Chapter 5

When Is a Proof a Proof?

In his famous talk at ICME 2 (Exeter 1972) the French mathematician R. Thom pointed out that any conception of mathematics teaching necessarily rests on a certain view of mathematics (Thom 1973, 204). As a consequence mathematics education cannot develop without close links to mathematics. However, “mathematics” must not be identified with the “official” picture of mathematics represented by lectures, mathematical journals and textbooks. What is needed is a fundamental and comprehensive view of mathematics as a cultural phenomenon including historic, sociological, philosophic and psychological aspects. Only this broader perspective permits us to recognize the “real” picture of mathematics and to use it for mathematics education.

In teacher training this extended perspective is necessary not only to provide prospective teachers in didactical courses with a sound meta-knowledge about mathematics but also with a productive relationship to school mathematics. During the past decade the investigations on the professional life of teachers have convincingly shown that school mathematics is not just a derivative of university mathematics but a relatively independent field, as it contains a variety of aspects which cannot be reduced to bare mathematical forms (Otte and Keitel 1979; Steinbring 1985, 1988; Dörfler and McLone 1986). As far as teacher training is concerned this new appreciation of school mathematics has influenced didactical and practical studies in pre-service and inservice training (Seeger and Steinbring 1986).

We believe, however, that in addition the mathematical training of student teachers has to be modified by including studies in elementary mathematics which emphasize meaning, process and informal means of representation, thus enabling student teachers to experience educational values of mathematics. Our position has developed through the past ten years while we have been strongly involved in reforming our teacher training programs. An example of what we have in mind is provided by a recent textbook on elementary geometry (Wittmann 1987).

The present paper aims at elaborating a central point of this kind of elementary mathematics, namely the notion of a sound informal proof. In German we use the term “inhaltlich-anschaulicher Beweis” in order to indicate that this method of demonstration calls upon the meaning of the terms employed, as distinct from abstract

methods, which dispense with the interpretation of the terms and employ only the abstract relations between them.

During the seventies and eighties “proof” has been the subject of extended research in mathematics education (cf. the carefully collected list of references in Stein 1981, as well as Winter 1983a and Stein 1985). A completely new line of research has been opened up by Gila Hanna’s book “Rigorous Proof in Mathematics Education” (Hanna 1983). This book was the first comprehensive attempt to transfer to mathematics education new tendencies in the philosophy of mathematics which originated in 1963 in Imre Lakatos’ famous thesis “Proofs and Refutations” (Lakatos 1979).

The present paper is intended as a further contribution to the demystification of formalism in mathematics education. The first section shows by means of case studies that formalistic views on proof are still widely spread among mathematics teachers and student teachers. The second section gives some examples which show that in mathematical research informal and social aspects are highly important not only for finding but also for checking proofs. For overcoming these formalistic views we offer two strategies. One of them consists of referring to papers of leading mathematicians who give an authentic account of what their work is about. The other strategy is to elaborate informal mathematics as an independent level of mathematical thought. The third section of this paper explains in more detail what is intended by this “elementary mathematics research program of mathematics education”.

1 Proofs and “Proofs”

The formalistic conception of mathematics, established during the first half of this century, defines mathematics as the science of “rigorous proof”, i.e. the purely logical derivation of concepts from basic concepts and of theorems from axioms. As an illustration we refer to Pickert (1957, 49):

Fortunately research into the foundations of mathematics – typical for this century – has developed a notion of mathematical proof which is independent of any imagination. I will start from this notion, explain it by means of a few examples and I will try to show by means of further examples what instruments are available in order to handle proofs in a more effective way, i.e. instruments which facilitate communication, retrieval and the discovery of proofs. It is the totality of these instruments which I would like to call “imagination” (Anschauung). In this way imagination is restricted to a certain domain: we use it as a guide, but we must not trust it. The validity of proofs depends only on what is left when imagination is completely removed. In my view this position is justified for the following reasons: first I do not see how generally accepted decisions on the validity of a proof could be made otherwise - i.e. by using imagination. Second I believe that this position reflects the view of contemporary mathematicians.

The following quotation from MacLane (1981, 465) is even more pointed:

This use of deductive and axiomatic methods focuses attention on an extraordinary accomplishment of fundamental interest: the formulation of an exact notion of absolute rigor. Such a notion rests on an explicit formulation of the rules of logic and their consequential and

meticulous use in deriving from the axioms at issue all subsequent properties, as strictly formulated in theorems. Moreover, each derivation can be tested and understood in its own terms, independent of any reference to examples of the activity or the reality for which the axioms were designed ... This formal character of mathematics may serve to distinguish it from all other types of science. Once the axioms and the rules are fully formulated, everything else is built up from them, without recourse to the outside world, or to Intuition, or to experiment. An absolutely rigorous proof is rarely given explicitly. Most verbal or written mathematical proofs are simply sketches which give enough detail to indicate how a full rigorous proof might be constructed. Such sketches thus serve to convey conviction – either the conviction that the result is correct or the conviction that a rigorous proof could be constructed. Because of the conviction that comes from sketchy proofs, many mathematicians think that mathematics does not need the notion of absolute rigor and that real understanding is not achieved by rigor.

