Chapter 6

Mathematics Education as a ‘Design Science’

Mathematics education (didactics of mathematics) cannot grow without close relationships to mathematics, psychology, pedagogy and other areas. However, there is the risk that by adopting standards, methods and research contexts from other well-established disciplines, the applied nature of mathematics education may be undermined. In order to preserve the specific status and the relative autonomy of mathematics education, the suggestion to conceive of mathematics education as a “design science” is made. In a paper presented to the twenty second Annual Meeting of German mathematics educators in 1988 Heinrich Bauersfeld presented some views on the perspectives and prospects of mathematics education. It was his intention to stimulate a critical reflection ‘among the members of the community’ on what they do and what they could and should do in the future (Bauersfeld 1988). The early seventies have witnessed a vivid programmatic discussion on the role and nature of mathematics education in the German speaking part of Europe (cf., the papers by Bigalke, Griesel, Wittmann, Freudenthal, Otte, Dress and Tietz in the special issue 74/3 of the *Zentralblatt für Didaktik der Mathematik* as well as Krygowska 1972). Since then the status of mathematics education has not been considered on a larger scale despite the contributions by Bigalke (1985) and Winter (1986). So the time is overdue for redefining the basic orientation for research; therefore, Bauersfeld’s talk could hardly have been more appropriate. In recent years the interest in a better understanding of the nature and role of mathematics education has also grown considerably at the international level as indicated, for example, by the ICMI-study on ‘What is research in mathematics education and what are its results?’ launched in 1992 (cf., Balacheff et al. 1992). The following considerations are intended both as a critical analysis of the present situation and an attempt to capture the specificity of mathematics education. Like Bauersfeld, the author presents them ‘in full subjectivity and in a concise way’ as a kind of ‘thinking aloud about our profession’. (The present paper concentrates on the didactics of mathematics although the line of

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argument pertains equally to the didactics of other subjects and also to education in general (cf., Clifford and Guthrie 1988, a detailed study on the identity crisis of the Schools of Education at the leading American universities.)

1 The 'Core' and the 'Related Areas' of Mathematics Education

The sciences should influence the outside world only by an enlightened practice; basically they all are esoteric and can become exoteric only by improving some practice. Any other participation leads to nowhere.

J.W. v. Goethe, *Maximen und Reflexionen*

Generally speaking, it is the task of mathematics education to investigate and to develop the teaching of mathematics at all levels including its premises, goals and societal environment. Like the didactics of other subjects mathematics education requires the crossing of boundaries between disciplines and depends on results and methods of considerably diverse fields, including mathematics, general didactics, pedagogy, sociology, psychology, history of science and others. Scientific knowledge about the teaching of mathematics, however, cannot be gained by simply combining results from these fields; rather it presupposes a *specific* didactic approach that *integrates* different aspects into a coherent and comprehensive picture of mathematics teaching and learning and then transposing it to practical use in a constructive way.

The specificity of this task necessitates, on the one hand, sound relationships to the disciplines related to mathematics education, and on the other hand, a balance between practical proximity and theoretical distance with respect to schools. Bauersfeld (1988, p. 15) refers here to the 'two cultures' of mathematics education. How we can integrate the variety of aspects, and at the same time, set weights and deal with the tensions that exist between theory and practice is not at all clear a priori. This is why it is so difficult to arrive at a generally shared conception of mathematics education.

In my view, the specific tasks of mathematics education can only be carried out if research and development have specific linkages with practice at their *core* and if the improvement of practice is merged with the progress of the field as a whole.

This *core* consists of a variety of components, including in particular:

- analysis of mathematical activity and of mathematical ways of thinking,
- development of local theories (for example, on mathematizing, problem solving, proof and practising skills),
- exploration of possible contents that focus on making them accessible to learners,
- critical examination and justification of contents in view of the general goals of mathematics teaching,
- research into the pre-requisites of learning and into the teaching/learning processes,

- development and evaluation of substantial teaching units, classes of teaching units and curricula,
- development of methods for planning, teaching, observing and analysing lessons, and
- inclusion of the history of mathematics education.

