A Small Remark on Hilbert's Finitist View of Divisibility and Kanovich-Okada-Scedrov's Logical Analysis of Real-Time Systems

Mitsuhiro Okada^(✉)

Keio University, 2-15-45 Mita, Minato-ku, Tokyo, Japan
`mitsu@abelard.flet.keio.ac.jp`

Abstract. Hilbert remarked in the introductory part of his most famous finitism address (1925 [1]) that “[t]he infinite divisibility of a continuum is an operation that is present only in our thought”, which means that no natural event or matter is infinitely divisible in reality. We recall that Scedrov's group including the author started logical analysis of real time systems with the principle similar to Hilbert's no-infinite divisibility claim, in [2]. The author would like to note some early history of the group's work on logical analysis of real time system as well as some remark related to Hilbert's claim of no-infinite divisibility.

Keywords: Real-time system · Hilbert · Finitism · Zeno paradox · Andre Scedrov · Multi-agent system · Formal verification

It was at a narrow corridor of the faculty building that a big whiteboard (with a photo-copying function of the board) was temporarily placed, and the three researchers discussed for the whole days and formed a starting point of the logical analysis of real-time systems. Since it was necessary for all faculty members on the floor to pass along the corridor quite frequently everyday, they were required to get sophisticated skills to find a way to go through some narrow space among the big board and the three men discussing including the big guy with big voice. It was how the collaboration began among the three lead by Andre Scedrov at Keio University in Tokyo.

1 Introduction

In his most famous finitism address, which was published in [1], Hilbert claimed that infinite divisibility of natural events and mater was not real but just our naïve impression; in the other words, infinite divisibility of a continuity was “an operation that is present only in our thought” and “merely an idea”, which is not in reality.

“The initial, naïve impression that we have of natural events (Geschehen) and of matter is one of uniformity, of continuity. If we have a piece of metal or a volume of liquid, the idea impresses itself upon us that it is divisible without limit, that any part of it, however small, would gain the same properties. But, wherever the methods of the research in the physics of matter were refined sufficiently, limits to divisibility were reached that are not due to the inadequacy of our experiments but to the nature of the subject matter....”

He continued, after mentioning the cases of electricity and quantum of energy, as follows.

“The infinite divisibility of a continuum is an operation that is present only in our thought; it is merely an idea, which is refused by our observations of nature, and by the experience gained in physics and chemistry (Hilbert 1925 [1]).”

Here, Hilbert expresses an aspect of the finitist distinction between reality and ideas as the distinction between no-infinite divisibility in reality and infinite divisibility only in our thought, in the introductory part of this paper on his finitist program. His finitist program was, very roughly speaking, a safety verification program, to prove consistency of ideal reasoning of classical mathematics by means of finitist methods, in other words, by means of concrete and contentual methods, so that the freedom of the use of classical mathematical reasoning and of its applications to physics and other mathematically described sciences were provably-safe without any possibility of critical state “contradiction” (even though classical mathematical reasoning using continuity, infinite divisibility, set-theoretical infinity, etc.). Although at the above quoted part Hilbert talked about no-infinite divisibility of natural events and matter in the spatial continuity sense, one could presume, from the finitist view, the same claim of no-infinite divisibility for events in wider and various senses) in a spatiotemporal continuum, e.g., to claim no-infinite temporal divisibility of any, either natural or artificial, events.

Hilbert himself made the remark on no-infinite divisibility in reality from the finitist view in the context of the foundational studies in mathematics in the early 20th century. We would like to remark that the similar no-infinite divisibility settings have appeared in more contemporary contexts of computer science. In this Note, we recall one example investigated by Andre Scedrov’s group including the author, initiated by [2], where some logical analyses of multi-agent real-time systems with dense-time were given in a logical way (using a subsystem of linear logic, or called multiset rewriting system). A different approach was given in using the timed automata setting.¹ This Note does not include the precise logical definitions and does not require the reader to know them, but the reader who wish to know more precise settings is invited to see the papers in the References.