Nevertheless, I claim that the notion of absolute rigor is present.

While MacLane like Pickert refer to intuition at least as a means for handling and evaluating proofs, the logician Rosser (1953, 7) takes an extreme position:

Thus, a person with simple arithmetical skills can check the proofs of the most difficult mathematical demonstrations, provided that the proofs are first expressed in symbolic logic. This is due to the fact that, in symbolic logic, demonstrations depend only on the forms of statements, and not at all on their meanings.

This does not mean that it is now any easier to discover a proof for a difficult theorem. This still requires the same high order of mathematical talent as before. However, once the proof is discovered, and stated in symbolic logic, it can be checked by a moron.

What role formalism has played in relationship with or in opposition to other philosophic positions and how it has developed into the “official” philosophy of mathematics is explained by Davis and Hersh (1983, Chap. 7). In addition these authors describe how during the past ten years quite different views have gained ground in which the possibility of absolutely rigorous and eternal proofs has been denied and in which proof is considered as a social process among mathematicians:

A proof only becomes a proof after the social act of “accepting it as a proof”. This is as true for mathematics as it is for physics, linguistics, or biology. The evolution of commonly accepted criteria for an argument’s being a proof is an almost untouched theme in the history of science. (Manin 1977, 48).

Going one step further leads to the possibility of different criteria for checking and evaluating proofs within different social contexts. An important example, applied mathematics, is analyzed by Blechman, Myschkis and Panovko (1984).

We will now show by means of four case studies, ranging from primary mathematics teaching to teacher training, how mathematical understanding can be inhibited by formalistic views on proof. Our experiences with student teachers are concentrated on the primary and the lower secondary level. However, our contacts with student teachers for the upper secondary level in didactical courses have convinced us that formalism is represented even more strongly in this group than in the two other groups—as one would expect.

Example 1 (*Chinese Remainder Theorem*) This theorem was part of a course in number theory for student primary teachers. Some students felt overwhelmed and

protested at the inclusion of this—in their view useless—topic into primary teacher programs. In defence the students were told that the Chinese Remainder Theorem could suggest some interesting work with 9-year old children. In order to substantiate this claim it was agreed to pose the following problem to third graders:

Find numbers which leave the remainder 1 when divided by 2, and the remainder 2 when divided by 3.

Of course this problem was not introduced in this compact form, but was explained by checking small numbers step by step. In particular 5 was stated as the smallest number with the required properties:

$$5 = 2 \cdot 2 + 1, \quad 5 = 1 \cdot 3 + 2.$$

Afterwards the children started their own search. Children who had found a series of solutions were stimulated to find all solutions.

The spectrum of achievements was considerable. Some students still had problems with the calculations, others paid attention only to the remainder 2. The best performance was by Henning, according to the teacher the “mathematician” of the class. It is represented here in facsimile (Fig. 1).

Obviously Henning had checked number by number and had found the solutions 11, 17, 23. Then he recognized a pattern which he summarised by the statement:

Fig. 1 Student solution

“Because $2 \cdot 3 = 6$ and $3 \cdot 2 = 6$.” When asked to explain this subtle argument Henning said something like this:

If I consider the remainder 1, I have to proceed in steps of 2 and I meet the odd numbers. If I consider the remainder 2, I have to proceed in steps of 3. The steps coincide only after 3 two-steps and 2 three-steps.

In our view this argument is a sound informal demonstration of the solution. The student teachers, however, were not willing to accept it as a proof. In their view a “real” proof had to be based on formal transformations of a system of congruences—something going obviously beyond the capabilities of primary children. Therefore they continued to reject the Chinese Remainder Theorem as a topic appropriate for primary teacher training.

Example 2 (*Irrationality of $\sqrt{2}$*) Pickert (1987, 212) presents the following proof of the irrationality of $\sqrt{2}$ by a 13-year-old student:

Let $a, b \in \mathbb{N}^*$ be relatively prime such that $(a/b)^2 = 2$. Then $a^2 = 2b^2$ and therefore b^2 is a common divisor for a^2 and b^2 . As a and b are relatively prime, so are a^2 and b^2 . As a consequence $b^2 = 1$ and so $a^2 = 2$, which is impossible for $a \in \mathbb{N}^*$.

According to Pickert this argument is a “proof”, because the student uses tacitly the inference

$$a, b \text{ relatively prime} \implies a^2, b^2 \text{ relatively prime},$$

which does not hold in rings in general. In our view this position is too formalistic. The inference used by the student is a direct consequence of the unique factorisation of natural numbers into prime numbers. The latter is well known to students in grade 7 because it has been treated in grade 5 and is used all the time for cancelling fractions. The student is right to use “socially shared” knowledge implicitly. The above mentioned inference becomes crucial only within ring theory—a context completely irrelevant at school.