Work in the core necessitates the researcher's interest and proximity to practical problems. A caveat is in order, however. The orientation of the core towards practice may easily lead to a narrow pragmatism that focuses on immediate applicability and may therefore become counterproductive. This hazard can only be avoided by connecting the core to a variety of related areas that bring about an exchange of ideas with related disciplines and that allow for investigating the different roots of the core in a systematic way (cf., Fig. 1). Of course, the core and the related areas overlap, and the ill-defined borders between them change over time. Thus, a strict separation is not possible.

Fig. 1 The core and the areas related to mathematics education, their links to the related disciplines and the fields of application

Although the related areas are indispensable for the whole entity to function in an optimal way, the specificity of mathematics education rests on the core, and therefore the core must be the central component. Actually, progress in the core is the crucial element by which to measure the improvement of the whole field. This situation is comparable to music, engineering and medicine. For example, the composition and performance of music must take precedence over the history, critique and theory of music; in mechanical engineering the construction and development of machines is paramount to mechanics, thermodynamics and research of new materials; and in medicine the cure of patients is of central importance when compared to medical sociology, history of medicine or cellular research.

However, the division between the core and the related areas does not imply that the core is restricted to practical applications since the related areas have to develop the necessary theory. In fact, building theories or theoretical frameworks related to the design and empirical investigation of teaching is an essential component of work in the core (cf., Freudenthal 1987).

As in engineering, medicine and art, the different status of the core and the related areas is also clearly indicated in mathematics education by the following facts:

1. The core is aimed at an *interdisciplinary, integrative view of different aspects and at constructive developments* whereby the ingenuity of mathematics educators is of crucial importance. The related areas are derived much more from the corresponding disciplines. Therefore research and development in didactics in general get their *specific* orientation from the requirements of the core. Theoretical studies in the related areas become significant only insofar as they are linked to the core and thus receive a specific meaning. In particular, the research problems listed in Bauersfeld (1988, pp. 16–18) can be tackled in a sufficiently concrete and productive way only from the core.
2. Teacher education oriented towards practice must be based on the core. The related areas are indispensable for more deeply understanding practical proposals and their applications in an appropriate way. However, in teacher education too, the related areas realize their full impact only if they are linked to the core.

The central position of the core is mainly an expression of the *applied status* of mathematics education. Emphasizing the core does not diminish the importance of the related areas, nor does it separate them from the core. As clearly indicated in Fig. 1, it is the core, the related areas and a lively interaction between them that represent the full picture of mathematics education and that also necessitate the common responsibility of mathematics educators independent of their special fields of interest.

Work in the core must start from mathematical activity as an original and natural element of human cognition. Further, it must conceive of “mathematics” as a broad societal phenomenon whose diversity of uses and modes of expression is only in part reflected by specialized mathematics as typically found in university departments of mathematics. I suggest a use of capital letters to describe MATHEMATICS as mathematical work in the broadest sense; this includes mathematics developed and used in science, engineering, economics, computer science, statistics, industry, commerce, craft, art, daily life, and so forth according to the customs and requirements specific to these contexts. Specialized mathematics is certainly an essential element of MATHEMATICS, and the broader interpretation cannot prosper without the work done by these specialists. However, the converse is equally true: Specialized mathematics owes a great deal of its ideas and dynamics to broader scientific and societal sources. By no means can it claim a monopoly for “mathematics”.

It should go without saying that MATHEMATICS, not specialized mathematics, forms the appropriate field of reference for mathematics education. In particular, the design of teaching units, coherent sets of teaching units and curricula has to be rooted in MATHEMATICS.

As a consequence, mathematics educators need a lively interaction with MATHEMATICS and they must devote an essential part of their professional life to stimulating, observing and analyzing genuine MATHEMATICAL activities of children, students and student teachers. Organizing and observing the fascinating encounter of human being with MATHEMATICS is the very heart of didactic expertise and forms a natural context for professional exchange with teachers.

