Chapter 7

Designing Teaching: The Pythagorean Theorem

1 Introduction

Here we consider a fundamental activity of teachers that has to be based on an integrated view of mathematics and pedagogy in order to be successful: namely, the design of teaching. As expressed in the title, a well-known topic of geometry, the Pythagorean theorem, is used for illustrating this integrative approach to student teachers. In other words, the emphasis of the paper is *less* on the Pythagorean theorem per se but *more* on general principles of a teacher's "design kit" that can be applied to other topics as well.

The design of teaching lies at the very heart of a teacher's professional activities. That is why some authors conceive of teaching as a design profession and correspondingly of mathematics education as a *design science* (Clark and Yinger 1987; Wittmann 1985, 1995). For this reason the paper can also be understood as an example of how to organize research and development in mathematics education along the lines of design.

Accordingly there are four sections that follow. The first section is to make the reader think about the Pythagorean theorem within the context of school by remembering personal experiences from school and university; by solving textbook problems; by looking at the treatment of the Pythagorean theorem in textbooks; and by interviewing students on what they have retained from teaching.

The second section introduces the reader to the framework of mathematical concepts behind and around this theorem and its proofs; problem contexts from which the theorem naturally arises; and research on students' psychological development in understanding and using these concepts.

The third section will demonstrate how the mathematical, heuristic, and psychological strands from the second section have to be related and tuned to one another, that is to be *integrated*, in the design of teaching units. The section contains teaching plans of introductory teaching units for the Pythagorean theorem.

The final section explains some key concepts that can be generalized from the three strands of Sect. 2: the notion of “informal” proof; the heuristic strategy “specializing”; and the so-called operative principle.

All three principles will also be illustrated by subject matter different from the Pythagorean theorem in order to stimulate the transfer of these key concepts to other topics.

2 Thinking About the Pythagorean Theorem within the School Context

But neither thirty years, nor thirty centuries, affect the clearness, or the charm, of geometric truths. Such a theorem as “the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the sides” is as dazzlingly beautiful now as it was in the days when Pythagoras first discovered it, and celebrated its advent, it is said, by sacrificing one hundred oxen – a method of doing honour to science that has always seemed to me slightly exaggerated and uncalled for. One can imagine oneself, even in these degenerate days, marking the epoch of some brilliant scientific discovery by inviting a convivial friend or two, to join one in a beefsteak and a bottle of wine. But one hundred oxen! It would produce a quite inconvenient supply of beef.

C.L. Dodgson

In any right triangle the area of the square described on its longest side (the hypotenuse) is equal to the sum of the areas of the squares described on the other two sides (the legs).

This theorem is named after the Greek philosopher Pythagoras who lived around 500 B.C. and was the spiritual leader of a kind of philosophic-religious sect (the Pythagorean brotherhood, see van der Waerden 1978). Historians are certain that the fact stated in the theorem was already known to the ancient Babylonians, Egyptians and Chinese. So Pythagoras did not discover it, but might have been one of the first to give a proof.

The Pythagorean theorem enables one to compute the length of the third side of a right triangle if the lengths of the other two sides are given. In elementary geometry and its applications this situation arises very frequently when informations about lengths of segments are near at hand and right triangles can easily be identified or introduced.

Because of its richness in mathematical relationships and applications the Pythagorean theorem and its generalizations form a cornerstone of geometry. Mathematicians do not hesitate to rank the theorem among the top 20 theorems of all times. Without any doubt the Pythagorean theorem is *the* outstanding theorem of school mathematics. Generations of students have learned it, willingly or unwillingly, and many of them have kept the “Pythagorean” in their mind throughout their lives as the incarnation of a mathematical theorem.

Before interacting with the views expressed in this paper it is absolutely necessary for you first to mobilize your knowledge about the Pythagorean theorem and to get some fresh first hand experiences about the Pythagorean theorem, its teaching and, most important, about the learners. The following six activities are intended as catalysts for “jumping in.”

Hints to solutions can be found in the appendix, but first try yourself.

Exploration 1

Write down your own “memories” of the Pythagorean theorem both from school and university. Do you remember how the theorem was introduced, proved, applied? Did you encounter the theorem later on? Discuss your notes with your fellow students.

Exploration 2

Fig. 1 shows a cartoon from the nineteenth century. Discuss it in terms of the Pythagorean theorem: What special case is represented and how can it be proved from the two shapes?

Pythagoras

before

and after the discovery of the theorem
named after him

Fig. 1

Exploration 3

The following three problems may serve as a test for your feeling about the appropriate use of the Pythagorean Theorem.

Solve the problems, and record whether or not you used the Pythagorean theorem.

1. How long is the spatial diagonal s in a rectangular solid with edges a , b , c (see Fig. 2)?

Fig. 2

2. The vertices of a square and the midpoints of its sides are connected as shown in Fig. 3. What part of the area is formed by the shaded figure?

Fig. 3

3. A car is jammed in a parking lot. Under which conditions is it possible for the car to move out of the lot? Represent cars by pieces of cardboard, do some tests and devise a geometric model (see Fig. 4).

Fig. 4

Exploration 4

Because of its prominent role in school mathematics the Pythagorean theorem provides a rich source for collecting data on what “remains” in students after they have been taught the Pythagorean theorem in school.

The following interview form (Fig. 5) may give you an idea of how to probe students’ thinking about the Pythagorean theorem. The interview starts with questions that scratch only the surface of the Pythagorean theorem and from there goes on to questions that test understanding.

Age: _____ *Intended profession:* _____

1. Do you remember the Pythagorean theorem and can you write it down?
2. Do you have an idea of what the Pythagorean theorem is good for?
3. Can you give an example for its use in some profession?
4. Can you relate Fig. 5 to the Pythagorean theorem?
5. Do you know a proof of the Pythagorean theorem?

Fig. 5

1. Use the above interview form (or make your own form) and interview some students from grades 9 to 12. You may also ask some students to give written responses.
2. Analyze your data. Are there recurring patterns in students’ responses?

Exploration 5

Select a sample of textbooks for grades 7 to 10 and investigate if and how the Pythagorean theorem is introduced, proved, and applied. Which approach do you find most convincing? Discuss your choice with your fellow students.

Exploration 6

If you had to design a teaching unit for introducing the Pythagorean theorem on the basis of your present knowledge about the Pythagorean theorem, what basic idea would you choose and why?

3 Understanding the Structure of the Pythagorean Theorem

The design of teaching units requires a thorough understanding of the subject matter and of the psychological premises for learning it as teaching is a continuous process of mediating between the mathematical structure of the subject matter and the cognitive structures of the learners.

The best way to understand the *mathematical* structure of the Pythagorean theorem consists of examining proofs of and heuristic approaches to it. In order to get information about the *psychological* structures on which the teaching of this theorem can be based we have to look into developmental research on students' thinking about basic notions relevant in this context.

Although the mathematical, the heuristic, and the psychological strands of the Pythagorean theorem will be investigated separately in this section and their proper integration is to be attacked in the next section within the design of teaching units, relationships between them will become apparent quite naturally without taking special effort.

3.1 Different Proofs of the Pythagorean Theorem

The main goal of all science is first to observe and then to explain phenomena. In mathematics the explanation is the proof.

D. Gale

The richness of the Pythagorean theorem in conceptual relationships is clearly demonstrated by a multitude of different proofs. Lietzmann (1912) lists about 20 proofs, Loomis (1968) in his classic "The Pythagorean Proposition" even 370, most of which, however, are obtained from a few basic proofs by slight variations. It is interesting to realize that the Pythagorean theorem is rooted in all cultures. A particularly nice ethnomathematical approach based on a special decorative motif was developed by Gerdes (1988).

The following four proofs and their variations are interesting for both historic and educational reasons, and they cover also the essential approaches to the Pythagorean theorem found in textbooks. These four proofs are presented here as they are typically met in the mathematical literature. Taken as they stand they certainly cannot serve as a model for lively teaching, and the reader might wonder why they have been included here. However, the proofs display the conceptual relationships behind and around the Pythagorean theorem in the most effective way, and so analyzing and comparing them is indispensable for integrating content and pedagogy. Moreover, it will be instructive for the reader to compare the "lecture style" of this section with the process-oriented style of the next section and to see by what means life can be brought into seemingly "dead" content.

Proof 1 (*Euclid's proof (Euclid, Book I, §47)*) Euclid's famous *Elements of Mathematics* (1926) represents the first systematic mathematical treatise ever written. The thirteen books develop elementary geometry and arithmetic through a deductively organized sequence of theorems and definitions starting from basic concepts and axioms. The *Elements* has been the most influential mathematical textbook of all times and up to the twentieth century has also determined the teaching of geometry at school.

At the end of Book I we find the Pythagorean theorem (Proposition 47, see Fig. 6):

Fig. 6

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right; I say that the square on BC is equal to the squares on BA , AC .

For let there be described on BC the square $BDEC$, and on BA , AC the squares GB , HC ; through A let AL be drawn parallel to either BD or CE , and let AD , FC be joined. Then, since each of the angles BAC , BAG is right, it follows that with a straight line BA , and at the point A on it, the two straight lines AC , AG not lying on the same side make the adjacent angles equal to two right angles; therefore CA is in a straight line with AG .

For the same reason BA is also in a straight line with AH . And, since the angle DBC is equal to the angle FBA : for each is right: let the angle ABC be added to each; therefore the whole angle DBA is equal to the whole angle FBC .

And, since DB is equal to BC , and FB to BA , the two sides AB , BD are equal to the two sides FB , BC respectively; and the angle ABD is equal to the angle FBC ; therefore the base AD is equal to the base FC , and the triangle ABD is equal to the triangle FBC .

Now the parallelogram BL is double of the triangle ABD , for they have the same base BD and are in the same parallels BD , AL .

And the square GB is double of the triangle FBC , for they again have the same base FB and are in the same parallels FB , GC . (But the doubles of equals are equal to one another.) Therefore the parallelogram BL is also equal to the square GB .

Similarly, if AE , BK be joined, the parallelogram CL can also be proved equal to the square HC ; therefore the whole square $BDEC$ is equal to the two squares GB , HC .

And the square $BDEC$ is described on BC , and the squares GB, HC on BA, AC . Therefore the square on the side BC is equal to the squares on the sides BA, AC . Therefore etc.

Q.E.D.

Proof 1* (*Dynamic Version of Euclid's Proof*) In order to seek a more palatable way of understanding a proof whose diagram is as complicated as Euclid's proof, it is necessary first of all to understand the essence of the proof. What is Euclid trying to do? He has two squares, $BAGF$ and $ACKH$, on the sides of the right triangle ABC . He wants to show that the sum of the areas of these squares is equal to square $BCED$, the one on the hypotenuse. How does he do that? We can reduce the number of lines considerably for the purpose of demonstrating what he is trying to show (Fig. 7).

Fig. 7

He is trying to demonstrate that the area of square $BDEC$ can actually be decomposed into two pieces, one equal in area to square $BAGF$ and the other equal in area to $ACKH$. He demonstrates that the two darker regions are equal in area and the two lighter regions are equal in area.