In order to see the effect of the no-infinite divisibility setting, which we call the no-infinite divisibility principle in this Note, we could compare two different ways

¹ We only consider the logical approach in this Note.

of resolving Zeno's paradoxes, such as the Paradox of Achilles and the Tortoise. One way is the usual way, namely the classical analytic way, and the other way is the way employing the no-divisibility principle (without analytic notion of "limit"). In the Hilbert's finitist terminology, the one by using ideal operations of thought, the other by finitistic description of the real world. First, we remind ourselves what was Zeno's argument: The starting point of the Tortoise is a little ahead of that of Achilles. Zeno argues that when one considers the event that Achilles reaches the starting point of the Tortoise, the Tortoise's position is already a little advanced. Then, one considers another event that Achilles reaches the position of the Tortoise at the previous event, and in the same way one could consider unlimited number of events, which are all sub-events of this whole race event. Zeno claims that since there are infinitely many events before Achilles reaches the Tortoise, Achilles would never catch up with the Tortoise, which is a paradox.

However, by means of the modern mathematical definition of "limit" and convergence of an infinite series, one can easily calculate exactly when and where in the spatiotemporal continuum Achilles catches up with the Tortoise and overtakes him. This usual resolution is consistent with Zeno's argument itself, without any contradiction assuming that modern mathematics has no contradiction.²

The other way to resolve the Paradox of Achilles and the Tortoise is to take the standpoint of the no-divisibility principle that there is no infinite divisibility of the race: Zeno's argument divides the race into infinite sub-events, which violates reality, namely the no-infinite divisibility principle. Hence, the principle blocks Zeno's argument itself, therefore no paradox occurs here. From the view point of Hilbert, the former analytic resolution is not real but ideal, while the latter real. This latter would suggest us some idea of finite states and events-based modeling of the outer world of finitely many agents on the spatiotemporal continuum with the no-infinite divisibility principle. For example, the race of Achilles and the Tortoise could be modeled as a two-agent state transition system in various ways by choosing primitive states and by respecting the no-infinite divisibility principle. An extremely simplest modeling, but with reality, is the following.

In "What the Tortoise Said to Achilles" (1895), Lewis Carroll emphasizes that reality of the race uses only a few state transitions; with starting state of the race, only after a single tick (a little time progress, expressed by a single application of the tick-axiom in the sense of [2], it gets the state that Achilles has overtaken the Tortoise, then, after another-tick, Achilles has seated himself on the back of the Tortoise. There would be various different ways to set the primitive states associated to each agent to describe the race or other events in a state transition-based multi-agent real-time systems depending on a focused critical states or events.³ One could consider more complexed multi-agent systems

² To show no contradiction of classical mathematical reasoning was the exact purpose of Hilbert's finitist program.

³ Note that Zeno, believing his argument of the paradox successful, concluded no state-change and no event in reality of the world, which is too simple description of the real world which is contradictory with our intuition.

where various time constraints are imposed on the time-continuum. For example, if we have one-time constraint that the race should be finished by 5 h after the starting-time (as such a time constraint is usually imposed to some citizen marathon events), then, Achilles who now sits on the back of the Tortoise automatically moving towards the goal, needs to think a scheduling until when he may remain on the back of the Tortoise and when he should leave the back to start running himself) (or just one jump) to the goal in the constraint deadline, namely safely with respect to the critical state. Adding time constraints would involve notions of “may”, “should”, “critical state”, “safety”, “scheduling” in multi-agent modeling with the no-infinite divisibility principle even for modeling the Achilles-the Tortoise race by a state transition system.