As a part of MATHEMATICS, specialized mathematics must be taken seriously by mathematics educators as one point of view that, however, has to be balanced with other points of view. The history of mathematics education clearly demonstrates the risks of following specialized mathematics too closely: On the one hand, subject matter and elements of mathematical language can be selected that do not make much sense outside specialized mathematics—perhaps a lasting example of this mistake is the New Maths movement. On the other hand, the educationally important fields of MATHEMATICS that are no longer alive in specialized research and teaching may lose the proper attention—perhaps the best example for this second mistake is elementary geometry.

Mathematics educators must be aware that school mathematics cannot be derived from specialized mathematics by a “transposition didactique du savoir savant au savoir enseigné” (cf., Freudenthal 1986). Instead, they must see school mathematics as an extension of pre-mathematical human capabilities which develop within the broader societal context provided by MATHEMATICS (cf., Schweiger 1994: p. 299 and Dörfler 1994, as well as the concept of “ethno-mathematics” in D’Ambrosio 1986). It is only from this perspective that the unity of mathematics teaching from the primary through the upper secondary level can be established and that reasonable mathematical courses in teacher training can be developed which deserve to be called a scientific background of teaching.¹

2 A Basic Problem in the Present Development of Mathematics Education: The Neglect of the Core

The ‘hard sciences’ are successful, as they deal with ‘soft problems’. The ‘soft sciences’ are badly off, as they are confronted with ‘hard problems’.

Heinz v. Foerster

An approach to the study of problems of learning and teaching in mathematics education requires a scientific framework that includes both research methods and standards. As a young discipline, mathematics education is under considerable pressure

¹I do not intend to give mathematical specialists advice not asked for. However, in my opinion it would also be beneficial for them to perceive themselves as partners in a larger mathematical system described by MATHEMATICS. Without some change of awareness on their part, all attempts to change the public image of “mathematics” are nothing but cosmetics and bound to fail.

from different directions. How to establish standards is as controversial as the status of mathematics education itself and can likewise be addressed in different ways.

One tempting approach is to adapt methods and standards from the hard sciences and the humanities. I dare say that all around the world quite a number of mathematics educators are taking this approach wherein the scientific background and their personal interests might be as influential as the wish to be recognized and supported by scientists in the related disciplines. However, approaches, methods and standards adopted from related disciplines are more easily applied to problems in the neighborhood of these disciplines than to problems in the core. Consequently, a great deal of didactic research adheres to mathematics, psychology, pedagogy, sociology, history of mathematics and so forth. Thus the holistic origin of didactic thinking, namely mathematical activity in social contexts, is dissolved into single strands, and the specific tasks of the core are neglected. In my view this is a big problem that presently inhibits major progress in mathematics education. The problem is by no means restricted to mathematics education, however. For example, Clifford and Guthrie (1988: p. 3) have identified it as a universal problem in education:

Our thesis is that schools of education, particularly those located on the campuses of prestigious research universities have become ensnared improvidently in the academic and political cultures of their institutions and have neglected their own worlds. They have seldom succeeded in satisfying the scholarly norms of their campus letters and science colleagues, and they are simultaneously estranged from their professional peers. The more they have rowed toward the shores of scholarly research the more distant they have become from the public schools they are bound to serve.

The movement away from the core and towards the related areas may also be problematic because very often the adoption of frameworks and standards from related disciplines is linked to the dogmatic claim that these frameworks and standards were the only ones possible for didactics. From this position follows a blindness towards the central tasks of mathematics education and a systematic underestimation of the *constructive achievements* brought about in the core. Sometimes the core is even denied a scientific status. Mathematics educators who retreat into a “mathematical garden” (H. Meschkowski) tend of course to trivialize the educational aspects of mathematics education; similarly, those working in the areas related to psychology and pedagogy neglect the mathematical aspects. These tendencies are reinforced by voices from the related disciplines that argue against the scientific status of didactics more or less publicly. As a result we have an unreasonable set-back into reductionist positions analyzed as unfounded many years ago (cf., Bigalke 1985; Winter 1985). It is ironic that mathematics education set out in the late sixties to overcome exactly these polarized positions. What is urgently needed therefore is a methodological framework that does justice to the core of mathematics education.