Example 3 (*Euler’s polyhedron theorem*) In a lecture “Geometry in 3-space” for primary students, Euler’s theorem was proved for convex polyhedra in the following informal way:

First the concept of a Schlegel diagram was explained and as an illustration, the Schlegel diagrams for some polyhedra were produced by means of rubber sheets. Then the relationship $v + f - e = 2$ was proved by showing that an arbitrary Schlegel diagram can be reconstructed by starting with one point ($v = 1, f = 1, e = 0$) and adding edge by edge such that $v + f - e$ does not change (cf. Wittmann 1987, 270ff.) Right after the demonstration a student asked “Was that really a proof?” The teacher, somewhat irritated by this unexpected question, asked back: “Why not?”, and received a very instructive answer: “Because I understood it!”

A conversation later on showed clearly that the student had difficulties with the formalistic teaching received at school and had consistently arrived at the conclusion that mathematical proofs were not accessible to her.

In our view this student is not a single case but represents a great number of students.

Example 4 (*Trapezoid numbers*) Experiences like that in example 3 have stimulated us to investigate the mathematical “world view” of our students more systematically. The method we have found most useful is to confront the students with informal and formal proofs and to ask them to evaluate the validity of each type.

In this way primary student teachers were introduced to the old Greek “arithmetic of dot patterns” (cf. Becker 1954, 34 ff.). For example starting with square numbers (Fig. 2) and the triangular numbers (Fig. 3) trapezoid numbers were defined as composition of square and triangular numbers (Fig. 4).

Fig. 2 Square numbers

Fig. 3 Triangular numbers

Fig. 4 Trapezoid numbers

By “playing” with patterns the students guessed that for all n the trapezoid number T_n and n leave the same remainder when divided by 3:

$$T_n \equiv n \pmod{3}.$$

The teacher offered the following “iconic” proof (Fig. 5):

Starting from the right the pattern T_n is decomposed into columns. Obviously each 3-column is a multiple of 3. If n itself is a multiple of 3 (case 1) then T_n splits

completely into 3-columns and is also a multiple of 3. If n leaves the remainder 1 (case 2) then T_n splits into 3-columns and a single column with n points at the left side and it is again obvious that T_n leaves the remainder 1, too.

Fig. 5 Decomposition of trapezoid numbers modulo 3

Finally, if n leaves the remainder 2 (case 3), T_n breaks up into 3-columns and two columns with n and $(n + 1)$ points at the left side, and it is easy to see that T_n like n leaves the remainder 2.

Right after this demonstration some students expressed their doubt on the validity of it. The teacher didn't intervene and the group agreed very quickly that the demonstration could achieve only the status of an illustration, and not the status of a proof.

The teacher then offered the following “symbolic” proof:

Case 1: Let $n = 3k$. Then $T_n = (3n^2 - n)/2 = n(3n - 1)/2 = 3k(9k - 1)/2$, and as k or $9k - 1$ are even, T_n (like n) is divisible by 3.

Case 2: Let $n = 3k + 1$. Then $T_n = (3(9k^2 + 6k + 1) - (3k + 1))/2 = (27k^2 + 15k + 2)/2 = (27k^2 + 15k)/2 + 1 = 3k(9k + 5)/2 + 1$,
i.e. T_n (like n) leaves the remainder 1.

Case 3: Let $n = 3k + 2$. Then $T_n = 3(9k^2 + 12k + 4) - (3k + 2))/2 = (27k^2 + 33k + 10)/2 = 3(9k^2 + 11k + 2)/2 + 2$,
i.e. T_n (like n) leaves the remainder 2.

The confrontation of the iconic and the symbolic argument aroused a lively discussion on the validity of each, in which the teacher defended the iconic argument as a sound proof. The students were then asked to give written comparisons of the two forms of proof. The papers showed very clearly how strongly the teaching received at school had predisposed the students towards formal proofs and how difficult it was for them to accept an “iconic” proof. As an illustration we quote from some papers:

The iconic proof is much more intuitive for me and it explains to me much better what the problem is. For me dot patterns are convincing and sufficient as a proof. Unfortunately we have not been made familiar with this type of proof at school ... Only symbolic proofs have been taught.

For me the iconic proof is easier and better to understand than the symbolic proof. The reason is that it is much more intuitive than the symbolic one. I have tried to follow the symbolic proof by verification. However, the calculation is somewhat abstract, and I cannot link anything concrete to it.

The symbolic proof is to be preferred, because it is more mathematical.

The iconic proof is highly intuitive. It shows very clearly that T_n is divisible by 3 if n is divisible by 3, and vice versa. It is true that the iconic proof does not allow to substitute arbitrarily large numbers for n and to prove the statement for them because so many dots cannot be drawn; but for smaller numbers the representation by means of 3-columns is very useful for the understanding.