2 A Case Study: Multi-agent Real-Time State Transition Model with the No-Infinite Divisibility Principle

2.1 Finitely-multi-agent Finite State Transition System with Finite Time Constraint Conditions on the Dense Real-Time

Kanovich-Okada-Scedrove [2] provided a case study of finitely modeling of real time system with the no-infinite divisibility principle, which is sometime also called the Zeno principle, which tells that there is no infinite events, actions, or state transitions in any bounded time interval of the dense or continuous time.⁴ Kanovich-Okada-Scedrov [2] took a state transition model of the external environments and any state is either a primitive state or a composite state of concurrent finite primitive states. Any primitive state is typically related to an agent. A composite state transition was expressed by a logical implication where a composite state is expressed by multiplicative connectives of atomic states, which has been also called a multiset rewrite rule by Scedrov’s group later (See References, where the reader could find the use of the name even in the title of later papers co-authored by Scedrov.). A time progress with unspecified duration is expressed by a time transition rule called tick-axiom. For example, instead of (classical) mathematical description of physical moving as to how a train approaches to a railway gate by means of continuous function on the real number, one takes a finite-states description, such as “far distance safe state”, “approaching to the railway (be-careful) state”, “train’s passing the railway gate (danger) state”, “train’s-passing away state” for the purpose of safety verification of a traffic system as in [2]. The distinction of the two ways of description of train states has a certain correspondence to the distinction between the classical mathematical-physical description with continuity and the finite description with the no-infinite divisibility of events, discussed in the previous Section. Scedrov’s group initiated by [2] showed that the apparent dense real-time is eliminable; the main purpose was to specify a certain class of dense-time multi-agent systems

⁴ Setting a time interval could be considered an event with the duration, such as the race of Achilles and the Tortoise.

with time-constraints, with the focuses on (1) showing formal specification ability of critical states for safety and (2) showing computational complexity of the reachability problem, which implies the complexity of decision problem of safety in the specification framework. The dense-time setting provided an interesting modeling of multi-agent system, where any agent could start its primitive state transition (or an agent’s action) at any moment on the dense time under finitely many must-time constraints and may-time constraints.⁵ This dense-time setting could allow us to have an interpretation that any participating agent apparently has freedom to trigger its action; e.g., a walker can change her state from the standing-state to the crossing-street-state whenever some may-constraints and must-constraints are satisfied; a may-constraint could be “one may start crossing 5s after the traffic-light changed to the Green state”, and a must-constraint could be “one must complete crossing the street before 5s passing of the Yellow state of the traffic light”, for example. [2] and the related works tell that in fact the dense time is eliminable with the state transition model and the reachability problem is PSPACE. This could be understood that if a given specific composite state happens or not is PSPACE decidable even if apparent free-triggering primitive state transitions by any agent under the constraints. There are some corresponding work in the timed-automata research community.

2.2 Global Clock and Local Alarm Stopwatches

The original work [2] was involved with set theoretical treatment to consider equivalent classes of states to eliminate the dense (infinite) time-states. During the author’s stay in Paris in 2000, the author had an opportunity to reformulate it by switching the use of global clocks to local alarm-stopwatches and gave lectures on it. The author shared this modified method with the co-authors of [2]. This switching gave the effect of putting the time constraint locally inside the linear logical implicational or multiset transition rules and the whole argument of [2] became more constrictive in the sense that one eliminates the dense-time without the set-theoretical equivalence classes treatment⁶.

To hint some of the specific characteristics of the work initiated by [2] it would be worth noting here various different reactions from the audience to the author’s series of lectures on this topic in 2000–2002. For example, when the author gave an invited talk at an interdisciplinary meeting organized by G. Longo at ENS-Paris, it was physicists who expressed most appreciations on our work, especially on the way how the continuous time in the physics sense could be reformed to finite state transition systems with PSPACE decidability. As for another example, when the author gave the lectures at the Coq group meetings in INRIA, they were interested in the way to switch the global clock to local

⁵ A transition of primitive state associated to an agent is called an action of the agent in this Note for simplicity of the argument. See [2] for more formal and precise setting.