3 Mathematics Education as a Systemic-Evolutionary ‘Design Science’

It is the yardstick that creates the phenomena... A religious phenomenon can only be revealed as such if it is captured in its own modality, i.e., if it is considered by means of a religious yardstick. To locate such a phenomenon by means of physiology, psychology, sociology, economics, linguistics, art, etc. means to deny it. It means to miss exactly its uniqueness and its irreducibility.

Mircea Eliade, *The Religions and the Sacred*

Establishing² scientific standards in mathematics education by adopting standards from related disciplines is, as mentioned, unwise because problems and tasks of mathematics education tend to be tackled only insofar and to the extent that they are accessible to the methods of the related disciplines. As a consequence, the core is not sufficiently recognized as a scientific field in its own right.

Fortunately there is a silver lining in this dilemma if one abandons the fixation on the traditional structures of the scientific disciplines and instead looks at the specific character of the core, namely the *constructive* development of and research into mathematics teaching. Here mathematics education is assigned to the larger class of “design sciences” (cf., Wittmann 1974) whose scientific status was clearly delineated from the scientific status of natural sciences by the Nobel Prize Winner Herb Simon. The following quotation from Simon (1970, pp. 55–58) explains also the resistance offered to the design sciences in academia. In this way the present situation of mathematics education is embedded into a wider context and becomes accessible to a rational evaluation.

Historically and traditionally, it has been the task of the science disciplines to teach about natural things: how they are and how they work. It has been the task of engineering schools to teach about artificial things: how to make artifacts that have desired properties and how to design ...

Design, so construed, is the core of all professional training; it is the principal mark that distinguishes the professions from the sciences. Schools of engineering, as well as schools of architecture, business, education, law and medicine, are all centrally concerned with the process of design.

In view of the key role of design in professional activity, it is ironic that in this century the natural sciences have almost driven the sciences of the artificial from professional school curricula. Engineering schools have become schools of biological science; business schools have become schools of finite mathematics ...

²The term “design” and related terms used subsequently in this paper might cause irritation, for in traditional understanding these terms are linked to mechanistic procedures of making tools and controlling systems (cf., Jackson 1968: 163 ff.). In part 3 of this paper we will show, however, that in striking contrast to the “mechanistic” paradigm of design and management there is a new “systemic-evolutionary” paradigm based on the appreciation of the complexity and self-organisation of living systems. It is in the context of this new paradigm that the term “design” and similar ones are used in the present paper.

The movement toward natural science and away from the sciences of the artificial has proceeded further and faster in engineering, business and medicine than in the other professional fields I have mentioned, though it has by no means been absent from schools of law, journalism and library science ...

Such a universal phenomenon must have a basic cause. It does have a very obvious one. As professional schools ...are more and more absorbed into the general culture of the university, they hanker after academic respectability. In terms of the prevailing norms, academic respectability calls for subject matter that is intellectually tough, analytic, formalizable and teachable. In the past, much, if not most, of what we knew about design and about the artificial sciences was intellectually soft, intuitive, informal and cookbooky. Why would anyone in a university stoop to teach or learn about designing machines or planning market strategies when he could concern himself with solid-state physics? The answer has been clear: he usually wouldn't ...

The older kind of professional school did not know how to educate for professional design at an intellectual level appropriate to a university; the new kind of school has nearly abdicated responsibility for training in the core professional skills ...

The professional schools will reassume their professional responsibilities just to the degree that they can discover a science of design, a body of intellectually tough, analytic, partly formalizable, partly empirical, teachable doctrine about the design process.

It is the thesis of this chapter that such a science of design not only is possible but is actually emerging at the present time.³

In the writer's opinion the framework of a design science opens up to mathematics education a promising perspective for fulfilling its tasks and also for developing an unbroken self-concept of mathematics educators. This framework supports the position described in part 2, for the core of mathematics education concentrates on constructing "*artificial objects*", namely teaching units, sets of coherent teaching units and curricula as well as the investigation of their possible effects in different educational "*ecologies*". *Indeed the quality of these constructions depends on the theory-based constructive fantasy, the "ingenium", of the designers, and on systematic evaluation, both typical for design sciences.* How well this conception of mathematics education as a design science reflects the professional tasks of teachers

³The underestimation of the "skills of designing and making" is deeply rooted in our culture. Cf. A. Smith, A coherent set of decisions, the Stanley Lecture, Manchester Polytechnic 1980: p. 22:

Throughout the whole of our society we show little respect for the skills of designing and making. Indeed in many of our schools these very skills are looked down upon and are referred to as the noddy subjects, fit only for the less able in our community.