The iconic proof is very intuitive. One understands the connection from which the statement flows. I can't imagine how a counterexample could be found, because it does not matter how many 3-columns are constructed. In my opinion this is nevertheless no proof, but only a demonstration which, however, holds for all n . At school I have learnt that only a symbolic proof is a proof. Therefore I trust such proofs more. As symbolic proofs are more or less just "calculating to and fro" one easily loses sight of what is to be proved. The confrontation of the two types of proof seems very instructive to me.

The iconic proof is much easier to understand and more intuitive than the symbolic one. At school there were mainly symbolic proofs. Iconic proofs were only means for finding symbolic proofs. I still have this feeling.

I prefer the symbolic proof, as school has confronted me only with this type of proof. These proofs guarantee generality. The iconic proof is more intuitive, which is surely an advantage for primary teaching. I for myself see the iconic proof more as an illustration and concretisation of the symbolic proof.

The symbolic proof is more mathematical. This proof is more demanding, as some formulae are involved which you have to know and to retrieve. The iconic proof can be followed step by step, and each is immediately clear. However, I wonder if an iconic proof would be accepted in examinations.

Personally I like the iconic proof as it is more intuitive. You can see at once what's going on whereas the symbolic proof forces you to think in an abstract way. You know the formula and you develop through abstract thinking (?) the proof, but you don't have a direct reference to numbers. I prefer the symbolic proof as I have been confronted with such proofs at school and as it is more mathematical.

I am familiar with the symbolic proof. Therefore it is easier for me to handle.

I prefer the iconic proof because of its intuitive character. To me the symbolic proof is too abstract. Possibly I could have discovered the iconic proof for myself. Nevertheless I always try to find a symbolic proof, presumably because of my former mathematics teaching.

Influenced by the mathematics teaching at school and at the university I would prefer the symbolic proof. However, the iconic proof is much more convincing, as it is less abstract and easier to verify. Up to now iconic proofs were unknown to me.

Normally I trust symbolic proofs more, as they use general "numbers" (variables), i.e. they cover any number in any case. However, in this example I trust the iconic proof, too. It is more intuitive and the idea of the proof is clear and obvious.

To me the iconic proof is mathematical enough – I do not mistrust it!! One important point for me is that children at the primary level can understand things better through intuition.

Judging by the objectives of my study as a prospective primary teacher I cannot see very much reason for symbolic proofs – in particular when they become even “more mathematical”. I miss the practical impact.

What these statements show is reinforced by our experiences in teacher training: the vast majority of student teachers holds a definite formalistic view on mathematical proof (cf. also Aner et al. 1979). Obviously this fact is only a reflex of the “official” picture of mathematics prevailing at school and in teacher training for a very long time. Branford (1913, 328) has recognized this problem very clearly at the beginning of this century:

I think it a fact that the vast majority of teachers is firmly convinced that mathematics does not differ so much from other sciences by the measure of rigour but by the absolute rigour of mathematical proofs in contrast to the approximate rigour of other proofs.

And Branford continues:

The disaster caused by this belief at all levels of mathematics teaching is, I think, terrible.

The negative consequences of a formalistic understanding of proof by teachers can be quite different:

Teachers who see themselves as pedagogues, pragmatists and teachers with a negative attitude towards mathematics refrain from introducing their students to proof because they think “mathematical” proofs are too difficult for their children. Instead they offer pictures, plausible arguments, verifications, examples and rules related to certain types of tasks. Lenné (1969, 51) called this didactical position the “didactics of tasks” (Aufgabendidaktik). On the other side teachers who care for “mathematical rigour” try to bring their teaching to the level of formal definitions and proofs, even if in practice they do not get very far. Systematic attempts in this direction were made by the “New Maths” movement in the sixties and seventies. Here elements of formal university mathematics were more or less directly “mapped down” to the level of teaching. This approach has therefore rightly been called the “didactics of mapping down” (Andelfinger and Voigt 1986, 3).

Contrary to the “didactics of tasks” and the “didactics of mapping down” there is a third branch of didactics, centered around the genetic principle, in which the concept of informal proof has been developed. Branford, one of the most important representatives of genetic didactics distinguishes three types of proof (Branford 1913, 100 ff., 239 ff.):

- experimental “proofs” (typical of the “didactics of tasks”)
- intuitive proofs
- scientific proofs (typical of university mathematics and the “didactics of mapping down”)

Branford thinks that the middle type of proof, intuitive proof, is indispensable for the development of mathematical understanding and characterises it in the following way:

This level of proof establishes general and rigorously valid truths. However it refers, if necessary, to postulates of sensual perception. It puts truth on a basis of its own by immediate recall to first principles. It does not represent truth as a mere link in a systematic chain of arguments where the effective strength of the connection is weakened by the number of previously stated truths forming the links of the chain (103) ...