⁶ Among our agreement Max Kanovich was supposed to write the version, but unfortunately without realization soon.

watches to make the argument more constructively, and the author collaborated with them on the topic including an implementation work. When the author gave talks to the audience of the AI and multi-agent system research community, some of them told us that it was such a rare good work in the AI-multi-agent system research community that the primitive concepts, ontology, syntax, technical proofs of computational complexity, semantics, applications and implementation altogether were given by one group: they appreciated our work. The part of our local-watches-based linear logical transition systems were implemented by our master students on the Coq group under the author's supervision with Jean-Pierre Jouannaud, as FATALIS System. Some part of FATALIS was presented in [4]. Some versions were presented at [3] and [5] by the author. The linear logical (multiset rewrite) system framework is easy to express dynamic changes of the system, such as new agents participations, changing the time constraints, on the rewrite rule level.

The framework proposed in [2] was one of the earliest works for modeling timed multi-agent systems, either with dense-time or with discrete time, which opened a wide range of applications. Andre Scedrov and his colleagues have developed sophisticated logical techniques on this line for various different settings, including time-related security, scheduling and others. See [6–9] for some of the recent developments of the group. These show wide range of descriptibility and logical analyzability by means of the timed multiset rewriting framework and its further potentials.

2.3 Back to Finitism

We already remarked that the no-infinite divisibility claim, in Hilbert's finitism [1], also occurred in the setting of [2] as the basic principle. Hence it would be interesting to comparing the two shortly to conclude this Note. Hilbert distinguishes reality and ideality, and he anticipated a finitist or realist (in his sense) safety verification proof of classical or ideal (in his sense) mathematics which was essentially needed for our describing the physical world. On the other hand, the standpoint of the framework of the timed multiset rewriting (and its linear logical version) is rather neutral with respect to the reality-vs-ideality distinction of the description level. It could be considered a combined description. For example, the timeless version of the discrete system description of [2] is a finitely modeling (hence idealized modeling) for some focused aspect of the external world, such as a part of train-traffic system for the purpose of safety proofs or safe scheduling etc. But, when the dense time enters to the discrete model, the no-infinite divisibility is required as a realist requirement. (As we reminded in Introduction, unless one takes the continuous physical description of the agents' moving, it would go into Zeno's too idealistic world view.) One could even relativize the real-time; in certain finitely described cases discussed in this Note the no-infinite divisibility is the real requirement, while in certain analytically-physically described cases the continuity is the real requirement. The finitely described real-time modeling framework used in [2] and others provided typical cases of the former. A discrete state transition modeling could

be natural as a modeling of the physical real train traffic in “reality”. In fact, when we define which area is the “approaching (danger) state” area, for example, we need some physical calculation; then based on the physical calculation, one takes the view of train traffic system as a discrete state transition system. In that case, one might think that the description by continuous functions with the time-continuum as relatively real and the description by discrete modeling relatively with the no-infinite divisibility relatively ideal. However, one could easily imagine the other way around, too, for certain cases. After designing and formally verifying safety with the timed multiset rewrite system, the physical area and the physical speed bound of the approaching-state can be defined; in this way, one could consider the discrete safe system is relatively real and the implementation in the physical world is a simulation of the symbolic safe system. In fact, both views are interactive. Time-continuum and no-infinite divisibility of time co-exist. One could say the interactive view of the two aspects important, beyond the reality-ideality issue. In practice, we often use the view of hybrid systems accommodating both.

Note that we have not discussed another way of description, besides the finite-discrete way and the analytic mathematical -physical way of description (say, for safety verification): one missing was the probabilistic way of description. For example, security properties of cryptographic protocols are often investigated and proven by the discrete modeling, such as using finitely Dolev-Yao model description, which can be understood as a state transition model, and by the cryptographic computational description, such as using probabilistic poly-time Turing machine description and others. The interactive view of the two ways of description is important in these cases, too. One of the author’s views may be found in [12] where a variant of the forcing model construction, Fitting-embedding, method was used for connecting the logical-discrete verification proofs to the poly-time probabilistic Turing machine based-verification proofs, by means of giving “computational soundness” of logical proofs with respect to the computational models.⁷

Acknowledgements. This work was supported by JSPS KAKENHI Grant Numbers 19KK0006, 17H02265, 17H02263. The author would like to express his sincere thanks to the anonymous referee for helpful comments at the preparation stage of this manuscript.