I remember, during my years as chairman of the Schools Council, visiting a school where, after I had been shown the fairly conventional range of school work, I was taken into the workshops and there on the bench was a most beautiful and competent piece of metal work. It was a joy to look at, but it was described to me as a piece of work "by one of our less able pupils". It was an extraordinary description, which spoke volumes about our distorted scale of values. There was a piece of work which expressed ability, as fine in its way as the best essay written by the highest flyer in English, but never seen by academic people as such. To write things with pen on paper is an up-marked, respectable activity; to conceive pattern in your mind and to make them with your hands is a down-marked activity, less worthy of respect.

is shown, for example, by Clark and Yinger (1987, pp. 97–99) who have identified teaching as a “design profession”.

The clear structural delineation of mathematics education as a *design science* from the related sciences underlines its specific character and its relative independence. Mathematics education is not an appendix to mathematics, nor to psychology, nor to pedagogy for the same reason that any other design science is not an appendix to any of its related disciplines. Attempts to organize mathematics education by using related disciplines as models miss the point *because they overlook the overriding importance of creative design for conceptual and practical innovations*.

As far as research frameworks and standards are concerned, mathematics educators working in the core should primarily start from the achievements in the core already available. There is no doubt that during the past 25 years significant progress, that includes the creation of theoretical frameworks, has been made within the core and that standards have been set which are well-suited as an orientation for the future. “Developmental research” as suggested by Freudenthal and elaborated by Dutch mathematics educators is a typical example (cf., Freudenthal 1991, pp. 160–161; and Gravemeijer 1994). Of course, it is reasonable also to adopt methods and standards from the related disciplines to the extent that they are appropriate to the problems of the core.

It is no surprise that there objections to the view of mathematics education as a “design science” emerge, for the simple reason that the design sciences have traditionally followed—and are still widely following—a mechanistic paradigm whose harmful side effects are becoming more and more visible. This approach would certainly be detrimental to education. However, we are presently witness to the rise of a new paradigm for the design sciences that is based on the “systemic-evolutionary” development of living systems and takes the complexity and self-organization of these systems into account (cf., Malik 1986). Even if researchers in the design sciences in general hesitate to adopt this new paradigm, there is no reason why mathematics educators should not follow it, even more so since this paradigm corresponds to recent developments in the field. The systemic-evolutionary view on the teacher-student and the theorist-practitioner relationships differs greatly from the traditional view. Knowledge is no longer seen as the result of a transmission from the teacher to a passive student, but is conceived of as the productive achievement of the student who learns in social interaction with other students and the teacher. Therefore the materials developed by mathematics educators must be construed so as to acknowledge and allow for this interactive approach. In particular, they must provide teachers and students the freedom to make choices of their own. In order to facilitate and stimulate a flexible use of the materials designed in this way, teachers have to be trained and regarded as partners in research and development and not as mere recipients of results (cf. Schupp 1979; Schwab 1983; Fischer and Malle 1983, and the papers by Brown/Cooney, Seeger/Steinbring, Voigt, and others in *Zentralblatt für Didaktik der Mathematik* (4/91 and 5/91)). As a consequence, teacher training receives a new quality. An important orientation for innovations along these lines is the approach developed by Schön (1987) for the training of engineers that is based upon the idea of the “reflective practitioner”.

As a systemic-evolutionary design science mathematics education can follow different paths. It is certainly not reasonable to develop it into a "monoparadigmatic" form as postulated, for example, for the natural sciences. In a design science the simultaneous appearance of different approaches is a sign of progress and not of retardation as stated by Thommen (1983, p. 227) for management theory:

Because of a continuously changing economic world it is possible to (re-)construct an economic context within different formal frameworks, or models. These need not be mutually exclusive, on the contrary, they can even be complementary, for no model can take all problems and aspects into account as well as consider and weigh them equally. The more models exist, the more problems and aspects are studied, the greater is the chance for mutual correction. Therefore we consider the variety of models in management theory as an indicator for an advanced development of this field moving on in an evolutionary, not a revolutionary process in which new models emerge and old ones disappear.