Opposite to experimental “proof” we find the two other types of proof, the scientific and the intuitive proof, the latter being a more preliminary and less rigorous kind of the ideal scientific proof; in reality there is no sharp border between these two types, they differ only in the degree of logical rigour. The truths derived by each of the two types are generally valid as far as we can judge. Otherwise sensual perception would show us exceptions (108 f.)

With perfect clarity Branford points out here that the border between “proof” and proof does not lie between “experimental” and “intuitive” proof on the one side and purely logical proofs on the other side but is to be drawn between experimental proofs on the one side and the two other types on the other side. Later on this position was elaborated in more detail, in particular by H. Freudenthal’s contributions (Freudenthal 1963, 1973, Chap. 8, 1979). The decisive distinction between experimental “proofs” and intuitive, informal proofs has been clarified by didactical research on operative proofs and on “pre-mathematics” (Semadeni 1974; Kirsch 1979; Winter 1983b; Walther 1984): Experimental “proofs” consist of the verification of a finite number of examples guaranteeing of course no generality. Informal operative proofs are based on constructions and operations which by intuition are seen as applicable to a whole class of examples and as leading to certain consequences. For example the decomposition of a trapezoid point pattern into 3-columns is a universal operation which generates insight into the remainder. The dot pattern is not a picture here but a symbol (cf. Jahnke 1984).

In spite of its advanced development the genetic position has not received much attention as yet in mathematics teaching at any level. The main reason might be that informal explanations of concepts and informal proofs seem inhomogeneous, unsystematic, shaky and invalid when considered from the point of view of formalism. Many teachers, textbook authors and teacher trainers refrain from representations which might be interpreted as a sign of mathematical incompetency. A change in this unfavourable situation can be expected only to the extent that formalism is overcome as the “official” philosophy of mathematics and that comprehensive informal conceptions of elementary theories of mathematics are developed. These two points are considered in the following sections.

2 Formalism as a Fiction: The Indispensability of Intuition and Social Agreement in Checking Proofs

As already mentioned in the introduction of this paper at present the philosophy of mathematics is undergoing a dramatic change arising from a growing awareness of working mathematicians that formalism is in contradiction to their experiences and

that the ideal of an “absolutely rigorous” proof can no longer be maintained (Davis and Hersh 1983, Chap. 7; Hanna 1983).

We would like to illustrate the new views on proof by quoting from papers of leading mathematicians.

Hardy (1929, 18 f.):

I have myself always thought of a mathematician as in the first instance an *observer*, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees *A* sharply, while of *B* he can obtain only transitory glimpses. At last he makes out a ridge which leads from *A*, and following it to its end he discovers that it culminates in *B*. *B* is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he *points to it*, either directly or through the chain of summits which led him to recognise it himself. When his pupil also sees it, the research, the argument, the *proof* is finished.

The analogy is a rough one, but I am sure it is not altogether misleading. If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point*; that proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. This is plainly not the whole truth, but there is a good deal in it. The image gives us a genuine approximation to the processes of mathematical pedagogy on the one hand and of mathematical discovery on the other; it is only the very unsophisticated outsider who imagines that mathematicians make discoveries by turning the handle of some miraculous machine ...

On the other hand it is not disputed that mathematics is full of proofs, of undeniable interest and importance, whose purpose is not in the least to secure conviction. Our interest in these proofs depends on their formal and aesthetic properties ... Here we are interested in the pattern of proof *only*. In our practice as mathematicians, of course, we cannot distinguish so sharply, and our proofs are neither the one thing nor the other, but a more or less rational compromise between the two. Our object is *both* to exhibit the pattern and to obtain assent. We cannot exhibit the pattern completely, since it is far too elaborate; and we cannot be content with mere assent from a hearer blind to its beauty.

Wilder (1944, 319):

In conclusion, then, I wish to repeat my belief that what we call “proof” in mathematics is nothing but a testing of the products of our intuition. Obviously we don’t possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think.

Thom (1973, 202 ff.):

The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of “meaning”, of the “existence” of mathematical objects.

This leads me to deal with the old war-horse of the modernists (of the Continental European variety): rigour and axiomatics. One knows that any hope of giving mathematics a rigorously formal basis was irreparably shattered by Gödel’s theorem. However, it does not seem as

if mathematicians suffer greatly in their professional activities from this. Why? Because in practice, a mathematician's thought is never a formalised one. The mathematician gives a meaning to every proposition, one which allows him to forget the formal statement of this proposition within any existing formalised theory (the meaning confers on the proposition an ontological status independent of all formalisation). One can, I believe, affirm in all sincerity, that the only formal processes in mathematics are those of numerical and algebraic computation. Now can one reduce mathematics to calculation? Certainly not, for even in a situation which is entirely concerned with calculation, the step of the *calculation* must be chosen from a very large number of possibilities. And one's choice is guided only by the intuitive interpretation of the quantities involved. Thus the emphasis placed by modernists on axiomatics is not only a pedagogical aberration (which is obvious enough) but also a truly mathematical one.

