynamic pressure of the solar wind, in particular, when a solar wind pressure pulse or interplanetary shock compresses the dayside configuration, or the nightside field stretches during the substorm growth phase.

For a distorted dipole the L -parameter can be generalized by defining

$$L^* = \frac{2\pi k_0}{\Phi R_E} . \quad (2.85)$$

L^* is sometimes called *Roederer's L -parameter* to distinguish it from the original McIlwain's L -parameter of the dipole field. L^* is inversely proportional to the magnetic flux (Φ) enclosed by particle's drift contour. Thus L^* is an alternative way to express the third action integral. If changes in the field are slower than the drift period of the particle, L^* remains invariant.

L^* is equal to McIlwain's L -parameter only in a purely dipolar field. Otherwise, L^* corresponds to the radial distance to the equatorial points of the symmetric L -shell on which the particle would be found if all non-dipolar contributions to the magnetic field were turned off adiabatically. This method can also be applied if internal field perturbations closer to the Earth are taken into account.

2.6.1 Bounce and Drift Loss Cones

Charged particles that complete one or more drift cycles around the Earth are said to be *stably-trapped*. Trapping is lost if the particle reaches an altitude where it is lost through collisions with atmospheric particles, or if the particle's drift shell encounters the magnetopause and the particle is lost there. Particles that are able to perform a number of bounces but are lost before a complete drift cycle are said to be *pseudo-trapped*.

If the particle does not fulfil a complete bounce cycle in the magnetic bottle of the inner magnetosphere before it is lost into the atmosphere, it is said to be in the *atmospheric* (or *bounce*) *loss cone*. In the dipole field the width of the atmospheric loss cone depends on L and the magnetic latitude (Fig. 2.6). As bounce periods of radiation belt electrons are only fractions of a second (Table 2.2), electrons close to the edge of the bounce loss cone can be lost rapidly from the belts due to wave-particle interactions whereas scattering from larger equatorial pitch angles takes more time (Chap. 6).

The asymmetric deviations from the dipole introduce complications to this picture. Although the field inside of about $3\text{--}4 R_E$ is very close to that of a dipole, the dipole is displaced from the center of the Earth, which introduces an asymmetry in the Earth-centered frame of reference. Now the distance from the dipole to the upper atmosphere varies as a function of geographic latitude and longitude.

Let us consider an electron in the Earth's displaced dipole field, freshly injected to a field line where it is not in a bounce loss cone. Assume that the electron mirrors from the field B_m at an altitude somewhat above 100 km, and drifts around the Earth. If the dipole field were symmetric, the electron would stay trapped unless something (e.g., wave-particle interaction) lowers its pitch angle letting it to escape from the magnetic bottle. But in the case of a displaced dipole the electron can drift to a

field line that is connected to the atmosphere above a region that is farther from the dipole. The field at 100 km is now smaller than B_m and the particle may reach a lower altitude and be lost from the magnetic bottle. The particle has performed a number of bounces but was not able to complete a full drift around the Earth. It is said to have been in the *drift loss cone*. The drift loss cone is as wide as the widest bounce loss cone on a given drift shell. The effect is strongest above the South Atlantic Anomaly, where the minimum surface magnetic field is $22 \mu\text{T}$, which is much smaller than $35 \mu\text{T}$ of Earth-centered 11° -tilted dipole in the same region where dipole latitude is about -20° .

2.6.2 Drift Shell Splitting and Magnetopause Shadowing

In an azimuthally symmetric static magnetic field particles with different pitch angles, which at a given longitude are on a common drift shell, will remain on the same shell. This is called *shell degeneracy*. It is a consequence of condition (2.84), according to which drift shells are defined by constant mirror magnetic field magnitude B_m and constant integral I . In a pure dipole field I can be replaced by the L -parameter.

In an azimuthally asymmetric field charged particles with different pitch angles, whose guiding centers are on a joint magnetic field line at some longitude, do not stay on a common field line when they move to another longitude. The reason is that they mirror at different field strengths B_m and, consequently, their I -integrals are different. This is known as *drift shell splitting*. The magnetosphere is always compressed on the dayside and stretched on the nightside, and the asymmetry increases during increased solar wind forcing and geomagnetic storms. Figure 2.8 illustrates the shell splitting in the noon–midnight cross section under weakly asymmetric configuration. The particles with different equatorial pitch angles on a common drift shell in the nightside are found on different shells in the dayside (top). Vice versa, particles on a common drift shell in the dayside are on different shells in the nightside (bottom).

In Fig. 2.8 the drift shell splitting is most pronounced at equatorial distances beyond about $6\text{--}7 R_E$. The upper panel indicates that the drift shells of particles with large equatorial pitch angles (small $\cos \alpha_0$) extend in the dayside further out than of particles with smaller pitch angles. Depending on the amount of compression of the dayside magnetosphere these particles may hit the magnetopause and be lost to the magnetosheath before they pass the sub-solar direction. The phenomenon where particles are lost at the magnetopause is known as *magnetopause shadowing*. Particles lost this way are said to *pseudo-trapped*, as they remain trapped during a part of their drift around the Earth.