4 The Design of Teaching Units and Empirical Research

That, in concrete operation, education is an art, either a mechanical art or a fine art, is unquestionable. If there were an opposition between science and art, I should be compelled to side with those who assert that education is an art. But there is no opposition, although there is a distinction.

John Dewey, *On the sources of a science of education*

For developing mathematics education as a design science it is crucial to find ways how design on the one hand and empirical research on the other can be related to one another. In the following the writer proposes a specific approach to empirical research, namely empirical research centered around teaching units.

It cannot be denied that teaching units, and on a wider scale curricula, have found attention in mathematics education in the past. In fact, curriculum development held a prominent place in the late sixties and early seventies. Nevertheless, the writer contends that the design of teaching units has never been a focus of research. At best teaching units have been used as more or less incidental examples in investigating and presenting theoretical ideas. Many of the best units were published in teachers' journals, not in research journals, and were hardly noticed by the research community. For this phenomenon the following explanation is offered: In contrast to "research", the design of teaching has been considered as a mediocre task normally done by teachers and textbook authors. To rephrase Herb Simon: Why should anyone anxious for academic respectability stoop to designing teaching and put him- or herself on one level with teachers? The answer has been clear: He or she usually wouldn't.

In order to overcome this fundamentally incorrect view we have to recognize that in all fields of design there is—by the very nature of design—a wide spectrum of competence and experience ranging from the amateur, to the novice, the less or more skilled worker, the experienced master, up to the creative inventor. Typically, the bulk of design on a larger scale is done in special centers for research and development.

As a design science mathematics education can be no exception from this rule. That teachers take part in design can be no excuse for mathematics educators to refrain from this task. On the contrary: The design of substantial teaching units, and particularly of substantial curricula, is a most difficult task that must be carried out by the experts in the field. By no means can it be left to teachers, although teachers can certainly make important contributions within the framework of design provided by experts, particularly when they are members of or in close connection with a research team. Also, the adaptation of teaching units to the conditions of a special classroom requires design on a minor scale. Nevertheless, a teacher can be compared more to a conductor than to a composer or perhaps better to a director (“*metteur en scène*”) than to a writer of a play. For this reason there should exist strong reservations about “teachers’ centers” wherein teachers meet to make their own curriculum.

We should be anxious to delineate teaching units of the highest quality from the mass of units developed at various levels for various purposes. These “substantial” teaching units can be characterized by the following properties:

1. They represent central objectives, contents and principles of mathematics teaching.
2. They provide rich sources for mathematical activities.
3. They are flexible and can easily be adapted to the conditions of a special classroom.
4. They involve mathematical, psychological and pedagogical aspects of teaching and learning in a holistic way, and therefore they offer a wide potential for empirical research.

Typically, a substantial teaching unit always carries a name. As examples I mention “Arithmogons” by Alistair McIntosh and Douglas Quadling, “Mirror cards” by Marion Walter, “Giant Egbert” and other units developed in the Dutch Wiskobas project, and Gerd Walther’s unit “Number of hours in a year” (Walther 1984, 72–78). Other examples and a systematic discussion of the role of substantial teaching units in mathematics education are given by Wittmann (1984).

For the sake of clarity, one example of a substantial teaching unit is sketched below. In our primary project Mathe 2000 the following setting of arithmogons is used in grade 1:

A triangle is divided in three fields by connecting its midpoint to the midpoints of its sides. We put counters or write numbers in the fields. The simple rule is as follows: Add the numbers in two adjacent fields and write the sum in the box of the corresponding side (cf. Fig. 2).

Fig. 2 Modified representation of arithmogons

Various problems arise: When starting from the numbers inside, the numbers outside can be obtained by addition. When one or two numbers inside and respectively two or one number outside are given, the missing numbers can be calculated by addition and subtraction. When the three numbers outside are given, we have a problem that does not allow for direct calculation but requires some thinking. It turns out that there is always exactly one solution. However, it may be necessary to use fractions or negative numbers.