Once you have convinced yourself that the above description is an accurate rendition of Euclid's proof, then you are in a position to create a proof that has more visual appeal. One such proof involves transforming each of the small squares on the sides of the original triangle into something more dynamic than the triangles as intermediaries. We can actually imagine the original small squares being transformed progressively into several parallelograms before actually forming the shaded rectangles that compose square $BCDE$.

Once we have the essence of the proof, we are still left with the pedagogically interesting task of transforming something quite technical and nonintuitive into something that is dynamic and intuitive. Euclid's proof shows that the two lighter regions

are equal in area by introducing an intermediary figure: $\triangle FBC$. He shows that the two darker regions are equal in area by introducing another intermediary figure: $\triangle BCK$. Each member of the pair of similarly shaded regions is equal to twice the area of that corresponding triangle.

Below is a description of the stages of successive transformation (see Fig. 8).

Fig. 8

1. Square $BAGF$ is sheared into parallelogram $BCMF$.
2. Parallelogram $BCMF$ is rotated into parallelogram $BDNA$.
3. Parallelogram $BDNA$ is sheared into rectangle $BDLQ$.

All three transformations preserve area. Therefore square $BAGF$ and rectangle $BDLQ$ have equal areas. In an analogous way square $ACKH$ is transformed into rectangle $QLEC$. As a consequence the area of $CBDE$ is equal to the sum of the areas of squares $ACKH$ and $BAGF$.

It is tempting to reduce the whole argument to a film simply “showing” the equality of areas. However, this would give a distorted view of proof. A visual demonstration can certainly support, but not replace, a proof. The proof hinges upon a conceptual framework that explains *why* these transformations can be applied and *why* they lead to the properties in question.

Exploration 7

Compare proofs 1 and 1*. Which parts of proof 1 correspond to which parts of 1*? Are there details in proof 1 that are missing in proof 1*? What are the advantages and the disadvantages of the formal language of “signs” in proof 1 and the informal language of “pictures” in proof 1*?

While Proofs 1 and 1* employ transformations, the following Proofs 2 and 3 depend on dissecting figures and rearranging the parts in clever ways.

The reference to dissections (decompositions) is quite natural as the measure “area” has the following properties:

1. Squares are used as units.
2. Congruent shapes have equal area.
(Formally formulated: Area is invariant under rigid motions.)
3. If a polygon is dissected in disjunct parts the sum of the areas of the parts is equal to the area of the whole polygon.
(Formally formulated: The area measure is additive.)

As a consequence of 1. and 2. equi-decomposable polygons have the same area. This relationship is also basic for the derivation of formulae for the areas of special polygons. So decomposition proofs are well embedded in the curriculum.

Obviously the following three proofs are the result of playing with shapes with the intention to get closed figures.

Proof 2 (*Indian Decomposition Proof*) This proof comes to us from the ancient Indians. It gives a direct solution of the problem to construct a square whose area is equal to the sum of the areas of two given squares.

Fig. 9

Construction 1 (Fig. 9): Draw right triangle ABC with sides $a = BC$, $b = AC$, $c = AB$. Describe square $CEDB$ on side BC , extend CA and draw square $EFGH$ (side b). Extend EH such that $HK = a$ and draw quadrilateral $AGKB$.

Statement 1 The sum of the areas of squares $CEDB$ and $EFGH$ is equal to the area of square $AGKB$.

Proof Let α and β be the acute angles in the right triangle ABC . As the sum of angles in all triangles is 180° we have the basic (and frequently used!) relation $\alpha + \beta = 180^\circ - 90^\circ = 90^\circ$.

By construction $AF = CE + EF - CA = a + b - b = a$ and $DK = EH + HK - ED = b + a - a = b$. Therefore all triangles ABC , GAF , GKH , and KBD have sides a , b subtending a right angle and so are congruent. As a consequence all sides of $AGKB$ have equal length c and all angles have measure $\alpha + \beta = 90^\circ$, that is, $AGKB$ is a square.

The area c^2 of $AGKB$ is equal to the sum $a^2 + b^2$ as $AGKB$ is composed of the shaded polygon and two triangles and the original squares are covered by the same polygon and two congruent triangles.

On Sect. 3.2 we will meet Fig. 19 which turns out as nothing but Fig. 9, rotated by 180° .

Proof 3 (*Geometric-Algebraic Proof*) This proof relates the Pythagorean theorem to the binomial formula $(a + b)^2 = a^2 + 2ab + b^2$, another fundamental topic of school mathematics.

Fig. 10

Construction 2 (Fig. 10): Given lengths a , b we construct a square with side $a + b$ and inscribe a quadrilateral $ADCB$ which is a square (why?). As the area of each of the right triangles surrounding $ADCB$ is $\frac{1}{2} \cdot ab$ we get

$$c^2 = (a + b)^2 - 4 \cdot \frac{1}{2} ab = a^2 + 2ab + b^2 - 2ab = a^2 + b^2.$$

Exploration 8

Cut a square frame (side $a + b$) and four right triangles with legs a, b out of a piece of cardboard. The triangles can be put into the frame in different ways (see Figures 11a, 11b, and 11c).

1. Derive the Pythagorean theorem from Figures 11a and 11b and also from Figures 11c and 11b without using algebra. Compare these geometric proofs with proof 3.
2. Compare Figures 11b and 11c with Fig. 9 (proof 2). Can you extend Fig. 10 such that both Fig. 11b and Fig. 11c are visible in the extended figure?

Fig. 10

Fig. 11

Proof 3* (*Bhaskara's Proof*) This proof is credited to the Hindu mathematician Bhaskara, who lived in the twelfth century, but it is much older and likely to have been known to the Chinese before the time of Christ.

Fig. 12

The “Bhaskara” Fig. 12 arises from Fig. 10 (Proof 3) by folding the four right triangles inside the square. A careful check of lengths and angles reveals that the small quadrilateral inside is a square with side $b - a$. Therefore

$$c^2 = 4 \cdot \frac{1}{2}ab + (b - a)^2 = 2ab + b^2 - 2ba + a^2 = a^2 + b^2.$$

In this case there is no immediate purely geometric interpretation as before. However, we will come back to this problem later.

Proof 4 (Similarity Proof) It is an interesting question for historians which proof might have been given by Pythagoras himself. van der Waerden (1978) concludes from the context in which the Pythagoreans lived and worked that they might have used the self-similarity of a right triangle, that is its decomposability into two triangles similar to it. This proof runs as follows (see Fig. 13):

Fig. 13

The altitude dropped from vertex C divides the right triangle ABC into two right triangles with angles equal to the original triangle (why?). Therefore BCD and CAD are similar to ABC . This gives the proportions

$$\frac{p}{a} = \frac{a}{c}, \quad \frac{q}{c} = \frac{b}{c}$$

that can be transformed into

$$p = \frac{a^2}{c}, \quad q = \frac{b^2}{c}.$$

As $p + q = c$ we get $c = p + q = \frac{a^2}{c} + \frac{b^2}{c}$ and finally $c^2 = a^2 + b^2$.

Note that area doesn't play any role in this proof. The geometric basis is provided by proportions of lengths arising from similarity. The squares are the result of a purely algebraic manipulation of symbols standing for lengths. However, it is possible to interpret Fig. 13 in terms of area. This leads us to

Proof 4* (*Similarity/Area Proof*) Consider Fig. 13 once more. Triangles BCD and CAD are small copies of triangle ABC . Therefore the lengths of the sides of BCD and CAD can be obtained by reducing the lengths of the corresponding sides of ABC by the factor $\frac{a}{c}$ and respectively the factor $\frac{b}{c}$. So we have

$$\text{Area}(BCD) = \frac{a^2}{c^2} \cdot \text{Area}(ABC)$$

$$\text{Area}(CAD) = \frac{b^2}{c^2} \cdot \text{Area}(ABC).$$

As the sum of the areas of BCD and CAD is equal to the area of ABC we arrive at

$$\frac{a^2}{c^2} \cdot \text{Area}(ABC) + \frac{b^2}{c^2} \cdot \text{Area}(ABC) = \text{Area}(ABC)$$

$$\left(\frac{a^2}{c^2} + \frac{b^2}{c^2} \right) \cdot \text{Area}(ABC) = \text{Area}(ABC)$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$a^2 + b^2 = c^2.$$

Note: If a dilatation with scale factor k is applied, areas are transformed by the *square* k^2 . For example, area is multiplied by 4 if lengths are doubled, and multiplied by $\frac{1}{4}$ (that is, divided by 4) if lengths are halved.

Exploration 9

Compare Proofs 4 and 4*: In both proofs each of the two small triangles is first related to the big triangle separately. How? Then all three triangles are brought together. What is the crucial relation combining the three triangles and leading to the Pythagorean theorem in each proof? In other words: the Pythagorean theorem expresses an equality of areas. On what relationship is this equality based in each proof?

Reflective Problem 1

Analyze the proofs in this section: Where in the proof is the existence of a right angle crucial?

On which geometric or algebraic concepts is each of them based? Which geometric transformations are used? How do these affect area, length, measure of angles? Which algebraic formulae are used? Which step establishes the “equals” sign inherent in the theorem?

Evaluate the proofs: Which of them do you find easiest, which one most demanding? List them in order of increasing difficulty. Do you find them equally sound? If not, why? Which proof do you find most convincing, which one most interesting? Why? Do you prefer the algebraic or the geometric proofs?

Discuss your views with your fellow students; in particular compare your “difficulty” lists.

3.2 *Heuristic Approaches to the Pythagorean Theorem*

We should orientate our teaching more on problems than on theories; a theory should be taught just as far as necessary for framing a certain class of problems.

Giovanni Prodi

From the point of view of mathematical learning the mere study of proofs is not satisfactory, as it presents mathematics as a corpse laid down for an autopsy. Certainly logical analyses have their merit for recognizing conceptual relationships. However, in order to design teaching units that stimulate students to explore, describe, explain, and apply patterns we have to go back to the source of mathematical activity, that is, to *mathematical problems* inside and outside of mathematics. It is of central importance that students are offered the opportunity to experience mathematical concepts, theorems and techniques as *answers to problems* and as starting points for new problems. Otherwise it will be almost impossible for them to grasp the meaning of mathematics and to develop confidence in the use of it.

Our next task will be then to find appropriate problems that can lead to the discovery of the Pythagorean theorem and to explanations, that is proofs, of it.

The general direction of search is clear: We have to investigate situations in which the Pythagorean theorem is naturally used and examine if the context is strong enough in order to “generate” the theorem and to establish a proof.

Two approaches are offered below.

Approach 1 Clairaut’s Approach (Clairaut 1741, sections 16,17)

A.C. Clairaut (1713–1765) was one of the most famous French mathematicians of the 18th century. He was a mathematical prodigy and wrote his first published mathematical paper on four spatial curves discovered by him as a twelve-year-old. Another paper of his attracted the attention of members of the French Academy

of Science who at first couldn't believe that a sixteen-year-old had written such an ingenious and profound paper of 127 pages. By special order of the King Clairaut was appointed a member of the Academy at the age of 18. It remained the only exception ever made to admit a person under 20 to the Academy.