Fig. 2.8 Illustration of drift shell splitting from nightside to dayside (top) and from dayside to nightside (bottom) in a weakly compressed/stretched magnetic field configuration. The dots indicate the mirror points for different equatorial pitch-angle cosines. (From Roederer 1970, the colors of dots have been added to guide the eye) (Reprinted by permission from Springer Nature)

The magnetopause shadowing, where the drift shell splitting plays a role, thus leads to a loss of particles nearly 90° pitch angles, while the scattering of particles to the atmosphere results in the loss of particles with small pitch angles. In two-dimensional velocity space ($v_{\parallel}$, $v_{\perp}$) this leads to a butterfly shape of the particle distribution function (Sect. 3.4.2).

Magnetopause shadowing is an important particle loss mechanism from the outer electron belt (Sect. 6.5.1). The shadowing is most important during strong magnetospheric activity when the dayside magnetopause is most compressed/eroded and the nightside field is stretched already at the distance of the outer radiation belt. Note that although the bulk plasma is frozen-in the magnetospheric field, the gyro radii of the energetic radiation belt particles are large enough to break the freezing close to the magnetopause boundary layer.

Correspondingly, common drift shells in the dayside are split when particles drift toward the nightside (Fig. 2.8, bottom). In this case the drift shells of particles with larger equatorial pitch angles are closer to the Earth in the nightside and the particles remain stably trapped. However, now particles with *small pitch angles*, reaching farther out to the magnetotail, may lose their magnetic field guidance, when crossing the current sheet, before they drift through the midnight meridian and again become pseudo-trapped.

Note that particles observed at a given point on the dayside come from different locations in the nightside. As there may well be more high-energy particles with large pitch angles in the nightside closer to the Earth than farther away, the observed pitch-angle distribution in the dayside may have a shape of a pancake (Sect. 3.4.1)

The compression of the dayside magnetopause due to solar wind pressure produces further local quasi-trapping regions in the high-latitude dayside magnetosphere. While the pure dipole field along a given field line has a minimum on the equatorial plane, the compression of the dayside magnetopause enhances the equatorial magnetic field causing a local maximum at the dayside equator. Now the equatorial minimum bifurcates to off-equatorial local minima (Fig. 2.9).

These local magnetic bottles in the northern and southern hemispheres are continually reformed due to the changing angle between of the dipole axis and the Sun-Earth line and due to the changes in the solar wind pressure. This leads to complicated charged particle orbits known as *Shabansky orbits* (Antonova and Shabansky 1968). For example, a fraction of the particles drifts across the noon sector bouncing in the northern hemisphere, another fraction bouncing in the southern hemisphere. (For examples of Shabansky orbits, see, e.g, McCollough et al. 2012).

2.7 Adiabatic Drift Motion in Time-Dependent Nearly-Dipolar Field

As an example of drift betatron acceleration (Sect. 2.4.4) we consider the effects of slow temporal changes of the quasi-dipolar field, examples of which are due to changes in solar wind pressure or approaching dipolarization fronts from the magnetotail. To keep the presentation simple we limit the discussion to bounce-averaged motion in the equatorial plane. Furthermore, we assume that there is no background electric field.

Fig. 2.9 Bifurcation of the local magnetic minima in the compressed dayside dipole field. The crosses indicate the local minima on different field lines in the noon–midnight meridional plane. (From McCollough et al. 2012, reprinted by permission from American Geophysical Union)

We start with the *fully adiabatic motion*² conserving all three adiabatic invariants, i.e., we assume that the temporal changes in the magnetic field are slower than the drift motion around the dipole

$$\frac{B}{dB/dt} \gg \tau_d, \quad (2.86)$$

where τ_d is the bounce-averaged drift period (2.83). Temporal increase or decrease of the magnetic field is associated with an azimuthal inductive electric field $\mathbf{E}_i$. In the equatorial plane this leads to an outward or inward $\mathbf{E}_i \times \mathbf{B}$ drift across the magnetic field lines, corresponding to positive or negative dL^*/dt .

² The term “adiabatic motion” is often used to refer to conservation of the first adiabatic invariant without the requirement of conserving the other action integrals.

Particles can thus gain (lose) energy when they are transported by the $\mathbf{E}_i \times \mathbf{B}$ drift across the region of increasing (decreasing) magnetic field magnitude. The radial location and energy of the particles change during such a slow contraction or expansion of the magnetic field but the fully adiabatic process is reversible and the initial state of the particles is recovered when the field returns to its original strength. Particles that were on a common drift shell at the beginning of the magnetic field perturbation are transferred to an outer drift shell and return to the original shell when the magnetic configuration recovers its configuration before the perturbation. Due to the slow temporal changes all particles experience the same amount of inductive electric field in their orbit around the Earth, and therefore the same amount of inward/outward displacement. This kind of reversible process is observed during weak and moderate magnetic storms when the ring current enhances slowly in the main phase and then recovers in the recovery phase. This causes weakening and subsequent strengthening of the equatorial magnetic field. As the changes of the equatorial magnetic field strength are displayed in the Dst index, the reversible process is called the *Dst effect*.