The mathematics behind arithmogons is quite advanced: The three numbers inside form a vector as well as the three numbers outside. The rule of adding numbers in adjacent fields defines a linear mapping from the three-dimensional vectorspace over the reals into itself. The corresponding matrix is non-singular. One can generalize the structure to n -gons as shown in McIntosh and Quadling (1975).

The teaching unit based upon arithmogons consists of a sequence of tasks and problems that arise naturally from the mathematical context. The script for the teacher may be structured as follows:

1. Introduce the rule by means of examples and make sure that the rule is clearly understood.
2. Present some examples in which the numbers inside are given.
3. Present some examples in which some numbers inside and some numbers outside are given.
4. Present a problem in which the numbers outside are given.
5. Present other problems of this kind.

As can be seen, a substantial teaching unit is essentially open. Only the key problems are fixed. During each episode the teacher has to follow the students' ideas in trying to solve the problems. This role of the teacher is completely different from traditional views of teaching. Teaching a substantial unit is basically analogous to conducting a clinical interview during which only the key questions are defined and the interviewer's task is to follow the child's thinking.

The structural similarity between substantial teaching units on the one hand, and clinical interviews on the other, suggests an adaptation of Piaget's method for studying children's cognitive development to empirical research on teaching units (cf., Fig. 3). As a result we arrive at "clinical teaching experiments" in which teaching units can be used not only as research tools, but also as objects of study.

	Tools	Method
Piagetian Psychology	Structured sets of tasks	Clinical interviews
Mathematics Education	Teaching units	Clinical teaching experiments

Fig. 3 Comparison of clinical interviews with teaching experiments

The data collected in these experiments have multiple uses: They tell us something about the teaching/learning processes, individual and social outcomes of learning, children's productive thinking, and children's difficulties. They also help us to evaluate the unit and to revise it in order to make teaching and learning more efficient.

The Piagetian experiments were repeated many times by other researchers. Many became a focus of extended psychological research. Some even established special lines of study; for example, the "conservation" experiments. It is no exaggeration to say that Piaget's experiments and the patterns he observed in children's thinking survived much longer than his theories, in many cases until the present. In the same way, clinical teaching experiments can be repeated and thereby varied. By comparing the data we can identify basic patterns of teaching and learning and derive well-founded specific knowledge on teaching certain units. Much can be learned here from Japanese research in mathematics education (cf., Becker and Miwa 1989).

In conducting such studies, existing methods of qualitative research can effectively be used, particularly those developed by French mathematics educators in connection with "didactical situations" and with "didactic engineering" (cf. Brousseau 1986; Artigue and Perrin-Glorian 1991; Arzac et al. 1992). Concerning the reproducibility of results it is very instructive to look at the social sciences. Friedrich von Hayek, another Nobel prize winner in economics, has convincingly pointed out that empirical research on highly complex social phenomena yields reproducible results if directed towards revealing general patterns beyond special data (von Hayek 1956). To admit that the results of teaching and learning depend on the students and on the teacher does not preclude the existence of patterns related to the mathematical content of a specific teaching unit (cf., also, Kilpatrick 1993, pp. 27–29 and Sierpinska 1993, pp. 69–71). Of course, we must not expect all these patterns to arise on any occasion nor under all circumstances. It is quite natural that patterns will occur, varying with the educational ecologies. One should be reminded here of the well-known fact that

Piagetian interviews also reveal recurring content-specific patterns which, however, do not occur with every individual child.

Research centered around teaching units is useful for several reasons. First, it is related to the subject matter of teaching (cf., the postulate of “relatedness” in Kilpatrick 1993, p. 30). Second, knowledge obtained from clinical teaching experiments is “local”. Here we need to be more careful in generalizing over contents than we have been in the past. In the future we can certainly expect to derive theories covering a wide range of teaching and learning. But these theories cannot emerge before a variety of individual teaching units has been investigated in detail. For studying the mathematical theory of groups the English mathematician Graham Higman stated in the fifties “that progress in group theory depends primarily on an intimate knowledge of a large number of special groups”. The striking results achieved in the eighties in the classification of finite simple groups showed that he was right. In a similar way, the detailed empirical study of a large number of substantial teaching units could prove equally helpful for mathematics education.