Clairaut was also very much interested in teaching mathematics and as he strongly objected to the formalistic style of the textbooks used at his time, including Euclid's *Elements of Mathematics*, he set out to write books on elementary geometry and algebra in a quite different style. In the preface of his *Elémens de Géometrie* (Clairaut 1743) he explains his views on learning and teaching as follows:

Although geometry is an abstract field of knowledge, nobody can deny that the difficulties facing beginners are mostly due to how geometry is taught in elementary textbooks. The books always start from a large number of definitions, postulates, axioms and some preliminary explanations that appear to the reader as nothing but dry stuff. The theorems coming first do not direct the students' mind to the interesting aspects of geometry at all, and, moreover, they are hard to understand. As a result the beginners are bored and rejected before they have got only the slightest idea of what they are expected to learn.

In order to avoid this dullness attached to geometry some authors included applications in such a way that right after the theoretical treatment of the theorems their practical use is illustrated. However, in this way only the applicability of geometry is shown without facilitating the learning of it. As any theorem precedes its applications the mind is brought into contact with meaningful situations only after having taken great pains in learning the abstract concepts.

Some thoughts on the origins of geometry made me hope to avoid these unpleasant difficulties and to take students' interests seriously into account. It occurred to me that geometry as well as other fields of study must have grown gradually; that the first steps were suggested by certain needs, and that these could hardly have been too high as it were beginners who made them for the first time. Fascinated by this idea I decided to go back to the possible places where geometric ideas might have been born and to try to develop the principles of geometry by means of a method natural enough to be accepted as possibly used by the first inventors. My only addendum was to avoid the erroneous attempts these people necessarily had to make.

Exploration 10

Compare Clairaut's view on problem-oriented teaching with statements on "Mathematics as Problem Solving" in the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989, pp. 7, 66, 75-77, 125, 137-139). What arguments are put forward in favor of problem-oriented teaching?

The problem chosen by Clairaut for introducing the Pythagorean theorem was this:

Determine the side c of a square whose area is the sum of the areas of two given squares with sides a and b .

In section 16 of his book he considers first the special case $a = b$ by asking how to construct a square whose area is twice the area of a given square.

The solution of this special case is fairly easy if one takes two copies of the given square, draws the diagonals, and rearranges the four triangles (see Fig. 14a and b). In concrete form four congruent isosceles triangles can be cut from cardboard and arranged in two ways corresponding to Fig. 14a and b ("square puzzle").

Fig. 14

In section 17 of his book Clairaut addresses the general case:
How to construct a square whose area is the sum of the areas of two different given squares?

The straightforward transfer from the special to the general case (see Fig. 15) is not successful, however, at least not immediately. Figure 16 does not “close”.

Fig. 15

Fig. 16

But the construction can be adapted: If one dissects Fig. 16 by starting from a different point H (see Fig. 17) the new Fig. 18 is an “improvement.”

Fig. 17

Fig. 18

Clairaut continues: “Following this idea it is quite natural to ask if it is possible to find a point H on DF such that

1. the triangles ADH and EFH if rotated around A resp E into the positions $AD'H'$ and $EF'H'$ meet in H' ,
2. the four sides AH , HE , EH' and $H'A$ are equal and form right angles.

It is easy to see that H is determined by $DH = CF (= b)$ or $HF = DC (= a)$. (See Fig. 19).”

Fig. 19

The problem is now solved and all that remains to do is to introduce the sides a , b , c and to state that by construction $c^2 = a^2 + b^2$. The figure is determined by the right triangle AHD and can be drawn by starting from an arbitrary right triangle AHD . Therefore $c^2 = a^2 + b^2$ holds for the sides of *any* right triangle.

Figure 19 is well known to us: It is nothing but Fig. 9 of the “Indian decomposition proof” (Proof 2). While this figure came out of the blue in Sect. 1 it appears here *within the solution of a problem*, and the Pythagorean theorem gives the answer to this problem. We have in this example a good illustration for the difference between a proof embedded solely into a net of logical relationships and a proof embedded into a meaningful context.

Exploration 11

Use the software *The Geometer's Sketchpad* or *Geogebra* for representing Clairaut's approach in a dynamic way.

Special case: First draw figure 14a. Rotate AED around A by 270° and triangle BCE around B by -270° (or 90°). You get a combination of Figures 14a and 14b.

General case: Draw Figure 17 starting with segment DF and choose H as a (moving) point on DF . Rotate AHD around A by 270° and EFH around E by -270° (or 90°). You get Fig. 18. By moving H on segment DF points H' and H'' move on line CD' , and you can easily find the position of H when the figure “closes” (see Fig. 19).

Approach 2 The Diagonal of a Rectangle

Our second approach starts from the following problem:

How long is the diagonal of a rectangle with sides a and b ?

This problem is interesting from the mathematical point of view, but it has also a reasonable real interpretation: A rectangular frame with sides a , b is to be stabilized by means of a diagonal lath. How long should the lath be (see Fig. 20)?

Fig. 20

If the Pythagorean theorem is known the answer is obvious: $c = \sqrt{a^2 + b^2}$. However, our aim is again to use this problem for “generating” the Pythagorean theorem.

How can we approach this problem? For example, we can compare a , b and c and find that c is longer than both a and b and smaller than $a + b$. We also can draw rectangles of different shapes, measure c , and establish a table.

a in cm	10	8	4	8	7.5	9
b in cm	5	5	3	6	7.5	7.5
c in cm	11.2	9.4	5	10	10.6	11.7

But how to *calculate* c ? The heuristic strategy “Specializing” used by Clairaut is a reasonable strategy here, too. So let us consider first the special case of a square (see Fig. 21).

Fig. 21

How is the diagonal c of a square related to its side a ?

Reflective Problem 2

Think about this problem. Note that one diagonal divides the square into two congruent right triangles with hypotenuse c and altitude $c/2$. So there are two ways of calculating the area that can be used to derive the relationships $c^2 = 2a^2$ and $c = \sqrt{2} \cdot a$.

$c^2 = 2a^2$ “cries” for a geometric interpretation. It is provided by the “square puzzle” from approach 1: Four congruent right isosceles triangles can be put together to form either one big square or two small squares (see Fig. 22a and b).

Fig. 22

As before we try to generalize this result to rectangles, that is, we look for a generalized “puzzle” establishing the Pythagorean theorem for arbitrary right triangles.

Is it possible to recombine the four halves of two congruent rectangles to make a square whose side is the diagonal of the rectangle?

Exploration 12

Cut four congruent right triangles from cardboard (see Fig. 23) and think about this problem first for yourself. Can you make a square shape with side c with these pieces?

Fig. 23

Fig. 24

A first attempt leads to Fig. 24 which, however, is not a square, but only a rhombus: All sides are equal, but the angles are different—two of them are 2α and two of them are 2β .

However, because of the basic relation $\alpha + \beta = 90^\circ$ we could try to combine the four right triangles in a slightly different way (see Fig. 25).

We arrive at three equal sides, two right angles, and an isolated right triangle. The question is:

Does the fourth triangle really fit in? The dotted line indicates a square “hole” with side $a - b$. Because of $a - (a - b) = b$ and $b + (a - b) = a$ the gap is exactly filled indeed by the fourth triangle. So we get a square with side c but, alas, with a square “hole” inside (see Fig. 26).

That the angles of the “hole” are right angles follows from the right angles of the triangles.

Fig. 25

Fig. 26

Nevertheless we can calculate c :

$$c^2 = 4 \cdot \frac{1}{2}ab + (a - b)^2$$

$$c^2 = 2ab + a^2 - 2ab + b^2$$

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

This is the formula we were looking for: The side c is expressed as a function of a and b .

Again, Fig. 26 is well known to us: It is exactly Fig. 12 used by Bhaskara (Proof 3*). In marked contrast to that presentation, the figure appears here within the solution of a problem. So we have another illustration of the difference between a formal proof within a deductive structure and an informal proof arising from a meaningful context.

As in the special case we want to understand $c^2 = a^2 + b^2$ in purely geometric terms.

The square with side c can be formed by means of a puzzle consisting of five pieces (“Bhaskara-Puzzle”): four congruent rectangular pieces with sides a , b and a square piece with side $a - b$. Can these five pieces be recombined to form a shape composed of a square with side a and a square with side b ?

Exploration 13

Cut the five pieces of the “Bhaskara-Puzzle” from cardboard and show geometrically that $c^2 = a^2 + b^2$. You have to arrange the five pieces such that they cover the union of a square with side a and of a square with side b .

Hint: Fig. 9 or Fig. 19.

Exploration 14

Reexamine the logical line in approaches 1 and 2: At what places is the assumption of right angles crucial?

3.3 Exploring Students’ Understanding of Area and Similarity

Concepts are the backbone of our cognitive structures. But in everyday matters concepts are not considered as a teaching subject. Though children learn what is a chair, what is food, what is health, they are not taught the *concepts* of chair, food, health. Mathematics is no different. Children learn what is number, what are circles, what is adding, what is plotting a graph. They grasp them as *mental objects* and carry them out as *mental activities*. It is a fact that the concepts of number and circle, of adding and graphing are susceptible to

more precision and clarity than those of chair, food, and health. Is this the reason why the protagonists of concept attainment prefer to teach the number concept rather than number, and, in general, concepts rather than mental objects and activities? Whatever the reason may be, it is an example of what I called the anti-didactical inversion.

Hans Freudenthal

The mathematical and heuristic structure of subject matter form only two of the three strands that have to be twisted in the design of teaching. The third equally important one is knowledge of the students' cognitive structures as far as they are relevant for the topic to be learned.

Our mathematical analyses have shown that the Pythagorean theorem is fundamentally related to the concepts of area and similarity. Therefore it is necessary to provide data on the psychological development of these concepts. We cannot give a systematic and coherent review of research here. Instead we concentrate on a few interesting studies that give a first orientation and—what is even more important—also provide a basis for doing similar studies. The central part of this section is “Clinical interviews on area and on doubling a square,” where the reader is stimulated to do some study of his or her own.

The basic message of this chapter is this: Mathematical concepts are neither innate nor readily acquired through experience and teaching. Instead the learner has to reconstruct them in a continued social process where primitive and only partly effective cognitive structures that are checkered with misconceptions and errors gradually develop into more differentiated, articulated and coordinated structures that are better and better adapted to solving problems. For teachers this message is of paramount importance: Concepts must not be presupposed as trivially available in students nor as readily transferable from teacher to student. On the contrary, the teacher must be prepared that students often will misunderstand or not understand what he or she is talking about. To have a feeling for students' misconceptions, to be able to dig into students' thinking until some solid ground appears that may serve as a basis for helping the students to reconstruct their conceptual structures on a higher level, to interact with students particularly in seemingly hopeless situations—that is the supreme mark and criterion of a competent teacher.

Doubling a Square: Plato's dialogue Meno

The Greek philosopher Plato (ca. 429–348 B.C.) is an important figure for mathematics education as in his philosophic system mathematics played a fundamental role. Relevant for teaching and learning and therefore frequently referred to is his dialogue *Meno* that centers around the fundamental questions if virtue can be taught and where knowledge does come from.

One part of this dialogue is particularly interesting as perhaps the oldest recorded lesson in mathematics: Socrates teaches, or better interviews, a boy on how to double a square (Plato 1949).

The structure of the interview is as follows:

1. A 2×2 -square is presented and the boy is asked to find a square of double size (see Fig. 27)

Fig. 27

2. Although the boy predicts the area of this new square as 8 square feet, nevertheless his first suggestion is to double the sides. This leads to the 4×4 -square (see Fig. 28) that turns out as four times as big instead as twice as big—a cognitive conflict for the boy!