The change of the particle energy during the expansion and contraction of the drift shells can be calculated by bounce-averaging the energy equation (2.68)

$$\begin{aligned} \left\langle \frac{dW}{dt} \right\rangle_b &= \mu \left\langle \frac{\partial B}{\partial t} \right\rangle_b + q \langle \mathbf{V} \cdot \mathbf{E}_i \rangle_b \\ &= \mu \left\langle \frac{\partial B}{\partial t} \right\rangle_b + q \langle \mathbf{V}_0 \rangle_b \cdot \mathbf{E}_{i0}, \end{aligned} \quad (2.87)$$

where $\mathbf{V}_0$ is the drift velocity of the GC and $\mathbf{E}_{i0}$ is the inductive electric field, both at the equator. In a slowly changing magnetic field the inductive electric field increases the kinetic energy of the particle both in its gyro motion (gyro betatron acceleration) and in its drift motion (drift betatron acceleration). The particle has in this case time to gradient drift considerably around the Earth before $\mathbf{E}_i \times \mathbf{B}$ drift takes it significantly inward, i.e., the drift velocity of the GC consists now primarily on the azimuthal gradient drift velocity and thus, the drift betatron term in, i.e., the second term on the RHS of (2.87), is finite. The $\mathbf{E}_i \times \mathbf{B}$ drift, being perpendicular to $\mathbf{E}_i$, does not contribute to the betatron acceleration.

In the inner magnetosphere the temporal changes often are faster than the azimuthal drift motion but slower than the bounce motion ($\tau_b \ll B/(dB/dt) \ll \tau_d$). Bounce-averaging of the energy equation is still possible but now the $\mathbf{E}_i \times \mathbf{B}$ drift is faster than the azimuthal drift. Consequently, the charged particles follow the contracting or expanding field lines. Now in the second term on the RHS of (2.87) the velocity of the GC consist primarily of the $\mathbf{E}_i \times \mathbf{B}$ drift that is naturally perpendicular to the inductive electric field $\mathbf{E}_i$. The drift betatron acceleration is zero because the $\mathbf{E}_i \times \mathbf{B}$ drift displaces the particle inward/outward so quickly that it does not have time to gradient drift much around the Earth. The increase/decrease in particle's energy is therefore due to gyro-betatron acceleration/deceleration.

Let us then consider a sudden impulse (SI, Sect. 1.4.2) in which a solar wind shock suddenly compresses the magnetosphere and the field thereafter relaxes slowly without no major storm development in the ring current. Assume that the compression is fast enough to violate the third invariant. Since the compression is now also asymmetric, particles at different longitudes, i.e., at different phases in their drift motion around the Earth respond differently to the disturbance. The compression is strongest on the dayside and particles there experience largest E_i and are mostly transported toward the Earth by the $E_i \times B$ drift. Since compression is now fast compared to the drift motion and the frozen-in condition applies, this can be considered in terms of particles following field lines as they are compressed toward the Earth. On the other hand, particles in the nightside at the time of the shock arrival more or less retain their original drift orbits. As a consequence, particles originally on the same drift shell but at different longitudes are transported to different distances toward the Earth and gain, in irreversible manner, different amount of energy.

Assuming that after the initial rapid compression the inductive electric field effect vanishes adiabatically, the particles find themselves on different drift shells. The change of the drift shell due to the inductive electric field depends on the longitude, i.e., the magnetic local time, where the particles are at the time of the asymmetric compression, and dL^*/dt is positive for some particles and negative for others. After the return to the original state the original distribution has diffused to a wider range of L^* . This is an example of *radial diffusion* that will be discussed further in the context of ULF waves in Chap. 6. For violation of L^* the disturbance has thus to be both fast when compared to the drift motion around the Earth and azimuthally asymmetric. Otherwise, all particles initially at the same drift shell would experience the same amount of displacement due to $E_i \times B$ -drift and there would be no diffusion in L^* .

Let us finally consider the case where the shock compression is followed by a magnetic storm. If the main phase (the rapid decrease of the Dst index) is not too fast, particles are subject to the fully adiabatic Dst effect described above. Faster and asymmetric changes can result, e.g., from substorm related $\partial \mathbf{B}/\partial t$ and corresponding inductive electric field, and ULF waves. They can lead to further violations of L^* and further spreading of drift shells compared to the initial situation. Furthermore, the radiation belt particles are accelerated and scattered in pitch-angle by various wave-particle interactions (Chap. 6). Thus the reversible Dst effect can be, while partially identifiable in observations, obscured by other processes.

Another inductive electric field effect is related to the injection of particles from the magnetotail into the inner magnetosphere during substorms. If the dipolarization is faster than the azimuthal drift motion, the injected particles are transported by the $E_i \times B$ drift toward the Earth gaining energy. At some point their energy becomes large enough for the gradient and curvature drifts to take over and the particles become trapped in the ring current and radiation belts.

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