One has not, I believe, extracted from Hilbert's axiomatics the true lesson to be found there; it is this: one accedes to absolute rigour only by eliminating meaning; ... But if one must choose between rigour and meaning, I shall unhesitatingly choose the latter. It is this choice one has always made in mathematics, where one works almost always in a semi-formalised situation, with a metalanguage which is ordinary speech, not formalised. And the whole profession is happy with this bastard situation and does not ask for anything better. ...

A proof of a theorem (T) is like a path which, setting out from propositions derived from the common stern (and thus intelligible to all), leads by successive steps to a psychological state of affairs in which (T) appears obvious. The rigour of the proof – in the usual, not the formalised, sense – depends on the fact that each of the steps is perfectly clear to every reader, taking into account the extensions of meaning already effected in the previous stages. In mathematics, if one rejects a proof, it is more often because it is incomprehensible than because it is false. Generally this happens because the author, blinded in some way by the vision of his discovery, has made unduly optimistic assumptions about shared backgrounds. A little later his colleagues will make explicit that which the author had expressed implicitly, and by filling in the gaps will make the proof complete.

Atiyah (1984, 16 ff.):

If I'm interested in some topic then I just try to understand it; I just go on thinking about it and trying to dig down deeper and deeper. If I understand it, then I know what is right and what is not right.

Of course it is also possible that your understanding has been faulty, and you thought you understood it but it turns out eventually that you were wrong. Broadly speaking, once you really feel that you understand something and you have enough experience with that type of question through lots of examples and through connections with other things, you get a feeling for what is going on and what ought to be right. And then the question is: How do you actually prove it? That may take a long time ...

I don't pay very much attention to the importance of proofs. I think it is more important to understand something ...

I think ideally as you are trying to communicate mathematics, you ought to be trying to communicate understanding. It is relatively easy to do this in conversation. When I collaborate with people, we exchange ideas at this level of understanding – we understand topics and we ding to our intuition.

If I give talks, I try always to convey the essential ingredients of a topic. When it comes to writing papers or books, however, then it is much more difficult. I don't tend to write books. In papers I try to do as much as I can in writing an account and an introduction which gives the ideas. But you are committed to writing a proof in a paper, so you have to do that.

Most books nowadays tend to be too formal most of the time, they give too much in the way of formal proofs, and not nearly enough in the way of motivation and ideas. ...

I think it is very unfortunate that most books tend to be written in this overly abstract way and don't try to communicate understanding.

Long (1986, 616):

As far as the loss of certainty is concerned in itself (i.e., not as a historical-cultural phenomenon) it does not seem extraordinarily surprising or significant to me. I am far more puzzled by what "absolute certainty" might mean than by the fact that mathematics doesn't offer it. It seems to me that there is a similarity between this historical event (of 450 years duration) and the debating tactic which builds and then destroys a straw man.

Mathematics is a human creation. That does not mean that it is arbitrary, but it does mean that it would be immodest to expect it to be "certainly true" in the common sense of the phrase. It is incoherent to try and imagine mathematics as a source or body of absolutely certain knowledge. ...

Proof is a form of mathematical discourse. It functions to unite mathematicians as practitioners of one mathematics. ... a proof functions in mathematics only when it is accepted as a proof. This acceptance is a behavior of practicing mathematicians. ...

Fermat wrote that "the essence of proof is that it compels belief." To the extent that the compulsion operates via insight, (relatively) informal proofs will continue to play an important role in mathematics. Proofs that yield insight into the relevant concepts are more interesting and valuable to us as researchers and teachers than proofs that merely demonstrate the correctness of a result. We like a proof that brings out what seems to be essential. If the only available proof of a result is one that seems artificial or contrived it acts as an irritant. We keep looking and thinking. Instead of being able to move on, we are arrested. I mention these familiar facts only to emphasize that proof is not merely a system of links among various theorems, axioms, and definitions but also a system of discourse among people concerned with mathematics. As such it functions in a variety of ways.

From these first-hand informations we derive the following picture about the role of intuition and social agreement in elaborating and checking proofs in mathematics:

- (1) The validity of a proof does not depend, at least not only, on its formal presentation within a formal axiomatic-deductive setting, but on the intuitive coherence of conceptual relationships and their agreement with the experiences of the researchers.
- (2) The highly complex abstract theories of higher mathematics need a certain level of formal presentation for the sake of conceptual unambiguity and brevity. However, working with this formalism in a meaningful way presupposes an understanding of the communicative structures of the researchers working in the field and an intuitive understanding of the investigated objects. Any mathematical theory refers to a class of objects which can be represented in various ways and which via these representations become accessible for an operative study of their properties and relationships. Therefore mathematics is "quasi-empirical" (Lakatos 1963, 29 ff.; Jahnke 1978).
- (3) Proofs serve primarily for understanding *why* the theorem in question is true. During the process of creating and sharing understanding among researchers proofs (and theorems!) are elaborated, re-formulated, generalized, improved, formalized etc. Along this process criteria of rigour may change. "Absolutely rigorous" proofs do not exist.