Fig. 28

3. In order to correct his mistake the boy offers the 3×3 -square (lying between the 2×2 and the 4×4 -square) as the solution (see Fig. 29). Again Socrates arouses a cognitive conflict by having the boy calculate its area: to his own surprise the boy finds 9 square feet instead of the expected 8!

Fig. 29

4. Finally, it is Socrates who returns to the 4×4 -square (see Fig. 30), introduces the diagonals and guides the boy to discover that the square formed by the diagonals has the required area of 8 square feet and therefore is the solution of the problem.

Fig. 30

Clinical Interviews on Area and on Doubling a Square

Plato's dialogue is interesting in our context not only because it deals with a special case of the Pythagorean theorem, the doubling of a square, but also because it can be considered as the ancient version of a psychological method that was fully developed by the great Swiss psychologist and epistemologist Jean Piaget (1896–1980) in the thirties and is now widely used in research: the “clinical interview.” However, there is a basic difference in the views of Plato/Socrates and Piaget as far as the origin of knowledge is concerned. These Greek philosophers believed that knowledge was already *innate* in human beings. So they compared the teacher's task to that of a midwife: With a definite goal in mind the teacher has to make the student “remember” and bring his knowledge to light. In sharp contrast with this view Piaget sees knowledge not as something pre-fabricated inside or outside the learner but as the continued personal construction and reconstruction of the learner while interacting with and trying to adapt to the natural and social environment. Therefore the Piagetian interviewer unlike Socrates in Plato's dialogue is anxious *not* to guide the student to a definite end. The aim of the clinical interview is to uncover the student's authentic mental structures, not to subject him or her to any kind of “teaching”. Therefore the interviewer must be open to what the student has to offer, try as far as possible to put him- or herself in the student's place and make sense of the student's thinking—also in case of strange and contradictory answers. He or she must not be content with just listening to students, but has to stimulate them to express their mental processes with words or actions, always following their fugitive thoughts. Questions like “How do you know?” or “Why do you think so?” and cautious counter-arguments for arousing cognitive conflicts like “But some other child told me ...” are essential elements of clinical interviews. In short, the clinical interview is a kind of “mental auscultation” analogous to the physical auscultation in medical checkups. For this reason it was called *clinical*.

It is important for student teachers to realize that the clinical method is valuable not only from the psychological but also from the pedagogical point of view: In conducting clinical interviews the student teacher acquires insight into children's thinking and becomes familiar with essential habits of good teachers—introducing children into a mathematical situation with parsimonious means and with clear explanations, showing interest for what they are doing, observing them without interrupting, listening to them, accepting their intellectual level, giving them time to work and to think,

stimulating their thinking by sensitive questions and hints, and so forth (Wittmann 1985).

Of course, the clinical method is also extensively used in Piaget/Inhelder/Szeminska (1964), one of the major studies of children's geometric thinking. The book contains a chapter on doubling area and volume (Chap. XIII).

The following study done by student teachers was inspired by both Plato and Piaget. It may give a feeling for both Piaget-like studies into students' thinking and the clinical interview as a research method.

The following setting was used:

Material: 16 congruent squares ($3\text{ cm} \times 3\text{ cm}$) and 32 triangles (half of one 3×3 square), made of cardboard.

Technique: Students were interviewed individually according to the following scheme:

1. Involve the student in an informal chat as a kind of warm-up.
2. Show the student the geometric forms and ask: *Which different figures can you build with these?*
3. After phase 2 is finished take four squares, form a 2×2 -square, tell the student the following story, and conduct some "mental auscultation" on his or her understanding of area:

Imagine that this is a pasture surrounded by a fence. It is just big enough to give grass for exactly eight cows. Now the farmer buys eight more cows and wants to fence off a pasture that is twice as big. As he likes squares the bigger pasture should be a square as well.

Can you help the farmer and make a square of double size?

The above scheme is "half-standardized" in the sense that only some key informations and questions are prescribed that have to be reproduced in all interviews and so form the common core. All other interactions depend on the student's responses.

The following two interviews with the 11 year-old Dirk and with the fifteen-year-old Stefan give an impression of the wide range of students' abilities and thinking. Dirk

1. While playing around with the forms and laying out a variety of figures Dirk explicitly states that four triangles can be arranged to make a larger triangle (see Fig. 31).

Fig. 31

2. In order to solve the pasture problem Dirk adds four squares and produces Fig. 32:

Fig. 32

Dirk: Oh, no, that doesn't work. That gives me nine squares, but I need eight.

Next he tries to attach a triangle to a square (see Fig. 33a).

When seeing that this is possible he builds Fig. 33b and adds another triangle (see Fig. 33c)

Fig. 33

Dirk: Oh, no, that doesn't fit!

His next figure is Fig. 34.

Fig. 34

He counts the squares: These are only seven squares. Must all sides be equal?

Interviewer: Yes. Do you think you can do it?

Dirk's next step is Fig. 35, a figure with an area equivalent to eight squares, but not a square.

Fig. 35

Dirk, after thinking for a while: I could try it with triangles.

He first makes the original square (see Fig. 36a)

Then he supplements eight more triangles (see Fig. 36b).

Fig. 36

Dirk: I think, that's it!

Interviewer: Why do you think it is twice as big as the old square?

Dirk: Inside you have the old square with eight triangles, and in addition eight new triangles.

Interviewer: How did you hit upon the idea to arrange the triangles this way?

Dirk: I saw that two triangles make a larger triangle (see Fig. 37) that can be added to the square.

Fig. 37

Stefan

1. Stefan builds only few figures, for example a rectangle and a “house” (see Fig. 38a and b).

Fig. 38

2. Stefan immediately lays a 3×3 -square (see Fig. 39)

Fig. 39

Interviewer: Do you think this is right for twice as many cows? Is it really double the size?

Stefan: Yes.

Interviewer: How do you know that?

Stefan: These are four [*he points to the original square*] and these are ... five—there is room for more cows.

Interviewer: Can you also build a pasture for exactly sixteen cows?

Stefan: I do not see how.

He adds two squares to the original square (Fig. 40).

Fig. 40

Stefan: No. In this way I again get one more ... It is not possible.

Interviewer: What about using these triangles?

Stefan: No. Two triangles make a square again.

Interviewer: So you think, it's not possible?

Stefan: No, it's not possible. Either you have to take one square more or you must build a rectangle.

Exploration 15

Analyze Figures 33c, 34 and 35 in the interview with Dirk. How far are they away from the solution? Keep in mind that the solution has to meet two requirements: it has to be a square *and* it has to have an area of 8 unit squares. Fig. 34 can be extended to a square in two ways: 1. By adding four given triangles to the four longer sides of the octagon one gets a 3×3 square. 2. But compare Fig. 33c with Fig. 34. Obviously Dirk tries to add triangles to the smaller sides of the octagon. The given triangles are too big, as Dirk recognizes: What triangles would be necessary to extend the octagon to a square in this second way? How many unit squares would this second square have?

Do a similar analysis with the second interview. What is the biggest block in Stefan's thinking?

The main findings of some thirty interviews with students in the age range eleven to fifteen were the following:

1. Only a few students built first the 4×4 -square and they were aware *at once* that it was too big, that is, the first misconception in Plato's dialogue was not observed.
2. However, almost all students arrived at the 3×3 -square somewhere in the interview either taking it for the solution (as the boy in Plato's dialogue) or using it as a step towards the solution.
3. Students that flexibly operated with forms (like Dirk) had a much higher chance of finding the solution than students who were "fixed" to certain tracks (like Stefan). The variable "age" was of minor importance.

Reflective Problem 3

Form pairs or triples of student teachers and conduct clinical interviews with secondary students on doubling a square by using the material and the technique described above. One of you should conduct the interview, the other one or two should act as observer(s) and take written notes or serve the cassette recorder or the video camera. Don't forget to change roles.

In retrospect you can also examine how far you have fulfilled the requirements for clinical interviews listed at the beginning of this subsection or in Ginsburg 1983.

Exploration 16

Show secondary students the "Bhaskara-puzzle" (Exploration 13) and ask them to form "different figures". If they do not hit upon the square by themselves ask them to make one.

Student Conceptions and Misconceptions about Area Measure and Similarity

Piaget's research on area formed the starting point for many investigations that share the following general framework: The student is offered a series of items where

figures are to be constructed and transformed in various ways—cut from paper, moved around, reflected, decomposed, rearranged, enlarged and reduced. The student is always asked to describe, predict and explain how area “behaves” under these transformations. The answers indicate how well he or she has attained the concept of area.

We will see in section called “The Operative Prinziple” that in Piaget’s theory of cognitive development the flexible use of “operations” is considered as the cornerstone of intelligent behavior in mathematics and beyond. Of course the nature of operations differs from domain to domain: in arithmetic we deal with number operations, in calculus we use transformations of functions, in combinatorics we operate with combinatorial formulae, and so forth. Nevertheless in all these fields there is an operative structure of knowledge. The following analysis of operations connected to area and similarity has therefore far-reaching importance as an instructive special case.

The present subsection tries to give an idea of research that clearly follows the Piagetian paradigm and covers the age range from eight to fifteen. The aim is just to establish a feeling for area as a concept *developing in students’ minds*.

In clinical interviews with eight- to eleven-year-olds Wagman (1975) used the following tasks that directly reflect the basic properties of the concept of area: 1. Use of squares as units, 2. Invariance under rigid motions, 3. Additivity.

Unit Area Task. The investigator presents three polygons that can be covered by unit squares and asks the child to find out how many squares are needed in each case. The necessary squares are piled besides each polygon. In the second part the child is given a large number of triangular tiles each of which is equal to one half the square tile. The child is asked to find out how many of these triangular tiles are needed to cover the polygons.

Congruence Axiom Task: The investigator presents the child with two congruent isosceles right triangles, one blue, the other one green. The child is asked how many white triangles (of half linear dimensions) are necessary for tiling the blue triangle. After discovering the answer (4) the child is asked to guess without trying how many white triangles will be needed to tile the green triangle.

Additivity-of-Area Tasks

1. The child is presented with two polygonal regions a. with equal areas, b. with different areas. Given a set of smaller shapes the child is asked to cover each of the two polygons and to decide if they have the same or different amounts of space.
2. Polygons are decomposed and the pieces are rearranged to form another polygon. The child is asked to compare the areas.

In all these tasks the investigator encourages the child to work on the materials and to give explanations: “How do you know?”, “Why do you know?”, “Are you sure?”.

As we shall see, these tasks test the children’s ability to operate flexibly and effectively with shapes and their understanding of the concept of area.

In her study with 75 children from eight to eleven years, Wagman found that 6 of them were still in a “pre-measurement” stage, 31 showed some first understanding of area, 35 could use all properties of the area concept in simple cases, and only 4 displayed full mastery (Wagman 1975, 107).

In a study with large numbers of secondary students (twelve to fourteen years) Hart (1981, 14-16) used the following tasks that are similar to some of Wagman’s tasks and also test the mastery of the properties *invariance* and *additivity*:

1. A machine makes holes in two equal squares of tin in two different ways (see Fig. 41 A, B). Students are asked to compare the amount of tin in A and B.