Third, theory related to teaching experiments is meaningful and applicable. We should, however, be aware that, due to the inherent complexity of teaching and learning, the data and theories that research might provide may never provide complete information for teaching a certain unit. Only the teacher is in a position to determine the special conditions in his or her classroom. Therefore there should be no sharp separation between the researcher and the teacher as stated earlier. As a consequence, teachers have to be equipped with some basic competence in doing research on a small scale. The writer’s experience in teacher training indicates that introducing student teachers into the method of clinical interviews is an excellent way towards that end (Wittmann 1985).

In the writer’s opinion, the most important results of research in mathematics education are sets of carefully designed and empirically studied teaching units that are based on fundamental theoretical principles. It follows that these units should form a major part of the professional training of teachers. Teachers who leave the university should have in their baggage a set of substantial teaching units that represent the standards of teaching. From the experiences with our primary project Mathe 2000 it is clear that such units are the most efficient carriers of innovation and are well-suited to bridge the gap between theory and practice.

5 And the Future of Mathematics Education?

The frogs tend to forget that once they were tadpoles, too.
Korean Proverb

Generally speaking, it may be taken for granted that dealing in an intelligent way with complex systems on a scientific basis will become inevitable in all parts of human life. Very often the methods offered by the specialized disciplines are not sufficient. Riedel (1988) recently pleaded for a more context-related, more practical and less-formal “second philosophy”, in contrast to the traditional “first philosophy” that aims

at complete descriptions and deductions and that is bound to fail when applied to complex systems, because of its “ideology of self-restriction” (Fischer 1980). This seems to be a signal for a critical reflection in all sciences from which mathematics education as a systemic-evolutionary design science can take profit in the long range, since society will have to accept the fact that the development of human resources is at least as important for economic prosperity as is the development of new technologies and new marketing strategies.

In the short run the status of didactics in the universities will remain arduous. The resistance from the specialized sections within the related disciplines to establishing didactics in teacher training programs at *all* levels and to funding research in didactics is likely to continue. The history of the universities shows many instances in which scholars of established disciplines displayed their ignorance and acted in an unfair way towards newly evolving disciplines. The resistance of the old universities towards the technical schools at the end of the 19th century, the resistance of pure mathematicians towards applied ones at the beginning of this century and the vote of the German Philosophical Society against the establishment of chairs of pedagogy at the universities in the fifties are only a few examples. Obviously it is difficult, if not impossible, for specialists to understand and to appreciate new developments on the very borderline of their discipline.

In order to strengthen their position at the universities and to acquire funds from research foundations, mathematics educators need support from society. In this respect the relationships of mathematics education to the schools play a fundamental role. The use and the indispensability of didactic research for improving practice have to be convincingly demonstrated to teachers, supervisors, administrators, parents and the public. This can only be achieved from the core, that is, by concentrating on central tasks and by organizing design, empirical research and teacher education accordingly.

At the same line, there is potential in establishing a network of “Public—School—School Administration—Teachers’ Unions—Teacher Training—Design, Research, Development” people in which the core of mathematics education will naturally find its proper place. In other words, organizing a systemic effort involving all the constituent groups.

This is consistent with the advice given by Clifford and Guthrie to schools of education in general (cf., Clifford and Guthrie 1988: pp. 349–350):

The major mission of schools of education should be the enhancement of education through the preparation of educators, the study of the educative process, and the study of schooling as a social institution. As John Best has observed, the challenge before schools of education is quite different from that confronting the specialist in politics in a department of political science; concerned with building the discipline, he or she is under no obligation to train country clerks, city managers, and state legislators, and to improve their performance by conducting research directed toward that end. In order to accomplish *their* charter, however, schools of education must take the profession of education, not academia, as their main point of reference. It is not sufficient to say that the greatest strength of schools of education is that they are the only places available to look at fundamental issues from a variety of disciplinary perspectives. They have been doing so for more than half a century without appreciable effect on professional practice. It is time for many institutions to shift their gears.

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