⁷ Andre Scedrov and his colleagues are known as the experts on the domain of formal verification of time-related cryptographic protocols using the similar framework discussed in this Note. The author have been working on clarifying the relationship between the two ways of description for security verifications, including [12], especially with Gergei Bana, a former student of Andre Scedrov.

References

1. Hilbert, D.: On the infinite, in Frege to Gödel, pp. 369–392. Harvard University Press (1967). (Edited By van Heijenoort, J.) The original version: Über das Unendliche, *Mathematische Annalen* **95**, 161–190 (1926). *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **36**, 201–215 (1927)
2. Kanovich, M.I., Scedrov, A., Okada, M.: Specifying real-time finite-state systems in linear logic. In: 2-nd International Workshop on Constraint Programming for Time-Critical Applications and Multi-agent Systems (COTIC), *Electronic Notes in Theoretical Computer Science*, Nice, France, vol. 16, no. 1, September 1998 (1998). 15 pp
3. Okada, M.: Theory of formal specification and verification of concurrency systems and real-time systems based on linear logic, Report Meeting, Jan. 2002 Mext 12480075, also, Logical verification method for dynamic real-time system beyond the limit of model-checking method, Report Meeting, MEXT: 13224081
4. Hasebe, K., Cremet, V., Jouannaud, J.-P., Kremer, A., Okada, M.: FATALIS: real time processes as linear logic specifications. In: Second International Workshop on Automated Verification of Infinite-State Systems (2003)
5. Hasebe, K., Jouannaud, J.-P., Kremer, A., Okada, M., Zumkeller, R.: Formal verification of dynamic real-time state-transition systems using linear logic. In: The 20th Conference of Software Science Society of Japan, 2003, Proc., Japan (2003)
6. Alturki, M.A., Ban Kirigin, T., Kanovich, M., Nigam, V., Scedrov, A., Talcott, C.: A multiset rewriting model for specifying and verifying timing aspects of security protocols. In: Guttman, J.D., Landwehr, C.E., Meseguer, J., Pavlovic, D. (eds.) *Foundations of Security, Protocols, and Equational Reasoning*. LNCS, vol. 11565, pp. 192–213. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-19052-1_13
7. Kanovich, M.I., Ban Kirigin, T., Nigam, V., Scedrov, A., Talcott, C.L.: Compliance in Real Time Multiset Rewriting Models. CoRR abs/1811.04826 (2018)
8. Kanovich, M.I., Ban Kirigin, T., Nigam, V., Scedrov, A., Talcott, C.L., Ranko, P.: A rewriting framework and logic for activities subject to regulations. *Math. Struct. Comput. Sci.* **27**(3), 332–375 (2017)
9. Kanovich, M.I., Ban Kirigin, T., Nigam, V., Scedrov, A., Talcott, C.L.: Timed multiset rewriting and the verification of time-sensitive distributed systems. In: FOR-MATS, pp. 228–244 (2016)
10. Kanovich, M.I., Ban Kirigin, T., Nigam, V., Scedrov, A., Talcott, C.L.: Discrete vs. dense times in the analysis of cyber-physical security protocols. In: POST, pp. 259–279 (2015)
11. Kanovich, M.I., Ban Kirigin, T., Nigam, V., Scedrov, A., Talcott, C.L., Ranko, P.: A Rewriting Framework for Activities Subject to Regulations. In: RTA, pp. 305–322 (2012)
12. Bana, G., Okada, M.: Semantics for “Enough-Certainty” and Fitting’s embedding of classical logic in S4. In: *Computer Science Logic 2016, Proceedings*, article 23, 17 p. (2016)

Open Access This chapter is licensed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license and indicate if changes were made.

The images or other third party material in this chapter are included in the chapter's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the chapter's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.