Fig. 41

2. A square A is cut into three pieces and the pieces are arranged to make a new shape B (see Fig. 42). Students are asked to compare the areas of A and B.

Fig. 42

The result revealed that about 72 percent of the total population could successfully answer both questions. There were no major differences between the age groups.

In comparing Wagman’s findings with these results we recognize a substantial increase in understanding the concept of area for the majority of students. However, the concept of area is by no means mastered by all secondary students.

In the past decade research on the development of the similarity concept has been intensified. A typical problem used in the International Studies of Mathematical Achievement in 1964 and in 1982 for eight graders is the following one:

On level ground, a boy 5 units tall casts a shadow 3 units long. At the same time a nearby telephone pole 45 units high casts a shadow the length of which, in the same units, is

- A. 24
- B. 27
- C. 30
- D. 60
- E. 75

The results are remarkable: 56 percent of the students chose the correct answer (27) at the end of the school year 1963/64, whereas only 41 percent did so at the end of the school year 1981/82.

In examining students' thinking on ratio and proportion Hart (1981, 97–101) used a battery of items among them the following task:

The students are shown a 3 cm $\times$ 2 cm rectangle (Fig. 43) and a base line of 5 cm (Fig. 44) that is to be completed “so it is the same shape but larger” than the given rectangle.

Fig. 43

Fig. 44

This task requiring an enlargement in ratio 5 : 3 belongs to the highest level of difficulty. Hart (1981, 99) found that only 5 percent of the thirteen-year-olds, 9 percent of the fourteen-year-olds and, 15 percent of the fifteen-year-olds achieved this level and stated:

Similar triangles appear early in the introduction to ratio in most secondary textbooks and children are expected to recognize when triangles are similar to each other and the properties they possess. On interview it was found that the word “similar” had little meaning for many children. In everyday language the word is used in a non-technical sense to mean “approximately the same”. There was particular difficulty with the word when similar triangles or rectangles were under discussion. The test items dealing with similar figures (not triangles or rectangles) were some of the hardest on the test paper. Using ratio to share amounts between people “so that it is fair” seemed to be much easier than dealing with a comparison of two figures. In enlarging figures there is the danger of being so engrossed in the method to be used that the child ignores the fact that the resulting enlargement should be the same shape as the original ... The introduction of non-whole numbers into a problem does not make the question a little harder but a lot harder.

These findings are confirmed by a series of other papers (see, for example, the review by Lappan and Even 1988).

Summary:

Psychological research on the concepts of area and similarity suggests the following conclusions for teaching the Pythagorean theorem:

1. A satisfactory treatment of the Pythagorean theorem can only be reached within a long-term perspective of the curriculum. For coming to grips with the concepts of congruent and similar figures as well as of linear and area measure students need rich opportunities to operate with figures within meaningful contexts. Work has to start in the early grades with concrete materials, it has to be continued with concrete materials and drawings in the middle grades, and should gradually be extended to more symbolic settings in higher grades. It is only in this way that students can understand the properties of area and relationships between figures based on area and exploit them with mental flexibility for solving problems as well as for proving theorems.
2. The approach to the Pythagorean theorem via similarity is conceptually much more difficult for students than the approaches via area preserving dissections and recombinations of figures. Therefore similarity is not appropriate for introducing the Pythagorean theorem. However, it is a good context for taking up the Pythagorean theorem at a more advanced level. Because of the fundamental importance of the Pythagorean theorem the first encounter with this theorem should take place at latest in grade 7 or 8. In each of the subsequent years the students should meet the theorem in ever new contexts and with new proofs.

Exploration 17

The psychological findings on childrens' thinking as illustrated in the present section show that students of a given age range differ considerably in their understanding of basic concepts. The teacher is therefore confronted with the following fundamental problem: How to cope with this wide spectrum of student abilities?

List the major measures that are recommended to teachers in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) for addressing this problem.

4 Designing Teaching Units on the Pythagorean Theorem

The mathematical and psychological analyses and activities in the preceding sections have set the scene for attacking the central problem of this paper: the design of teaching units on the Pythagorean theorem.

From what has been said so far in this paper the following boundary conditions for designing introductory teaching units on the Pythagorean theorem should be clear:

1. Students should be faced with a *problem* that is typical for the use of the Pythagorean theorem and rich enough to derive and explain (prove) the theorem.
2. The conceptual underpinning of the unit should be as firmly rooted in students' basic knowledge as possible.
3. The setting should be as concrete as possible in order to account for different levels in the mastery of basic concepts, to stimulate students' ideas and to facilitate checking.

It is typical for all kinds of design that the designer cannot derive his “products” by means of logical chains of arguments from a scientific basis. Instead he or she has to invent them relying on his or her imagination and using the scientific basis for checks of validity, reliability and effectiveness. Therefore the above boundary conditions do not determine one teaching unit but leave room for the designer's preferences. It is important to keep this in mind and to interpret the following units as suggestions, not as dogmatic prescriptions.

Exploration 18

Before analyzing the following teaching units resume Exploration 6 and do some brainstorming on ideas how to introduce the Pythagorean theorem. Which mathematical or real problem situations do you think appropriate at which school level? What approaches are chosen in textbooks?

Two introductory teaching units on the Pythagorean theorem are offered below. One of them is based on ideas developed in the section on heuristic approaches (pages 15–24), the other one is taken from a Japanese source and puts strong emphasis on technology.

4.1 *Approaching the Pythagorean Theorem via the Diagonal of a Rectangle*

The problem of determining the diagonal in a rectangle with sides a, b ($a \geq b$) seems to be an appropriate context for introducing the Pythagorean theorem in grade 8. Students can explore this problem by first measuring diagonals, then considering the special case $a = b$, and finally trying to transfer the idea from this case to the general case. The solution and the proof depend on dissecting and recombining figures. These area-preserving operations can easily be illustrated by using two puzzles made of cardboard:

1. The “square puzzle” consisting of four semi-squares with side a .
2. The “Bhaskara puzzle” consisting of four right triangles (semi-rectangles) with legs a, b and a square of side $a - b$ (where $a > b$).

The language of puzzles is very powerful and allows for expressing Proof 3 of the Pythagorean theorem in the following way that seems to be a good orientation for

work with students (for other approaches to the Pythagorean theorem using puzzles see Eaves 1953; Spaulding 1974; Engle 1976 and Beamer 1989). We take an arbitrary square piece of cardboard with side a and cut it along one of its diagonals (length c) into two isosceles right triangles (see Fig. 45).

Fig. 45

We repeat this process with a second congruent square piece of cardboard and arrive at four isosceles triangles with legs a and acute angles of 45° (see Fig. 46). These triangles are pairwise congruent as the right angles and the legs are equal. The four pieces can be recombined to make a square with side c as the four right angles form one full angle at the midpoint and adjacent legs of lengths a fit together.

Fig. 46

Fig. 47

All angles at the four vertices are right angles as $45^\circ + 45^\circ = 90^\circ$. Figure 47 is really a square.

Obviously, the area of the square with side c is equal to the sum of the areas of two squares with side a : $c^2 = 2a^2$.

In a similar way we dissect two congruent rectangular pieces of cardboard with sides a and b ($a > b$) into four right triangles with legs a and b (see Fig. 48).

Fig. 48

All four triangles are congruent as they coincide in the right angles and the legs a and b . The sum of the acute angles α and β is 90° ($= 180^\circ - 90^\circ$).

The four triangles can be recombined to make a square with a small square hole. At each corner the triangles fit perfectly as $\alpha + \beta = 90^\circ$ (see Fig. 49).

Fig. 49

The square inside has four right angles and equal sides of length $a - b$. Hence it is a square.

We fill the hole by a square piece of cardboard and recombine the five pieces as in Fig. 50.

Fig. 50

The resulting figure consists of two rectangles with sides a , b and a square with side $a - b$. The dotted line decomposes the figure into two quadrilaterals: As $(a - b) + b = a$, $a - (a - b) = b$ and all angles are right angles, the quadrilaterals are squares with sides a and b . Now the square with side c is composed of the same five pieces as the two squares with sides a and b . Therefore we have proved that $c^2 = a^2 + b^2$.

This description may sound a little clumsy, but it describes a procedure students can perform and comment *orally* quite easily, and this procedure explains *why* the relationship $c^2 = a^2 + b^2$ must be true: the line of arguments is a sound proof in an informal setting centered around the solution of a problem.

It appears as instructive to round out the unit by comparing the measurements in the table with the values calculated by means of the formula. Also the heuristic use of the Pythagorean theorem should be derived from this special context: given the lengths of two sides of a right triangle the length of the third side can be calculated.

As a result we arrive at the following plan for a teaching unit. The plan is presented in a “half-standardized” way directly analogous to the scheme used in conducting clinical interviews (see Sect. 3.3). The unit is divided into “episodes”. At the beginning of each episode the teacher has to take the initiative. His or her crucial interventions (and only these!) are explicitly described. The further moves to be taken depend on students’ ideas and therefore they have to be left open.

Teaching Plan

1. Presenting the guiding problem

Rectangles of different shapes are drawn on the blackboard (Fig. 51).

Fig. 51

The teacher explains the problem of finding the length of the diagonal. As an example making a lath for stabilizing a rectangular frame is mentioned.

It is only natural that the students will also suggest to measure the diagonals. The teacher recommends to draw a variety of rectangles and to measure the diagonals, and fixing the results in a table (Fig. 52).

Fig. 52

At the end of this episode some data are collected in a common table on the blackboard.

2. *Redefining the problem*

The teacher redefines the problem as the typically mathematical problem of finding a formula for computing the diagonal c from the sides a and b . The advantage of a formula should be plausible to students.

Students are stimulated to guess what such a formula could be like. The suggested ideas are written up and tested against the values in the table.

At the end of this episode the students are informed about the steps to follow: Receiving some hints from the teacher they should try to discover and prove the formula as far as possible by themselves.

3. *Specializing the problem: Diagonal of a square*

Material: Congruent paper squares. As a first hint the teacher suggests to study squares as an easier special case.

Each student gets some congruent paper squares and diagonalizes them. The task is to find an arrangement of squares such that a relationship between diagonal c and side a can be deduced.

Figure 53 is almost inevitable and leads to the relationship $c^2 = 2a^2$, from which $c = \sqrt{2} \cdot a$ can be derived.

Fig. 53

The episode is concluded by a guided informal proof of the relationship $c^2 = 2a^2$ based on the transformation in Fig. 54.

Fig. 54

4. *Generalizing the solution: Diagonal of a rectangle*

Material: Congruent paper rectangles. The teacher suggests to adapt the solution from squares to rectangles.

Each student gets two paper rectangles, diagonalizes them and tries to make a square. Students are guided to discover the Bhaskara solution and to give an informal proof of the Pythagorean theorem (Fig. 55).

Fig. 55

$$c^2 = a \cdot \frac{ab}{2} + (a - b)^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

5. *Discussing the formula*

The teacher informs about the history of the Pythagorean theorem and about its importance. Students check the formula by comparing the measured values (episode 1) with the values obtained from the formula.