3 The Elementary-Mathematics-Research-Program of Mathematics Education

The changing views on a proof in mathematics must be reflected in mathematics education in, as we believe, the following way:

- (1) The teaching and learning of mathematics in the social context of school has to be based on a context of understanding and a frame of communication different from those in university mathematics. To appropriately transfer proof activities into the boundary conditions of school, we must abandon formal axiomatic-deductive presentations of relevant mathematical theories in favour of sound informal presentations. These are characterized by embeddings into meaningful contexts, by emphasis on motivation, by the use of heuristic strategies and pre-formal means of representation and by informal proofs. “Save the phenomena!” must be the maxim of mathematics education.
- (2) Above all informal proofs can further understanding and therefore they have to be included into the *process* of learning and communication among students. Lakatos’ “Proofs and Refutations” (Lakatos 1969) may serve as a model.
- (3) The mathematical training of student teachers must contain informal courses in elementary mathematics in order to create a useful background for teaching. Comprehensive informal presentations of elementary mathematical theories are much more effective professional tools than background knowledge derived from formal presentations.

Within mathematics, formal presentations of elementary mathematics have a great tradition (cf. e.g. Lenz 1967; Griffith and Hilton 1976–1978) which we explicitly appreciate. But even if those presentations provide a lot of insight into mathematics teaching, from the point of view of this paper they are not sufficient. The literature of elementary mathematics, school mathematics, didactics of mathematics and the history of mathematics is full of informal approaches to certain problems, fields or even theories (cf. Sawyer 1964; Engel 1973/1976). To unify and to systematize these approaches, particularly by developing a “grammar” of iconic representations and concrete models, is in our view an extraordinarily important research problem of mathematics education which we would like to call the “elementary-mathematics-research-program” in mathematics education. The availability of comprehensive informal presentations of arithmetic, elementary algebra, elementary geometry, elementary stochastics and elementary analysis would lead to integrating mathematical, pedagogic, psychological and practical components of mathematics education and open a new level of didactical research and development and teacher training.

In order to indicate that this program points far beyond mathematics education we would like to close by quoting the mathematician Nowoshilow, member of the Soviet Academy of Science:

The closed and sterile formal mathematics is not only a “luxury” which civilization can afford [as stated by Dieudonné] but also an inevitable consequence of civilization. From this point of view the fight against the spread of mathematical formalism among human beings around the world is an ecological task.

(Epilogue in Blechman, Myschkis and Panovko 1984, 326).

References

- Andelfinger, B., Voigt, J.: Vorführstunden und alltäglicher Mathematikunterricht. Zur Ausbildung von Referendaren im Fach Mathematik (SI/SII). *Zentralblatt für Didaktik der Mathematik* **18**, 2–9 (1986)
- Aner, B. et al.: Beweisen im Mathematikunterricht - nur ein kognitives Problem?, pp. 19–27. Dörfler/Fischer (1979)
- Atiyah, M.: Interview with Michael Atiyah. *Math. Intell.* **6**, 9–19 (1984)
- Becker, O.: Grundlagen der Mathematik in geschichtlicher Entwicklung. Alber, Freiburg (1954)
- Blechman, I.I., Myschkis, A.D., Panovko, J.G.: *Angewandte Mathematik. Gegenstand, Logik, Besonderheiten*. Deutscher Verlag der Wissenschaften, Berlin (DDR) (1984)
- Branford, B.: *Betrachtungen über mathematische Erziehung vom Kindergarten bis zur Universität*. Teubner, Berlin (1913)
- Davis, Ph, Hersh, R.: *Erfahrung Mathematik*. Birkhäuser, Basel (1983)
- Dörfler, W., McLone, R. R.: Mathematics as a School Subject. In: Christiansen, B., Howson, G., Otte, M. (Hrsg.) *Perspectives on Mathematics Education*, pp. 49–97. Reidel, Dordrecht (1986)
- Engel, A.: *Wahrscheinlichkeitsrechnung und Statistik*. 2 Bände. Klett, Stuttgart (1973/1976)
- Freudenthal, H.: Was ist axiomatik, und welchen Bildungswert kann sie haben? *Der Mathematikunterricht* **9**, Heft 4, 5–19 (1963)
- Freudenthal, H.: *Mathematik als pädagogische Aufgabe*, vol. 1. Klett, Stuttgart (1973)
- Freudenthal, H.: Konstruieren, Reflektieren, Beweisen in phänomenologischer Sicht, pp. 183–200. In: Dörfler, W. and Fischer, R. (Hrsg.): *Beweisen im Mathematikunterricht*. Hölder-Pichler-Tempsky, Stuttgart (1979)
- Griffiths, H.B., Hilton, P.J.: *Klassische Mathematik in zeitgemäßer Darstellung*. 3 Bände. Vandenhoeck und Ruprecht, Göttingen (1976–1978)
- Hanna, G.: *Rigorous Proof in Mathematics Education*. Ontario Institute for Studies, Toronto (1983). In *Education*
- Hardy, G.H.: Mathematical proof. *Mind* **38**, Heft 149, 1–25 (1929)
- Jahnke, H.N.: Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik – Beweisen als didaktisches Problem. *IDM Materialien und Studien*, vol. 10. Institut für Didaktik der Mathematik, Bielefeld (1978)
- Jahnke, H.N.: Anschauung und Begründung in der Schulmathematik. *Beiträge zum Mathematikunterricht 1984*, pp. 32–41. Bad Salzdetfurth, Franzbecker (1984)
- Keitel, Ch., Otte, M.: Probleme der Profession und des professionellen Wissens des Mathematiklehrers. *Mathematisch-physikalische Semesterberichte* **26**, 154–176 (1979)
- Kirsch, A.: Beispiele für prämathematische Beweise, pp. 261–274. Dörfler/Fischer (1979)
- Lakatos, I.: *Beweise und Widerlegungen*. Vieweg, Braunschweig (1979)
- Lakatos, I.: *Mathematik, empirische Wissenschaft und Erkenntnistheorie*. Philosophische Schriften Band 2. Braunschweig, Vieweg (1982)
- Lenné, H.: *Analyse der Mathematikdidaktik in Deutschland*. Klett, Stuttgart (1969)
- Lenz, H.: *Grundlagen der Elementarmathematik*. Deutscher Verlag der Wissenschaften, Berlin (DDR) (1967)