4.2 Japanese Approach to the Pythagorean Theorem

The Japanese volume *Mathematics Education and Personal Computers* contains a case study on the Pythagorean theorem as an example for improving the traditional format of teaching (Okamori 1989, 155-161). Instead of treating the whole class as one body the class was split up into small groups (four or five students) according to interests, academic abilities and social relationships. The idea was to offer students different approaches to the subject matter that might better serve the individual needs and preferences.

Each group was provided with a microcomputer that had been fed with an interactive software allowing for three different contexts to investigate and prove the Pythagorean theorem:

1. "Geometric-algebraic": The screen shows squares and dissections as presented in Proofs 3 and 3* (see Figs. 10 and 11a).
2. "Euclid dynamized": The screen shows a movie according to Proof 1* (see Fig. 8).
3. "Experimental": The screen shows a right triangle and the squares described on its sides. The medium size square is dissected according to Fig. 73 (see the dissection proof derived from problem 2 of Exploration 3 on page 4).

The following teaching plan shows a structure that is typical for Japanese mathematics education:

- The objectives are clearly defined.
- The steps are precisely described.
- Materials for students are carefully provided.
- At the end of the lesson the teacher summarizes what has been learned.

Teaching plan

1. General information

The class is divided in small groups. Students are told that they are expected to do a geometric investigation by means of the computer. Then they receive some instructions how to use the system and how to interact within the groups. The three contexts for approaching the theme are explained in general terms, and the groups are asked to decide for themselves which context they would like to choose.

2. Introducing the task

When the students start the program three triangles appear on the screen: an obtuse one, a right one and an acute one. The sides of each triangle carry squares: the longest side a square colored red, the smaller sides squares colored green. The students are stimulated to discuss the relationship between the area of the red square and the sum of the areas of the green squares in all three cases. The teacher suggests to draw the squares on graphic paper and to estimate the area. The discussion within the groups and with the whole class should lead to the conjecture of the Pythagorean theorem for right triangles.

3. *Defining the task*

The groups are given the following task: Try to find out from the figures and transformations offered by the computer program why the conjectured relationship must hold. Give a written account of your reasoning. Use the prepared worksheets.

The groups are handed out worksheets that present the essential figures and give some hints for the solution. Groups that have finished their task may switch over to another context.

Context (1): The group has to express the lengths of the relevant segments and the areas of the relevant figures by means of letters and to derive the Pythagorean theorem by means of algebraic formulae. The worksheet for the context is shown in Fig. 56.

Fig. 56

Context (2): The group has to describe and to explain the dynamic version of Euclid's proof (Proof 1*). The worksheet for this context is shown in Fig. 57.

Fig. 57

Context (3): Students are asked to fit the four parts of the medium size square and the small square into the big square and to prove that the five parts fill the big square exactly. The worksheet for the context is shown in Fig. 58.

Fig. 58

4. Consolidation

The results of the groups are corroborated. The teacher asks some questions that test students' understanding.

Reflective Problem 4

Compare the two teaching units. Discuss the advantages and disadvantages of

- elaborating one idea versus presenting a variety of ideas
- introducing the Pythagorean theorem in a problem context versus guiding the students quickly to a conjecture
- using concrete materials versus using dynamic pictures on a screen
- formulating a proof orally versus fixing it on a worksheet

Reflective Problem 5

Tom Apostol made a videotape *The Theorem of Pythagoras* that is available from the NCTM. The three main ideas are:

1. Lengths of corresponding sides of similar triangles have the same ratios.
2. Shearing a triangle does not change its area.
3. If the scale for measuring distances is multiplied by a factor k , the area of a figure is multiplied by k^2 .

If you have access to this videotape: analyse it in terms of the proofs discussed in this chapter. How are the conceptual relationships represented that are essential for sound proofs but cannot be replaced by pictures? In which grade would you use the videotape? How would you use it? As an introduction, as an illustration during learning or as a summary?

5 Reflecting on the Units: Some Key Generalizable Concepts

The outlines of each of the units on the Pythagorean theorem given in the preceding section are restricted to a short description of a sequence of phases. They state what is “on” in each phase, but no information is given about how to interact with the students *within* each phase.

It might be tempting for student teachers to fill these holes with a step-by-step script promising control and a reduction of the uncertainties of teaching. This, however, would run counter to the conditions of effective teaching and learning as described in the introduction. Instead of a *straitjacket* the teacher needs a *concept* for his or her teaching, and this can be provided only by a professional tool-kit consisting of appropriate general principles. By their very nature these principles go beyond individual teaching units. They secure that present learning is rooted in past learning

and oriented towards future learning, and thus they provide teaching and learning with a direction, locally and globally.

The present section will examine three general principles that are rooted in the analyses and explorations before: the notion of informal proof, heuristic strategies, and the “operative principle.”

5.1 Informal Proofs

A proof becomes a proof only after the social act of “accepting it as a proof.” This is as true for mathematics as it is for physics, linguistics or biology. The evolution of commonly accepted criteria for an argument’s being a proof is an almost untouched theme in the history of science.

Yuri. I. Manin

A proof of a theorem is a pattern of conceptual relationships linking the statement to the premises in a logically stringent way. In an earlier section we have met a number of proofs of the Pythagorean theorem that vary in the conceptual relationships employed and—even more important for mathematics education—also in their representations. Some of them consist mainly of a text and use a figure just for supporting the text. Others rely heavily upon figures and transformations and contain only a few explaining lines. The proof aimed at in the first teaching unit even uses pieces of cardboard, real displacements and rearrangements of these pieces, and a comment that may be given only orally.

It is of paramount importance for appreciating new developments in the teaching of proofs to understand that the evaluation of different types of proof has been controversial in mathematics *and* in mathematical education over history, particularly in the twentieth century.

For almost two thousand years Euclid’s “Elements of Mathematics” dominated mathematics and the teaching of it, and the notion of mathematical proof established in this book was the celebrated peak of mathematical activity. In mathematics education, too, it was admired, emulated as far as possible, and hardly ever questioned, apart from a few outsiders (see, for example, Clairaut 1743).

At the end of the nineteenth century the situation changed fundamentally. Mathematicians and a growing minority of mathematics teachers became dissatisfied with the Euclidean standard for quite different reasons and initiated opposing developments. Mathematicians working in the foundations of mathematics discovered that Euclid unexpectedly had used intuitive assumptions in his logical chains of arguments—for example the assumption that any line intersecting a side of a triangle also intersects at least another side—and they set out to establish a purported level of “absolute” rigor that was to reduce reasoning to a manipulation of symbols and statements according to formal rules. No room was left for intuition. Hilbert’s famous book, *Foundations of Geometry*, became the model for the new standard that is perfectly described, for example, in MacLane (1981, 465):

This use of deductive and axiomatic methods focuses attention on an extraordinary accomplishment of fundamental interest: the formulation of an exact notion of *absolute rigor*. Such a notion rests on an explicit formulation of the rules of logic and their consequential and meticulous use in deriving from the axioms at issue all subsequent properties, as strictly formulated in theorems. ... Once the axioms and the rules are fully formulated, everything else is built up from them, without recourse to the outside world, or to intuition, or to experiment ... An absolutely rigorous proof is rarely given explicitly. Most verbal or written mathematical proofs are simply sketches which give enough detail to indicate how a full rigorous proof might be constructed. Such sketches thus serve to convey conviction—either the conviction that the result is correct or the conviction that a rigorous proof could be constructed. Because of the conviction that comes from sketchy proofs, many mathematicians think that mathematics does not need the notion of absolute rigor and that real understanding is not achieved by rigor. Nevertheless, I claim that the notion of absolute rigor is present.

In mathematics education, on the contrary, a growing number of teachers, supported by a few eminent mathematicians like F. Klein and H. Poincaré, recognized the educational inadequacy of formal systems in general and looked for more natural (“genetic”) ways of teaching. Although this movement brought about very nice pieces of “informal” geometry its influence remained quite limited as it failed to develop a global approach to the teaching of geometry comparable in consistency and systematics with the usual programs derived from Euclid. The main difficulty was to conceive a notion of an informal and at the same time sound proof, convincing the mass of teachers.

While up to the 1950s extreme forms of mathematical formalism were mitigated by the pedagogic sensitivity of many teachers who used informal proofs in their teaching, and if only as a didactic concession to their students, the movement of New Maths, influential around the world from the late fifties to the early seventies, sought to introduce mathematical standards of rigor into the classroom without any reduction (see, for example, the excellent analysis in Hanna 1983). This program eventually failed not only because it proved as impracticable, but also, and even more, because mathematical formalism and the idea of “absolute” rigor turned out as mere fictions. Mathematicians became more and more aware that a proof is part of the social interaction of mathematicians, that is of human beings, and therefore not only the discovery but also the check of proofs greatly depend on shared intuitions developed by working in a special field (Davis and Hersh 1983, Chap. 7). The validity of a proof does not depend on a formal presentation within a more-or-less axiomatic-deductive setting, and not on the written form but on the logical coherence of conceptual relationships that are not only to *convince that* the theorem is true, but are to *explain why* it is true. Informal representations of the objects in question are a legitimate means of communication and can greatly contribute to making the proof meaningful.

In a letter submitted to the working group on proof at the 7th International Congress on Mathematical Education, Québec 1992, Yuri Manin, a leading Russian mathematician, described the new view on proof very neatly:

Many working mathematicians feel that their occupation is discovery rather than invention. My mental eye sees something like a landscape; let me call it a “mathscape.” I can place myself at various vantage points and change the scale of my vision; when I start looking into a new domain, I first try a bird’s eye view, then strive to see more details with better clarity.

I try to adjust my perception to guess at a grand design in the chaos of small details; and afterwards plunge again into lovely tiny chaotic bits and pieces.

Any written text is a description of a part of the mathscape, blurred by the combined imperfections of vision and expression. Every period has its own social conventions, and the aesthetics of the mathematical text belong to this domain. The building blocks of a modern paper (ever since Euclid) are basically axioms, definitions, theorems and proofs, plus whatever informal explanations the author can think of.

Axioms, definitions and theorems are spots in a mathscape, local attractions and crossroads. Proofs are the roads themselves, the paths and highways. Every itinerary has its own sight-seeing qualities, which may be more important than the fact that it leads from A to B .

With this metaphor, the perception of the basic goal of a proof, which is purportedly that of establishing “truth,” is shifted. A proof becomes just one of many ways to increase the awareness of a mathscape...

Any chain of argument is a one-dimensional path in a mathscape of infinite dimensions. Sometimes it leads to the discovery of its end-point, but as often as not we have already perceived this end-point, with all the surrounding terrain, and just did not know how to get there.

We are lucky if our route leads us through a fertile land, and if we can lure other travellers to follow us.

The consequences of this new view for mathematics education can hardly be overestimated (Wittmann and Müller 1990, pp. 36–39). While in the past unjustified emphasis was put on the formal setting of proofs mathematics education is now in a position to exploit the rich repertoire of informal representations without distorting the nature of proof.

In this new framework the use of puzzles in proving the Pythagorean theorem as suggested in the first teaching unit is quite natural. However, it is essential for the soundness of the proof that the decomposition of figures into parts and their rearrangement is accompanied by explanations of *why* the figures fit together in different ways and *what* this means for area. It is the task of the teacher to ensure that the necessary questions are asked and answered by the students. For this interaction with the students the teacher needs a clear understanding of what an informal proof is about.

The use of informal proofs is by no means restricted to geometry. In order to enlighten the difference between formal and informal proofs a bit more we consider the famous theorem on the infinity of primes.

Formal Proof:

Let us assume that the set of prime numbers is finite: $p_1, p_2, \dots, p_r$.

The natural number

$$n = p_1 \cdot p_2 \cdot \dots \cdot p_r + 1$$

has a divisor p that is a prime number, that is, n is divisible by one of the numbers $p_1, \dots, p_r$. From $p|n$ and $p|p_1 \cdot \dots \cdot p_r$ we conclude that p also divides the differ-

ence $n = p_1 \cdot \dots \cdot p_r = 1$. However, $p|1$ is a contradiction to the fact that 1 is not divisible by a prime number. Therefore our assumption was wrong.

Informal Proof:

We start from the representation of natural numbers on the numberline and apply the sieve of Eratosthenes (Fig. 59). The number 2 as the first prime number is encircled, and all multiples of 2 are cancelled as they certainly are not prime numbers. The smallest number neither encircled nor cancelled is 3. The number 3 must be a prime as it is no multiple of a smaller prime. Therefore 3 is encircled and again all multiples of 3 are cancelled. For the same reason as before the first number neither encircled nor cancelled, namely 5, is a prime number. Thus 5 is encircled and all multiples of 5 are cancelled. This procedure is iterated and yields a series 2, 3, 5, 7, 11, ... of prime numbers.

Fig. 59

The infinity of prime numbers will be demonstrated if we can explain why the iterative procedure does not stop. Assume that we have arrived at a prime number p . Then p is encircled and all multiples of p are cancelled. The product $n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot \dots \cdot p$ of all prime numbers found so far is a common multiple of all of them. So it was cancelled *at every step*. As no cancellation can hit adjacent numbers the number $n + 1$ has not been cancelled so far. Therefore numbers must be left and the smallest of them is a new prime number.

Comparison of the Two Proofs

First it has to be stated that both proofs are based both are based on similar conceptual relationships. In particular a product of prime numbers increased by 1 plays the crucial role in both proofs. Contrary to the formal proof that works with symbolic descriptions of numbers the informal proof is based on a *visual* representation of numbers on the numberline and on operations on it. In this way the *formal* apparatus can be reduced as some of the necessary conceptual relationships are inbuilt into this representation.

Consequences for Mathematics Teaching

In the past concrete and visual representations of mathematical objects were almost exclusively used for the formation of concepts and for illustrating relationships. Our analyses have shown, however, that appropriate representations are powerful enough to carry sound proofs. This fact opens up to mathematics education a new approach to the teaching of proofs: instead of postponing the activity of proving to higher grades where the students are expected to be mature for some level of formal argument, informal proofs with concrete representations of numbers and geometric figures

can be developed from grade 1. Students can gradually learn to express conceptual relationships more and more formally.

This view on proofs is closely related to Jean Piaget's psychology in which several stages from concrete to formal ones are delineated. Although Piaget's theories have been criticized in many respects the basics of his genetic epistemology are still valid. His emphasis on "operations" as the motor of thinking is of extreme importance for teaching and learning. In Sect. 3 we will investigate the "operative principle".

Although Eratosthenes lived after Euclid it could well be that the sieve was already known to Euclid. As we have seen above that sieve naturally leads to the formal description in the term

$$n = 2 \cdot 3 \cdot 5 \cdot \dots \cdot p + 1.$$

Exploration 19

The sequence in Fig. 60 indicates a transformation of the squares described on the smaller sides of a right triangle into the square described on the hypotenuse is sometimes offered as a "proof without words". The reader is only invited to look at the figure ("Behold!"). Of course, without any explanations the transformation is nothing but an experimental verification. Elaborate an informal proof by describing the transformations, explaining why they are possible and why area does not change. Hint: See proof 1* for comparison.

Fig. 60

Exploration 20

Give two proofs of the formula $1 + 3 + \dots + 2n - 1 = n^2$: an *informal* one based on Fig. 61 and a *formal* one based on mathematical induction.

Fig. 61

5.2 “Specializing”—A Fundamental Heuristic Strategy

When we study the methods for solving problems, we notice a different face of mathematics. Actually, mathematics has two aspects; it is the rigorous science of Euclid, but it is also something else. According to Euclid mathematics appears as a systematic, deductive science; but mathematics in the making appears as experimental and inductive. Both aspects are as old as mathematics itself.

G. Polya

The heart of our teaching unit stimulating and controlling all activities is a mathematical problem: *How long is the diagonal of a rectangle?*

The essential step in solving this problem consists of considering a special case—*How long is the diagonal of a square?*—and in generalizing the special solution. It is important to understand this approach not just as a clever trick in the context of the Pythagorean theorem but as a fundamental heuristic strategy widely used in solving mathematical problems.

We owe to G. Polya (1887–1985), the great master of mathematical discovery in this century, a basic revival of *heuristics*, the study of means and methods of problem solving (Polya 1981). Polya’s work was taken up and extended by mathematics educators (Mason 1982; Brown and I. and Walter, M. I. 1983; Schoenfeld 1985) and is clearly visible in curriculum developments all over the world (see for example, the items “mathematics as problem solving and as reasoning” in the NCTM-Standards).

Heuristic strategies operate on two levels: They serve to generate new problems out of given ones, and they help to construct solutions of problems out of known results. The two levels, however, are inseparably intertwined: The art of problem posing and the art of problem solving are sides of one and the same medal.

Schoenfeld (1985, 76, 80–81) describes and differentiates the strategy “Specializing” (Strategy S) as follows:

To better understand an unfamiliar problem, you may wish to exemplify the problem by various special cases. This may suggest the direction of, or perhaps the plausibility of, a solution ... the description of Strategy S given above is merely a summary description of five closely related strategies, each with its own particular characteristics:

Strategy S_1 : If there is an integer parameter n in a problem statement, it might be appropriate to calculate the *special cases* when $n = 1, 2, 3, 4$ (and maybe a few more). One may see a pattern that suggests an answer, which can be verified by induction. The calculations themselves may suggest the inductive mechanism.

Strategy S_2 : One can gain insight into questions about the roots of complex algebraic expressions by choosing as *special cases* those expressions whose roots are easy to keep track of (e.g., easily factored polynomials with integer roots).

Strategy S_3 : In iterated computations or recursions, substituting the particular values of 0 (unless it causes loss of generality) and/or 1 often allows one to see patterns. Such *special cases* allow one to observe regularities that might otherwise be obscured by a morass of symbols.

Strategy S_4 : When dealing with geometric figures, we should first examine the *special cases* that have minimal complexity. Consider regular polygons, for example; or isosceles or right or equilateral rather than “general” triangles; or semi- or quarter-circles rather than arbitrary sectors, and so forth.

Strategy S_{5a} : For geometric arguments, convenient values for computation can often be chosen without loss of generality (e.g., setting the radius of an arbitrary circle to be 1). Such *special cases* make subsequent computations much easier.

Strategy S_{5b} : Calculating (or when easier, approximating) values over a range of cases may suggest the nature of an extremum, which once thus “determined”, may be justified in any of a variety of ways. *Special cases* of symmetric objects are often prime candidates for examination.

The heuristic pattern related to “specializing” can be described as follows:

For further illustration of Strategy S_4 , which obviously has been applied in our proof of the Pythagorean theorem, let us consider another example from geometry, Viviani’s theorem. This theorem states that the sum of distances of an arbitrary point inside or on the boundary of an equilateral triangle from the three sides has a constant value independent of the position of P (Fig. 62a–c).

Fig. 62

Students asked to measure the distances and to add them for different points will quickly conjecture this fact. As a direct proof is not near at hand a heuristic approach using Strategy S_4 seems natural.

The complexity is least if P is one of the vertices as then two of the three distances are 0 and the third distance is an altitude of the triangle. In an equilateral triangle all altitudes have equal length h (case 1, Fig. 62b).

The next level of complexity (case 2) is provided by points P on one of the three sides as in this case one distance is 0 (Fig. 62c).

Our goal is to show that the two other distances add up to h .

If P is the midpoint of the side the two distances are equal by way of symmetry (Fig. 63).

Fig. 63

The reflection at line AB maps PD onto PD' . The 30° -angles around P ensure that D' is on line PE . Because of the right angles at E and D resp. D' lines BD' and AC are parallel and $D'E = 2h_1$ is the distance between them. But this distance is also h . Therefore $2h_1 = h$.

This line of arguments holds also if P is an arbitrary point on a side (Fig. 64).

Fig. 64

The general case (P inside of ABC) can be reduced to case 2 (Fig. 65): $A'B'$ is the line through P parallel to AB .

Fig. 65

As the angles at A' and B' are equal to the angles at A and B , triangle $A'B'C$ is equilateral and, according to case 2, $h_1 + h_2 = h' =$ altitude of $A'B'C$. Obviously h' and h_3 add up to h , the altitude of ABC . Therefore $h_1 + h_2 + h_3 = h$. (For an alternative approach following the “What if not ...?” strategy, see Jones/Shaw 1988.)

Our example shows that “specialization” at the level of problems and “generalization” at the level of solutions can be performed in steps: the solution of the problem for an extremely special case is step-by-step transferred to less special cases up to the general case.

The interaction between “specialization” and “generalization” is often used to generate new knowledge in the following way: One tries to generalize a problem that has been solved (possibly in different ways). If a reasonable generalization has been found one attempts to generalize the solution(s).

Pattern of generalization:

?	← generalization	Problem
?	← generalization	Solution(s)

This heuristic strategy is particularly fruitful for the Pythagorean theorem and leads to the discovery of two important generalizations:

Law of cosines:

The sides a, b, c and the angles α, β, γ of an arbitrary triangle are related by the following formulae:

$$a^2 + b^2 = c^2 + 2ab \cos \gamma$$

$$b^2 + c^2 = a^2 + 2bc \cos \alpha$$

$$c^2 + a^2 = b^2 + 2ca \cos \beta$$

General Pythagorean theorem:

If similar figures are described on the sides of a right triangle then the sum of the areas of the two smaller figures is equal to the area of the third figure.

For an excellent heuristic analysis of these generalizations the reader is referred to *The Art of Problem Posing* by Stephen Brown and Marion Walter (1983, 44–61, 112–116).

Exploration 21

The midpoints of the sides of an arbitrary quadrilateral are the vertices of a new quadrilateral. What do you conjecture about the shape of this midpoint-quadrilateral? How long are its sides?

Prove your conjecture by applying the strategy “specialization”.

Exploration 22

Generalize Problem 2 of Exploration 3.

5.3 The Operative Principle

It would be a great mistake, particularly in mathematics education, to neglect the role of operations and always to remain on the level of language. ... The initial role of operations and logico-mathematical experience, far from hindering the later development of deductive thought, constitutes a necessary preparation.

J. Piaget

In discovering the Pythagorean theorem and in establishing a proof as envisaged in the teaching unit students have to “play around” with figures: Squares and rectangles are dissected, the pieces are arranged in various ways, a hole is filled, and so forth.