- Long, R.L.: Remarks on the history and philosophy of mathematics. *Amer. Math. Mon.* **93**, 609–619 (1986)
- MacLane, S.: Mathematical models: a sketch for the philosophy of mathematics. *Amer. Math. Mon.* **88**, 462–472 (1981)
- Manin, J.I.: *A Course in Mathematical Logic*. Springer, New York (1977)
- Pickert, G.: Die Bedeutung der Anschauung für den mathematischen Beweis. *Der Mathematikunterricht* **3**, Heft 4, 49–62 (1957)
- Pickert, G.: Zum Irrationalitätsbeweis für Quadratwurzeln aus natürlichen Zahlen. *Der mathematische und naturwissenschaftliche Unterricht* **40**, 212–213 (1987)
- Rosser, J.B.: *Logic for Mathematicians*. McGraw-Hill, New York (1953)
- Sawyer, W.W.: *Vision in Mathematics*. Penguin Books, London (1964)
- Seeger, F., Steinbring, H.: Neue Anforderungen an die Tätigkeit der Fachleiter zwischen Allgemeinbildung, Wissenschaft und Unterricht. *Zentralblatt für Didaktik der Mathematik* **18**, 9–14 (1986)
- Semadeni, Z.: The Concept of Premathematics as a Theoretical Background for Primary Mathematics Teaching. *Polnische Akademie der mathematischen Wissenschaften, Warschau* (1974)
- Stein, M.: Kommentierte Bibliographie: Zum Beweisen im Mathematikunterricht. *Zentralblatt für Didaktik der Mathematik* **13**, 63–75 (1981)
- Stein, M.: Didaktische Beweiskonzepte. *Zentralblatt für Didaktik der Mathematik* **17**, 120–133 (1985)
- Steinbring, H.: Mathematische Begriffe in didaktischen Situationen: Das Beispiel der Wahrscheinlichkeit. *Journal für Mathematikdidaktik* **6**, 85–118 (1985)
- Steinbring, H.: Nature du savoir mathématique dans le pratique de l'enseignement. In: Laborde, C. (ed.), *Actes du Premier Colloque Franco-Allemand de Didactique des Mathématiques et de l'Informatique*. Ed., pp. 307–316. La Pensée Sauvage, Grenoble (1988)
- Thom, R.: Modern mathematics. Does it Exist? In: Howson, G. (Hrsg.) *Developments in Mathematical Education*, pp. 194–212. University Press, Cambridge (1973)
- Walther, G.: Action Proof versus Illuminating Examples? *Learn. Math.* **4**, Heft 3, 10–12 (1984)
- Wilder, R.L.: The nature of mathematical proof. *Amer. Math. Mon.* **51**, 309–323 (1944)
- Winter, H.: Zur Problematik des Beweisbedürfnisses. *Journal für Mathematikdidaktik* **4**, 59–95 (1983a)
- Winter, H.: Prämathematische Beweise der Teilbarkeitsregeln. *Mathematica Didactica* **6**, 177–187 (1983b)
- Wittmann, ECh.: *Elementargeometrie und Wirklichkeit. Einführung in geometrisches Denken*. Vieweg, Braunschweig (1987)

Open Access This chapter is licensed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license and indicate if changes were made.

The images or other third party material in this chapter are included in the chapter's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the chapter's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.