The teacher of mathematics must be aware that this activity offers by no means just an ad hoc approach to the Pythagorean theorem but that it reflects the natural functioning of our cognitive system. According to the constructivist view of learning, knowledge is neither received from environmental sources (that is from structures considered as inherent in reality or structures offered by the teacher) nor unfolds simply from inside. Knowledge is *constructed* by the individual through interacting

with the environment: the individual operates upon the environment and tries both to assimilate the environment to his or her mental structures and to accommodate the latter to the external requirements.

Let us illustrate this goal-directed “playing around” by means of some examples.

Episode 1: During a christmas party a 1.5-year-old is sitting on the legs of his father at a table with candlelights. He gazes at a candle burning in front of him, but out of his reach. Suddenly a child on the other side of the table bends over the table and blows the candle out. The boy observes the event carefully and notes how somebody else lights the candle again. Now it is he who wants to blow the candle out: he hisses—the candle is still burning, he reinforces his hissing sound, again without success, he growls, he moves his body, first towards the candle, then aside, he hits the table with his hands and moves them around and so forth. All cognitive schemas available to him are tested, however, without success. After 15 minutes the boy loses his interest.

Episode 2: Two twelve-year-olds play the following game of strategy (Fig. 66).

Fig. 66

One of the players has red counters, the other blue ones. They take turns to fill the row from 1 to 10 successively with counters. Each player may add one or two counters of his colour. The player first arriving at 10 is the winner.

First the students play more or less randomly. Then they discover that 7 is a favourable position: the player arriving at 7 can also arrive at 10: If the opponent adds 1 counter, then 2 counters lead to 10. If the opponent adds 2 counters, then 1 counter is sufficient to cover 10. By trying out different moves and by evaluating them the students discover that 4 and 1 are also favourable positions, and that the player starting the game has a winning strategy.

Episode 3: A student teacher tries to solve the following geometric problem by means of The Geometer’s Sketchpad or Geogebra: Given lines g , h and circle k construct a square $ABCD$ such that A lies on g , B and D on h , and C on k (Fig. 67). First she draws g , h and k . Then she chooses A on g as a moving point. She recognizes that the choice of A determines B and D on h as the foot F of the perpendicular l dropped from A to h must be the midpoint of the square. The student constructs points B and C as images of A under rotations with center F and angles 90° and -90° . Next she recognizes that A is mapped to C by means of a rotation with center F and angle 180° . But C does not lie on circle k . In order to fulfill this requirement the student moves A along g , back and forth. B , C and D move correspondingly and it is easy for her to maneuver C on k .

Fig. 67

Actually, there are two solutions in this case. When performing the movement a second time the student suddenly observes that A and C move symmetrically with respect to h (Fig. 68). This leads her to the following solution of the problem: Line g is reflected on h into g' . The intersections of g' and circle k are possible positions for vertex C . Dropping the perpendicular from C to h and intersecting it with g gives the corresponding vertex A . B and D can be constructed as above.

Fig. 68

Each of the three episodes illustrates an important aspect of Piaget's view: The searching individual acts upon objects and observes the effects of his or her actions (episode 1). Known effects are used for anticipating paths to certain goals (episode 3). Knowledge is not a ready-made matter, but it is constructed by the individual through interaction with reality (episode 2).

This "operative" approach ranges from everyday situations to more and more abstract and complex mathematical situations, from concrete objects to symbolically represented objects, and thus it is essential for the whole mathematical curriculum.

For illustration, again a few examples.

Example 1 (*Primary level: Addition and Subtraction*) Problem: The sum of two numbers is 32, the difference is 8. Which are the numbers?

To solve this problem the numbers are represented by counters of different colours (Fig. 69).

Fig. 69

Here 16 red and 16 blue counters make 32, but the difference is 0. Replacing a blue counter by a red one leads to $17 + 15 = 32$, $17 - 15 = 2$. Repeating this operation two more times gives

$$19 + 13 = 32, \quad 19 - 13 = 6, \quad 20 + 12 = 32, \quad 20 - 12 = 8$$

Example 2 (*Secondary level: Symmetric figures*)

Fig. 70

A rectangular piece of paper (Fig. 70) is folded along a line of symmetry and cut along the dotted lines. The shaded triangle is unfolded and leads to a special quadrilateral, a kite.

Questions:

Which properties are imprinted into the kite by this generating process?

Which forms can a kite have?

How to cut in order to make all sides equally long?

Can a square be generated in this way?

To answer these questions students will have to fold, cut, check, vary the attempts, check again until they arrive at the answers.

Example 3 (*Secondary level: Quadratic functions*) The graph of a quadratic function is typically derived from the standard parabola, the graph of the function $y = x^2$, by means of four basic *geometric* transformations that model *algebraic* transformations of the functions:

<i>Algebraic transformation</i>		<i>Geometric transformation</i>
$y = x^2$	into $y = ax^2$	Affine dilatation of the standard parabola with factor a along the y -axis
$y = ax^2$	into $y = -ax^2$	Reflection at x -axis
$y = ax^2$	into $y = a(x - c)^2$	Translation by c along the x -axis
$y = a(x - c)^2$	into $y = a(x - c)^2 + d$	Translation by d along the y -axis

Example 4 (*(College): Derivative*) The software program Supergraph developed by David Tall allows—among other interesting things—for representing graphs of functions on the screen and for pursuing the tangent on its way along the graph. The computer also fixes the slope of the tangent step by step (derivative). By observing this “movie” for various functions the student can find out how basic properties of a function are reflected in the derivative (domains of increase and decrease, maxima, minima, etc.).

The common kernel of these four examples has been termed the *operative principle* and described as follows (Wittmann 1987, 9):

To understand *objects* means to explore how they are *constructed* and how they *behave* if they are subjected to operations (transformations, actions, ...).

Therefore students must be stimulated in a systematic way

- (1) to explore which *operations* can be performed and how they are related with one another,
- (2) to find out which *properties* and relationships are imprinted into the objects through construction,
- (3) to observe which effects properties and relationships are brought about by the operations according to the guiding question “What happens with ..., if ...?”

In this formulation the nature of the “objects” has deliberately been left open. Therefore the operative principle has a wide range of applications.

It is not by chance that examples 3 and 4 employ the computer. In fact the computer, if properly used, is the ideal device for making the operative principle practical.

Through the lens of the operative principle the concept of area appears in the following operative setting:

The “objects” in question are geometric figures. These figures can be changed by a great variety of “operations,” for example, reflections, translations, rotations, dilatations, shearing motions, decompositions, extensions, reductions... The standard questions are: *What happens* with the area of a figure *if* the figure is reflected, translated, decomposed, extended ...?

Answers:

Area behaves *invariant* under rigid motions,
additive under decompositions,
monotone under extensions,
quadratic under dilatations and
invariant under shearing motions.

In other words:

Congruent figures have the *same* area.

The area of a composite figure is the *sum* of the areas of its parts.

If a figure F_1 is contained in figure F_2 , the area of F_1 is *not bigger than* that of F_2 .

If figure F is mapped on to F' by means of a dilatation with factor k , then $\text{Area}(F') = k^2 \cdot \text{Area}(F)$.

Shearing motions do *not change* the area.

In retrospect the reader will see that the tasks for studying the development of the concept of area involved exactly the above operations. The reader is perhaps surprised that the clear emphasis on psychology at the beginning of this section has given way to quite mathematical considerations. However, this change of perspective has not happened by chance. In Piaget’s view cognitive *psychology*, that is, the study of the growth of knowledge in individuals, is strongly related to *epistemology*, that is, the study of growth and structure of scientific knowledge.

The “operative” view at cognition, learning and teaching is also strongly related to the notion of proof. In Sect. 1 it was stated that a proof is a logical chain of conceptual relationships. Now we can put it a little more precisely: In a proof objects are presented and introduced that are *constructed* in characteristic ways, and these objects are subjected to certain *operations* such that known effects arise. It is from these constructions and operations that the essential conceptual relationships flow on which the proof is based.

For illustration let us consider Proof 2 of the Pythagorean theorem. The proof starts with constructing an appropriate figure (Fig. 71). Then certain parts of the figure are analyzed whereby at some places operations appear.

Fig. 71

*Objects*Triangle ABC :Segment AF :Segment DK :Triangles ABC , GAF , GKH , KBD :Quadrilateral $AGKB$:Hexagon $BCFGHD$ *Relationships imposed on the objects by construction or by operation*

$$\alpha + \beta = 90^\circ$$

$$AF = a$$

$$DK = b$$

all congruent (triangle ABC can be laid upon the others)

square

consisting of squares $BCED$ and $EFGH$ and square $AGKB$

The square and the hexagon are equidecomposable, and therefore of equal area (three parts covering the hexagon can be rearranged to cover the square)

Exploration 23

What are “objects,” “operations” and “effects” in

1. the Bhaskaran puzzle proof of the Pythagorean theorem (see p. 133, Fig. 50),
2. Clairaut’s approach (see Figures 15 to 19 and Exploration 11),
3. the three episodes and the four examples of the present section?

6 Appendix: Solutions to the Problems in Exploration 3

Problem 1 This is a typical example for using the Pythagorean theorem. First, the diagonal d of the rectangular base $ABCD$ is calculated, $d^2 = a^2 + b^2$. Then the Pythagorean theorem is applied once more: the triangle ACP with sides c , d and s is also right. Therefore

$$s^2 = d^2 + c^2 = a^2 + b^2 + c^2, \text{ or } s = \sqrt{a^2 + b^2 + c^2}.$$

Problem 2 The problem seems to call for the Pythagorean theorem, and in fact it is possible to solve it by calculating the side of the shaded square in several steps by means of the Pythagorean theorem and similarity arguments. However, the Pythagorean theorem is not necessary. The figure can be embedded into a square lattice (why?). By comparing the parts of the resulting dissection (see Fig. 72) one sees that the original square is five times the area of the shaded square.

Fig. 72

The new figure, however, is close to figures related to the Pythagorean theorem, for example to the figures in Proof 3 and in Exploration 8. If we combine the parts of the original square in an appropriate way (see Fig. 73), we touch the idea of a new dissection proof of the Pythagorean theorem: by starting from the midpoints of its sides a big square can be dissected into a small square and four congruent rectangles. The latter can be recombined to make a square whose sides are twice the length of the small square (see Fig. 74).

It is important to see the Fig. 71 not as static but as a dynamic network of elements that are connected by relationships.

Fig. 73

Fig. 74

The transition from Figs. 72, 72 and 74 can also be made with arbitrary squares. All one has to do is to decompose the sides of the larger square into two segments whose difference is the side of the smaller square. Check it and you have the idea of a new proof of the Pythagorean theorem! Explain Fig. 75, which is used by the Japanese teaching unit mentioned earlier (worksheet (3)).

Fig. 75

Problem 3 Assume that the car is 4.60 m long and that the distances to the adjacent cars are 0.30 m each. Then the “length” available for the car is $4.60\text{ m} + 2 \times 0.30\text{ m} = 5.20\text{ m}$. In order to move the car out of the lot without too much trouble the diagonal d of the car should be a little bit smaller than the available length 5.20 m (see Fig. 76).

Fig. 76

Application of the Pythagorean theorem leads to

$$d = \sqrt{(4.60)^2 + (1.70)^2} \text{ m} \approx 4.90 \text{ m}.$$

4.90 m is 30cm smaller than the available length. So it is possible to move the car out.

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